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THE GENERALIZED GEOMETRIC UNCERTAINTY PRINCIPLE FOR SPIN 1/2 SYSTEM

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ABSTRACT. Geometric Quantum Mechanics is a version of quantum theory that has been formulated in terms of Hamiltonian phase-space dynamics. The states in this framework belong to points in complex projective Hilbert space, the observables are real valued functions on the space, and the Hamiltonian flow is described by the Schrödinger equation. Besides, one has demonstrated that the stronger version of the uncertainty relation, namely the Robertson-Schrödinger uncertainty relation, may be stated using symplectic form and Riemannian metric. In this research, the generalized Robertson-Schrödinger uncertainty principle for spin $\frac{1}{2}$ system has been constructed by considering the operators corresponding to arbitrary direction.

1. INTRODUCTION

Quantum mechanics and classical mechanics are two main fundamental theories in physics that describe the behavior of physical object. While there is no question that quantum and classical descriptions are doing well in their own implementation scales, one would consider a smooth transition between these two descriptions to be feasible, at least at theoretically. However, there is a problem

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in order to carry out such transition since both theories are quite different in several aspects. For instance, the classical mechanics is based on geometry and most of the systems are non-linear whereas quantum mechanics is intrinsically formulated as algebraic and linear. The linearity seems to be necessary condition since none of standard quantum mechanics postulate can be stated without referring to it. This distinction is quite strange since in general, linear structure in physics arises as approximations to more accurate non-linear ones, but in this case the situation happens in opposite way.

This problem has motivate some physicists to develop a formulation that does not involve the quantization process as such but acknowledges quantum theory as provided. The research line referred as Geometric Quantum Mechanics was partly motivated by Kibble's work in 1979 [1] which demonstrates how quantum theory may be formulated in the language of Hamiltonian phase-space dynamics. Further examination in this framework reveals that the Hilbert space $\mathcal{H}$ is not a proper phase space, since any two state vectors $\Psi, \Phi \in \mathcal{H}$ are said to be physically equivalent $(\Psi \backsim \Phi)$ if $\Psi = \alpha\Phi$. Therefore, the true quantum phase space refers to the collection of rays that intersect at the origin in $\mathcal{H}$, i.e. $P(\mathcal{H}) := \mathcal{H}/ \backsim$ which is known as the complex projective Hilbert space for both finite and infinite dimensional $\mathcal{H}$. Besides, the fact that $\mathcal{H}$ is equipped with Hermitian inner product provides $P(\mathcal{H})$ with the structure of Kähler manifold (ω, g, j) where ω is non-degenerate, closed symplectic two-form, g is Riemannian metric and j is the compatible complex structure satisfying $j^2 = -1$ [2]. Therefore, similar to classical mechanics, the proper quantum phase space can also be considered as a symplectic manifold. In terms of observable, one can define a real valued function in $\mathcal{H}$ corresponds to a self adjoint operator through its expectation value that has well defined projection h to $P(\mathcal{H})$ [1]. This function then induces a flow along with its Hamiltonian vector field X_h [3] on Hilbert space. Deeper investigation shows that the flow is explicitly generated by the well known Schrödinger equation which means that the solution of this equation is perfectly represents the Hamiltonian flow on quantum phase space $P(\mathcal{H})$. Many researchers have contributed to formulate the geometric version of quantum mechanics and demonstrate the significant of this formulation in order to provides us with crucial information about quantum realm and various application in foundations of quantum mechanics such as uncertainty principle, entanglement and many others [4-6].

Despite the successful of quantum mechanics in terms of application, the true nature of this theory is still far from being understood. In other words, some of its principles and concepts are clearly counter-intuitive and very difficult to explain in simple language since most of them do not have classical analogue. One of the famous examples to describe the weirdness of quantum mechanics is the uncertainty principle. The effort to cast uncertainty principle in term of geometrical language appeared to become the subject of intense study in geometric quantum mechanics. One of earliest studies refers to the work of Anandan [7] who proposes a new geometric meaning of times-energy uncertainty principle for an arbitrary quantum system. After that, Ashtekar [2] has shown that for pure quantum state, the fact that the expectation values of observables correspond to the Riemannian and symplectic structure allow one to formulate a geometric version of Robertson-Schröodinger uncertainty relation. In this paper, we extend this work by constructing the generalized version of Robertson-Schrödinger uncertainty principle for spin $\frac{1}{2}$ system by considering the operators corresponding to arbitrary direction generated by rotation of the system operators. Specifically, in section 2 the derivation of the geometric Robertson-Schrödinger uncertainty relation is briefly discussed. The findings of the generalized geometric formulation of the uncertainty principle for the case of spin $\frac{1}{2}$ system is presented in Section 3. In section 4, the author's study results have been addressed and summarized.

2. ROBERTSON-SCHRÖDINGER UNCERTAINTY RELATION

One of the key ideas that demonstrates the strangeness of quantum mechanics is the uncertainty principle, which was initially established by German theoretical physicist Werner Heisenberg [8]. It imposed a restriction on the simultaneous measurement of complementary variables such as position and momentum with high accuracy. After that, Robertson [9] generalized the inequality to an arbitrary observables $\hat{A}$ and $\hat{B}$ given by

$$(\Delta\hat{A})(\Delta\hat{B}) \geq \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right| \tag{2.1}$$

and in the following year, Schrödinger [10] offered a stronger extension by including a covariance element in the formulation as follows

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|^2 + \left|\frac{1}{2}\langle[\hat{A},\hat{B}]_+\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2. \tag{2.2}$$

In geometrical version of quantum mechanics, Ashtekar [6] demonstrates that the Roberton-Schrödinger uncertainty principle (2.2) can be casted in terms of symplectic form Ω and Riemannian metric G in Hilbert space $\mathcal{H}$. Let us consider Ψ be a normalized state vector, then the inequality can be expressed as

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \left(\frac{\hbar}{2}\Omega(X_{\hat{A}},X_{\hat{B}})\right)^2 + \left(\frac{\hbar}{2}G(X_{\hat{A}},X_{\hat{B}}) - AB\right)^2, \tag{2.3}$$

where $(\Delta\hat{A})^2$ represents a function on $\mathcal{H}$ that is $(\Delta\hat{A})^2(\Psi) := (\Delta\hat{A})^2_\Psi$ and

$$X_{\hat{A}} = -\frac{i}{\hbar}\hat{A}\Psi, \quad X_{\hat{B}} = -\frac{i}{\hbar}\hat{B}\Psi \tag{2.4}$$

are Schrödinger vector fields. Besides, the Robertson-Schrödinger uncertainty principle may also be defined on the complex projective Hilbert space $P(\mathcal{H})$, i.e. the appropriate quantum phase space. Let consider the expectation values $A(\Psi)$ and $B(\Psi)$ of two quantum observables $\hat{A}$ and $\hat{B}$ respectively, and let a and b be the corresponding functions on $P(\mathcal{H})$, i.e.,

$$a \circ \Pi = \langle\hat{A}\rangle_\Psi = A(\Psi), \quad b \circ \Pi = \langle\hat{B}\rangle_\Psi = B(\Psi), \tag{2.5}$$

where Π is the canonical projection $\mathcal{H} \to P(\mathcal{H})$. Thus, for any $X_a = \Pi_*(X_{\hat{A}})$ and $X_b = \Pi_*(X_{\hat{B}})$ are elements of tangent vector at point ψ on $P(\mathcal{H})$ i.e. $T_\psi P(\mathcal{H})$, the uncertainty principle may be expressed in terms of symplectic form ω and Riemannian metric g defined on $P(\mathcal{H})$ as follows

$$(\Delta a)^2(\Delta b)^2 \geq \frac{\hbar^2}{4}(\omega(X_a,X_b)^2 + g(X_a,X_b)^2), \tag{2.6}$$

where $(\Delta a)^2(\psi) := (\Delta A)^2(\Psi)$ and $(\Delta b)^2(\psi) := (\Delta B)^2(\Psi)$.

3. The uncertainty principle of general spin $\frac{1}{2}$ operator

In this section, the generalized Robertson-Schrödinger uncertainty principle for spin $\frac{1}{2}$ system is constructed by considering the operators corresponding to arbitrary direction generated by rotation of operators:

$$\hat{S}_x = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}, \quad \hat{S}_y = \begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}, \quad \hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}.$$

Consider the operator $\hat{A}'$ which generated by a rotation by angle θ about the arbitrary axis $\hat{n}$ expressed as follows

$$\hat{A}' = U_{\hat{n}}(\theta)\hat{A}U_{\hat{n}}^{\dagger}(\theta), \tag{3.1}$$

where

$$U_{\hat{n}}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} - in_3\sin\frac{\theta}{2} & -in_1\sin\frac{\theta}{2} - n_2\sin\frac{\theta}{2} \\ -in_1\sin\frac{\theta}{2} + n_2\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_3\sin\frac{\theta}{2} \end{pmatrix}.$$

Applying equation (3.1) to operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ and we get

$$\hat{S}_u = U_{\hat{n}}(\theta)\hat{S}_x U_{\hat{n}}^{\dagger}(\theta) = \frac{\hbar}{2}\begin{pmatrix} \alpha_u & \beta_u \\ \overline{\beta}_u & -\alpha_u \end{pmatrix}, \tag{3.2}$$

$$\hat{S}_v = U_{\hat{n}}(\theta)\hat{S}_y U_{\hat{n}}^{\dagger}(\theta) = \frac{\hbar}{2}\begin{pmatrix} i\alpha_v & -i\beta_v \\ i\overline{\beta}_v & -i\alpha_v \end{pmatrix}, \tag{3.3}$$

$$\hat{S}_w = U_{\hat{n}}(\theta)\hat{S}_z U_{\hat{n}}^{\dagger}(\theta) = \frac{\hbar}{2}\begin{pmatrix} \alpha_w & \beta_w \\ \overline{\beta}_w & -\alpha_w \end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned}
\alpha_u &= -2n_2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 2n_1n_3\sin^2\frac{\theta}{2};\\
\alpha_v &= -2in_1\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 2in_2n_3\sin^2\frac{\theta}{2};\\
\alpha_w &= {n_3}^2\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - ({n_1}^2 + {n_2}^2)\sin^2\frac{\theta}{2};\\
\beta_u &= ({n_1}^2 - {n_2}^2 - {n_3}^2 - 2in_1n_2)\sin^2\frac{\theta}{2} - 2in_3\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2};\\
\beta_v &= [-({n_1}^2 - {n_2}^2 + {n_3}^2) + 2in_1n_2]\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2in_3\sin\frac{\theta}{2}\cos\frac{\theta}{2};\\
\beta_w &= 2(n_2 + in_1)\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 2(n_1n_3 - in_2n_3)\sin^2\frac{\theta}{2},
\end{aligned}$$

and $\overline{\beta}$ is refers to conjugate β. Note that, in this paper only one case of the Robertson-Schrödinger uncertainty principle with respect to $\hat{S}_u$ and $\hat{S}_v$ operators has been considered since other cases can be obtained under appropriate rotation. In order to calculate the uncertainty principle, firstly the Schrödinger vector field of these operators must have been calculated. Let us consider the

Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ and (e_1, e_2) represents the orthonormal basis in $\mathbb{C}^2$ satisfy

$$\langle e_\alpha | e_\beta \rangle = \delta_{\alpha\beta}.$$

Then the state of spin $\frac{1}{2}$ particle in $\mathcal{H}$ is expressed as

$$|\Psi\rangle = Z_1 |e_1\rangle + Z_2 |e_2\rangle \,. \tag{3.5}$$

The expectation value corresponding to these operators are

$$S_u(\Psi) = \langle \Psi | \hat{S}_u |\Psi\rangle = \frac{\hbar}{2}[\alpha_u(|Z_1|^2 - |Z_2|^2) + \beta_u Z_2 \bar{Z}_1 + \overline{\beta}_u Z_1 \bar{Z}_2]; \tag{3.6}$$

$$S_v(\Psi) = \langle \Psi | \hat{S}_v |\Psi\rangle = \frac{i\hbar}{2}[\alpha_v(|Z_1|^2 - |Z_2|^2) - \beta_v Z_2 \bar{Z}_1 + \overline{\beta}_v Z_1 \bar{Z}_2] \tag{3.7}$$

and according to equation (2.4) we get

$$X_{\hat{S}_u} |\Psi\rangle = \frac{dZ_1}{dt} |e_1\rangle + \frac{dZ_2}{dt} |e_2\rangle = -\frac{1}{i\hbar}(Z_1 \hat{S}_u |e_1\rangle + Z_2 \hat{S}_u |e_2\rangle); \tag{3.8}$$

$$X_{\hat{S}_v} |\Psi\rangle = \frac{dZ_1}{dt} |e_1\rangle + \frac{dZ_2}{dt} |e_2\rangle = -\frac{1}{i\hbar}(Z_1 \hat{S}_v |e_1\rangle + Z_2 \hat{S}_v |e_2\rangle), \tag{3.9}$$

where

$$\langle e_1 | \hat{S}_u |e_1\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_u & \beta_u \\ \overline{\beta}_u & -\alpha_u \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\alpha_u;$$

$$\langle e_1 | \hat{S}_u |e_2\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_u & \beta_u \\ \overline{\beta}_u & -\alpha_u \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\beta_u;$$

$$\langle e_1 | \hat{S}_v |e_1\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_v & -i\beta_v \\ \overline{\beta}_v & -i\alpha_v \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{i\hbar}{2}\alpha_v;$$

$$\langle e_1 | \hat{S}_v |e_2\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_v & -i\beta_v \\ \overline{\beta}_v & -i\alpha_v \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{i\hbar}{2}\beta_v.$$

Therefore

$$\langle e_1 | X_{\hat{S}_u} |\Psi\rangle = \frac{dZ_1}{dt} = -\frac{(\alpha_u Z_1 + \beta_u Z_2)}{2i}; \tag{3.10}$$

$$\langle e_1 | X_{\hat{S}_v} |\Psi\rangle = \frac{dZ_1}{dt} = \frac{\beta_v Z_2 - \alpha_v Z_1}{2}. \tag{3.11}$$

and in the same way one obtain

$$\langle e_2| X_{\hat{S}_u} |\Psi\rangle = \frac{dZ_2}{dt} = -\frac{(\overline{\beta}_u Z_1 - \alpha_u Z_2)}{2i}; \tag{3.12}$$

$$\langle e_2| X_{\hat{S}_v} |\Psi\rangle = \frac{dZ_2}{dt} = \frac{\alpha_v Z_2 - \overline{\beta}_v Z_1}{2}. \tag{3.13}$$

Besides $\frac{dZ_1}{dt}$ and $\frac{dZ_2}{dt}$, the calculation on $\frac{d\bar{Z}_1}{dt}$ and $\frac{d\bar{Z}_2}{dt}$ is needed since here $\mathbb{C}^2$ is assumed as a complexification of real vector space $\mathbb{R}^4$. Thus the complexified tangent space is spanned by 4 vectors; $\frac{\partial}{\partial Z_1}, \frac{\partial}{\partial Z_2}, \frac{\partial}{\partial \bar{Z}_1}, \frac{\partial}{\partial \bar{Z}_2}$. Now let $\langle \Psi| = \bar{Z}_1 \langle e_1| + \bar{Z}_2 \langle e_2|$ be a state in dual Hilbert space $\mathcal{H}^*$, then from equation (2.4) one finds that

$$X_{\hat{S}_u} \langle \Psi| = \frac{d\bar{Z}_1}{dt} \langle e_1| + \frac{d\bar{Z}_2}{dt} \langle e_2| = \frac{1}{i\hbar}(\bar{Z}_1 \hat{S}_u \langle e_1| + \bar{Z}_2 \hat{S}_u \langle e_2|); \tag{3.14}$$

$$X_{\hat{S}_v} \langle \Psi| = \frac{d\bar{Z}_1}{dt} \langle e_1| + \frac{d\bar{Z}_2}{dt} \langle e_2| = \frac{1}{i\hbar}(\bar{Z}_1 \hat{S}_v \langle e_1| + \bar{Z}_2 \hat{S}_v \langle e_2|). \tag{3.15}$$

Then it is clear

$$\langle \Psi| X_{\hat{S}_u} |e_1\rangle = \frac{d\bar{Z}_1}{dt} = \frac{(\alpha_u \bar{Z}_1 + \overline{\beta}_u \bar{Z}_2)}{2i}; \tag{3.16}$$

$$\langle \Psi| X_{\hat{S}_v} |e_1\rangle = \frac{d\bar{Z}_1}{dt} = \frac{(\alpha_v \bar{Z}_1 + \overline{\beta}_v \bar{Z}_2)}{2}. \tag{3.17}$$

and

$$\langle \Psi| X_{\hat{S}_u} |e_2\rangle = \frac{d\bar{Z}_2}{dt} = \frac{\beta_u \bar{Z}_1 - \alpha_u \bar{Z}_2}{2i}; \tag{3.18}$$

$$\langle \Psi| X_{\hat{S}_v} |e_2\rangle = \frac{d\bar{Z}_2}{dt} = -\frac{(\beta_v \bar{Z}_1 + \alpha_v \bar{Z}_2)}{2}. \tag{3.19}$$

The Schrödinger vector fields correspond to $\hat{S}_u$ and $\hat{S}_v$ are

$$\begin{aligned} X_{\hat{S}_u} = & -\frac{(\alpha_u Z_1 + \beta_u Z_2)}{2i}\frac{\partial}{\partial Z_1} - \frac{(\overline{\beta}_u Z_1 - \alpha_u Z_2)}{2i}\frac{\partial}{\partial Z_2} + \frac{\alpha_u \bar{Z}_1 + \overline{\beta}_u \bar{Z}_2}{2i}\frac{\partial}{\partial \bar{Z}_1} \\ & + \frac{\beta_u \bar{Z}_1 - \alpha_u \bar{Z}_2}{2i}\frac{\partial}{\partial \bar{Z}_2}; \end{aligned} \tag{3.20}$$

$$X_{\hat{S}_v} = -\frac{(\alpha_v Z_1 - \beta_v Z_2)}{2}\frac{\partial}{\partial Z_1} + \frac{\alpha_v Z_2 - \overline{\beta}_v Z_1}{2}\frac{\partial}{\partial Z_2} + \frac{\alpha_v \bar{Z}_1 + \overline{\beta}_v \bar{Z}_2}{2}\frac{\partial}{\partial \bar{Z}_1} - \frac{(\beta_v \bar{Z}_1 + \alpha_v \bar{Z}_2)}{2}\frac{\partial}{\partial \bar{Z}_2}. \tag{3.21}$$

Let us define the Robertson-Schrödinger uncertainty relation for operator $\hat{S}_u$ and $\hat{S}_v$ as

$$(\Delta \hat{S}_u)^2(\Delta \hat{S}_v)^2 \geq \left(\frac{\hbar}{2}\Omega(X_{\hat{S}_u}, X_{\hat{S}_v})\right)^2 + \left(\frac{\hbar}{2}G(X_{\hat{S}_u}, X_{\hat{S}_v}) - S_u S_v\right)^2, \tag{3.22}$$

where the contraction of Schrödinger vector fields $X_{\hat{S}_u}$ and $X_{\hat{S}_v}$ with symplectic form Ω is

$$\Omega(X_{\hat{S}_u}, X_{\hat{S}_v}) = \frac{1}{4}\left[(\overline{\beta}_u \beta_v + \beta_u \overline{\beta}_v)(|Z_2|^2 - |Z_1|^2) + 2(\alpha_u \overline{\beta}_v - \overline{\beta}_u \alpha_v) Z_1 \bar{Z}_2 + 2(\alpha_u \beta_v + \beta_u \alpha_v) Z_2 \bar{Z}_1\right] \tag{3.23}$$

and the component of Riemannian metric corresponds to $X_{\hat{S}_u}$ and $X_{\hat{S}_v}$ can be written as

$$G(X_{\hat{S}_u}, X_{\hat{S}_v}) = \frac{i\hbar}{4}(2\alpha_u \alpha_v - \overline{\beta}_u \beta_v + \beta_u \overline{\beta}_v)(|Z_1|^2 - |Z_2|^2). \tag{3.24}$$

Thus, the Robertson-Schrödinger uncertainty principle (3.22) can be expressed as

$$\begin{aligned}(\Delta \hat{S}_u)^2(\Delta \hat{S}_v)^2 \geq & \left[\frac{1}{8}(\overline{\beta}_u \beta_v + \beta_u \overline{\beta}_v)(|Z_2|^2 - |Z_1|^2)\right]^2 \\ & + \left[\frac{1}{4}(\alpha_u \overline{\beta}_v - \overline{\beta}_u \alpha_v) Z_1 \bar{Z}_2 + \frac{1}{4}(\alpha_u \beta_v + \beta_u \alpha_v) Z_2 \bar{Z}_1\right]^2 \\ & + \left[\frac{i\hbar^2}{8}(2\alpha_u \alpha_v - \overline{\beta}_u \beta_v + \beta_u \overline{\beta}_v)(|Z_1|^2 - |Z_2|^2)\right]^2.\end{aligned} \tag{3.25}$$

Note that, any state vector $\Psi, \Phi \in \mathcal{H}$ such that $\Phi = c\Psi, c \in \mathbb{C}$ defines the same physical state. Thus, it is necessary to find the expression of Robertson-Schrödinger uncertainty principle in $\mathbb{C}P^1$ which is the quantum phase space of spin $\frac{1}{2}$ particle.

In order to compute the Robertson-Schrödinger uncertainty principle on $\mathbb{C}P^1$, we need to find the pushforward vector fields of $X_{\hat{S}_u}$ and $X_{\hat{S}_v}$ under the map $\Pi_* : T_\Psi \mathcal{H} \to T_\psi P(\mathcal{H})$. Let $\Pi(Z_1, Z_2) = z = \frac{Z_2}{Z_1}$ be a local coordinate of $U_1 \subset \mathbb{C}P^1$

where $Z_1 \neq 0$. Then the pushforward vector fields correspond to these vector fields are

$$\Pi_* X_{\hat{S}_u} = \frac{1}{2i}(\beta_u z^2 + 2\alpha_u z - \overline{\beta}_u)\frac{\partial}{\partial z} - \frac{1}{2i}(\overline{\beta}_u \bar{z}^2 + 2\alpha_u \bar{z} - \beta_u)\frac{\partial}{\partial \bar{z}}; \tag{3.26}$$

$$\Pi_* X_{\hat{S}_v} = -\frac{1}{2}(\beta_v z^2 - 2\alpha_v z + \overline{\beta}_v)\frac{\partial}{\partial z} - \frac{1}{2}(\overline{\beta}_v \bar{z}^2 + 2\alpha_v \bar{z} + \beta_v)\frac{\partial}{\partial \bar{z}}. \tag{3.27}$$

Therefore, the Robertson-Schrödinger uncertainty relation for the case of $\hat{S}_u$ and $\hat{S}_v$ in $\mathbb{C}P^1$ is

$$(\Delta\hat{S}_u)^2(\Delta\hat{S}_v)^2 \geq \frac{\hbar^2}{4}[\omega(\Pi_* X_{\hat{S}_u}, \Pi_* X_{\hat{S}_v})^2 + g(\Pi_* X_{\hat{S}_u}, \Pi_* X_{\hat{S}_v})^2] \tag{3.28}$$

where the contraction of Schrödinger vector field $X_{\hat{S}_u}$ and $X_{\hat{S}_v}$ with symplectic form ω can be written as

$$\begin{aligned}\omega(\Pi_* X_{\hat{S}_u}, \Pi_* X_{\hat{S}_v}) = {} & \frac{\hbar[(\beta_u\overline{\beta}_v + \overline{\beta}_u\beta_v)(|z|^4 - 1) + 2(\alpha_u\beta_v + \beta_u\alpha_v)(z[1 + z\bar{z}])]}{4(1+|z|^2)^2} \\ & + \frac{\hbar[2(\alpha_u\overline{\beta}_v - \overline{\beta}_u\alpha_v)(\bar{z}[1 + z\bar{z}])]}{4(1+|z|^2)^2}\end{aligned} \tag{3.29}$$

and the component of Riemannian metric associated with $X_{\hat{S}_u}$ and $X_{\hat{S}_v}$ is denoted by

$$\begin{aligned}g(\Pi_* X_{\hat{S}_u}, \Pi_* X_{\hat{S}_v}) = {} & \frac{i\hbar[(\beta_u\overline{\beta}_v + \overline{\beta}_u\beta_v)|z|^4 - 2z(\alpha_u\beta_v + \beta_u\alpha_v)(z\bar{z} - 1)]}{4(1+|z|^2)^2} \\ & + \frac{i\hbar[2\bar{z}(\alpha_u\overline{\beta}_v - \overline{\beta}_u\alpha_v)(z\bar{z} - 1) - (\overline{\beta}_u\beta_v + \beta_u\overline{\beta}_v)]}{4(1+|z|^2)^2} \\ & + \frac{i\hbar[2\beta_u\beta_v z^2 - 2\overline{\beta}_u\overline{\beta}_v\bar{z}^2 + 8\alpha_u\alpha_v z\bar{z}]}{4(1+|z|^2)^2}.\end{aligned} \tag{3.30}$$

4. CONCLUSION

In this study, the generalized geometric Robertson-Schrödinger uncertainty relation for spin $\frac{1}{2}$ system has been constructed. We shows that, the uncertainty principle (3.28) may reduced to the standard expression of spin $\frac{1}{2}$'s uncertainty principle when rotated at certain angles. For example, let take $n_1 = 0$, $n_2 = 1$, $n_3 = 0$ and $\theta = 90^0$. Then, one gets $\alpha_u = -1$, $\beta_u = 0$, $\alpha_v = 0$ and $\beta_v = 1$ which implies that the equation (3.28) is identical to the uncertainty principle for the case of $\hat{S}_y$ and $\hat{S}_z$ operators.

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