Advances in Mathematics: Scientific Journal **10** (2021), no.9, 3263–3272
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.9.15

THREE-DIMENSIONAL STAGNATION-POINT FLOW OVER AN UNSTEADY PERMEABLE SHRINKING SURFACE

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ABSTRACT. An analysis is carried out to theoretically investigate the unsteady three dimensional stagnation-point of a viscous flow over a permeable stretching/shrinking sheet. A similarity transformation is used to reduce the governing system of nonlinear partial differential equations to a set of nonlinear ordinary (similarity) differential equations, which are then solved numerically using the `bvp4c` function in MATLAB. Results show that multiple solutions exist for a certain range of unsteadiness and stretching/shrinking parameters. The effects of the governing parameters on the skin friction coefficients and the velocity profiles are presented and discussed.

1. INTRODUCTION

A stagnation point is a point in a flow field where the local velocity of the fluid is brought to rest, where it encounters the highest pressure, the highest heat transfer and the highest rates of mass deposition. The study of the stagnation-point flow is important due to its wide range of applications in many industrial manufacturing processes, such as aerodynamic extrusions of plastic sheet, the

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2020 *Mathematics Subject Classification.* 34B15, 65L10, 76-10, 76D05.
Key words and phrases. Stagnation-point flow, Unsteady, Dual solutions, Permeable shrinking sheet.
Submitted: 23.07.2021; *Accepted:* 26.08.2021; *Published:* 30.09.2021.

cooling and drying of papers and textiles, glass blowing and continuous casting and spinning of fibers. Early classical works include; the two dimensional stagnation point flow impinging normally on a fixed flat plate by Hiemenz [1]; the three-dimensional flow near a stagnation point by Howarth [2]; and the unsteady viscous flow in the vicinity of a stagnation point by Rott [3].

Unlike the stretching sheet where the velocity is moving away from the origin, the velocity on the boundary is moving towards a fixed point. The flow induced by a shrinking sheet shows physical phenomena quite distinct from the forward stretching flow. This phenomena can be found, for example, on a rising and shrinking balloon. It is found that there are two conditions that the flow towards the shrinking sheet is likely to exist: by imposing an adequate suction on the boundary; or by considering the stagnation flow [4]. Since then, the problems of stagnation flow towards a shrinking sheet are extended and investigated for various type of fluids and various physical properties. Some of the published papers regarding the unsteady stagnation-point flow towards a stretching/shrinking sheet worth mentioning are; [5–9] and very recently by Anuar and Bachok [10].

In this paper, we extend the work done in [4] to the problem of three dimensional stagnation-point flow over an unsteady shrinking sheet with the addition of suction effect. The governing partial differential equations are first transformed into a system of ordinary differential equations, and then solved numerically by using the `bvp4c` function. Our attention is devoted to obtaining the numerical results for the critical points which define the range of the existence of the dual solutions.

2. Mathematical Formulation

Consider the unsteady forced convection stagnation-point flow of an incompressible viscous fluid over a permeable stretching/shrinking surface, where x, y and z are the Cartesian coordinates. We assume that the velocities on the stretching/shrinking surface are $u_w(x,t) = \frac{b(x+c)}{1-\alpha t}$ and $w(t) = \frac{w_0}{\sqrt{1-\alpha t}}$, where b is the stretching rate (shrinking if $(b < 0)$), $-c$ is the location of the stretching/shrinking origin, w_0 is the mass transfer velocity with $w_0 < 0$ for suction and $w_0 > 0$ for injection, and α is positive or negative constant. Notice that the stretching/shrinking axis and the stagnation flow are, in general, not aligned. It is also assumed that the components of the inviscid (potential) flow are $u_e(x) = \frac{ax}{1-\alpha t}$ and $w_e(z) = -\frac{az}{1-\alpha t}$, where a is the constant positive strength of the stagnation flow. Based on these assumptions, the basic unsteady Navier-Stokes equations of the problem in the usual notations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \tag{2.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v, \tag{2.3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w, \tag{2.4}$$

subject to the initial and boundary conditions

$$\begin{aligned} &t < 0: \quad v = 0, \quad u = 0, \quad w = 0 \text{ for any } x, y, z, \\ &t \geq 0: \quad \left\{ \begin{aligned} &v = v_w(t), \quad u = u_w(x,t) = \frac{1}{1-\alpha t} b\,(x + c)\,, \\ &w(t) = w_w(t) = \frac{w_0}{\sqrt{1-\alpha t}} \end{aligned} \right\} \quad \text{at } z = 0, \\ &\qquad\quad u \to 0, \quad w \to 0 \quad \text{as } z \to \infty. \end{aligned} \tag{2.5}$$

Here $(u,\, v,\, w)$ are the velocity components along $(x,\, y,\, z)$ directions, p is the pressure, ν is the kinematic viscosity, ρ is the density and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian in the Cartesian coordinates $(x,\, y,\, z)$.

Velocity of the inviscid free stream are defined as $u = u_e(x,t)$, $v = v_e(y,t)$ and $w = w_e(z,t)$ [5, 7]. Therefore, momentum equations (2.2)-(2.4) can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} + \nu\nabla^2 u, \tag{2.6}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v_e}{\partial t} + v_e\frac{\partial v_e}{\partial y} + \nu\nabla^2 v, \tag{2.7}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w_e}{\partial t} + w_e\frac{\partial w_e}{\partial z} + \nu\nabla^2 w. \tag{2.8}$$

We look now for a similarity solution of Eqs. (2.1) and (2.6)-(2.8) subjected to the initial and boundary condition (2.5) of the following form

$$\begin{aligned} u &= \frac{1}{1-\alpha t}\left[axf'(\eta) + bcg(\eta)\right], \ v = 0, \ w = -\sqrt{\frac{a\nu}{1-\alpha t}}f(\eta), \\ \eta &= \sqrt{\frac{a}{\nu(1-\alpha t)}}z, \end{aligned} \tag{2.9}$$

where prime denotes differentiation with respect to η. Thus, from (2.5) and (2.9), we have

$$w_w(t) = \sqrt{\frac{a\nu}{1-\alpha t}}s, \tag{2.10}$$

with the constant s being defined as $s = -w_0/\sqrt{a\nu}$. Thus, $s > 0$ and $s < 0$ correspond to suction and injection, respectively.

Using (2.9), (2.1) is automatically satisfied, while (2.6)-(2.8) are reduced to the following ordinary (similarity) differential equations

$$f''' + ff'' + 1 - f'^2 + \beta\left(1 - f' - \frac{\eta}{2}f''\right) = 0, \tag{2.11}$$

$$g'' + fg' - f'g - \beta\left(g + \frac{\eta}{2}\right) = 0, \tag{2.12}$$

while the boundary conditions (2.5) become

$$\begin{aligned} &f(0) = s, \ f'(0) = \frac{b}{a} = \lambda, \ g(0) = 1, \\ &f'(\eta) \to 1, \ g(\eta) \to 0 \quad \text{as } \eta \to \infty, \end{aligned} \tag{2.13}$$

where $\beta = \frac{\alpha}{a}$ is the unsteadiness parameter with $\beta < 0$ for decelerating stretching/shrinking sheet. It is worth mentioning that the equations (2.11)-(2.13) are identical with those reported in [4] for the cases of steady flow $(\beta = 0)$ and impermeable $(s = 0)$.

The skin friction coefficient is defined as $C_f = \frac{\tau_w}{\rho u_e^2}$, where $\tau_w = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0}$. Using these informations, we then have

$$Re_x^{1/2}C_f = f''(0) + \frac{bc}{ax}g'(0), \tag{2.14}$$

where $Re_x = u_e x/\nu$ is the local Reynolds number.

3. RESULTS AND DISCUSSION

The system of ordinary differential equations (2.11) and (2.12) along with the boundary conditions (2.13) was solved numerically using the bvp4c function in MATLAB for several values of the governing parameters.To verify the accuracy of the present method, we made a comparison for the initial values $f''(0)$ and $g'(0)$ with those reported in [4]. The comparison, which displayed in Table 1, are found to be in a very good agreement, and thus we are confident that the present method and results are accurate.

TABLE 1. Values of $f''(0)$, $g'(0)$ and $h'(0)$ for different values of λ when $\beta, s = 0$ [$h'(0)$ in [4] are equivalent to $g'(0)$ in this study]

	[4]		Present study	
λ	$f''(0)$	$h'(0)$	$f''(0)$	$g'(0)$
-1.24658	–	–	0.56934	0.97298
-1.15	1.08223	0.297995	1.08223	0.297995
	[0.116702]	[0.276345]	[0.116702]	[0.276345]
-1	1.32882	0	1.32882	0

Note: Lower branch solutions are given in parenthesis.

The existence of dual solutions in this problem are illustrated in Figs. 1 and 2. From these figures, it can be seen that dual (upper and lower branch) solutions exist for a certain range of $\beta(< 0)$. The upper and lower branch solutions are represented with solid and dashed lines, respectively. In addition, when β is equal to a certain value, say $\beta = \beta_c$, where β_c is the critical values (and turning points) of β, there is only one (unique) solution. There is no solution when $\beta < \beta_c$, due to the separation of boundary layer from the surface, and thus the solution based on boundary layer approximations is no longer applicable. To obtain further solutions, one need to consider the full Navier-Stokes equations, which is beyond the scope of this paper.

Figs. 1 and 2 show the variations of the reduced skin friction coefficients $f''(0)$ and $g'(0)$ with unsteadiness parameter β for some values of suction parameter s when $\lambda = -1$. It is observed that with the increase of s, the solution domain expands with the critical value β_c moving further to the left. The wall shear stress $f''(0)$ in Fig. 2 is seen to decrease with the increase of $|\beta|$ and

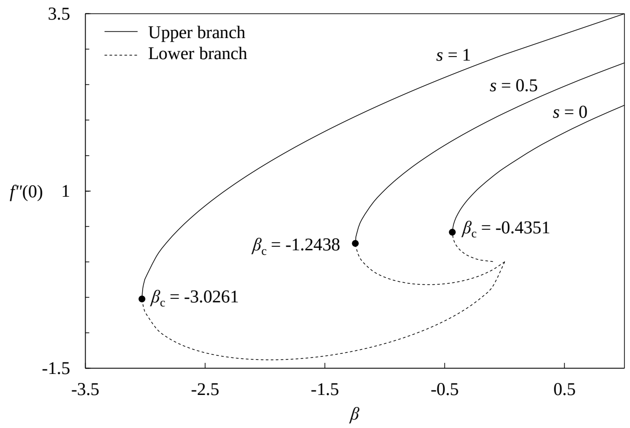

FIGURE 1. $f''(0)$ versus β for different values of s when $\lambda = -1$

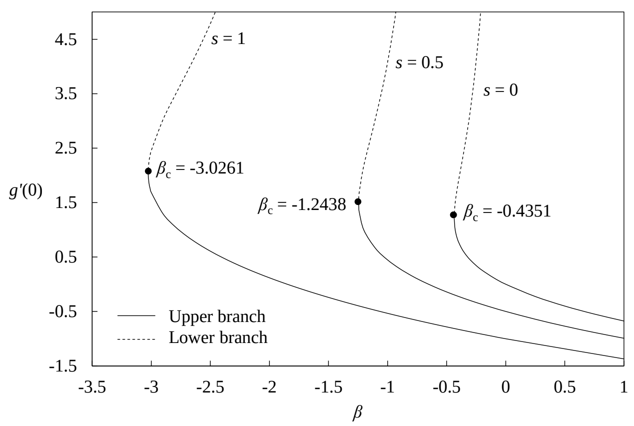

FIGURE 2. $g'(0)$ versus β for different values of s when $\lambda = -1$

s for the upper branch solution. At a certain value of β, $f''(0)$ becomes zero and continue decreases to negative, which implies that there is velocity overshoot near the shrinking sheet with the velocity in the fluid higher than the wall speed.However, in some cases of small s, $f''(0)$ can be positive for both of the solution branches, which in this case, is when $s = 0$ and $s = 0.5$. Meanwhile, in Fig. 2, it is seen that the values of $g'(0)$ for upper branch solution increase gradually with the increase of $|\beta|$ until it reaches the critical (turning) point β_c, and then the lower branch solution increases rapidly after the turning point. Also, it is seen that the values of $|\beta_c|$ increase with the increase of s. Hence, suction or blowing parameter widens the range of λ for which solutions exist.

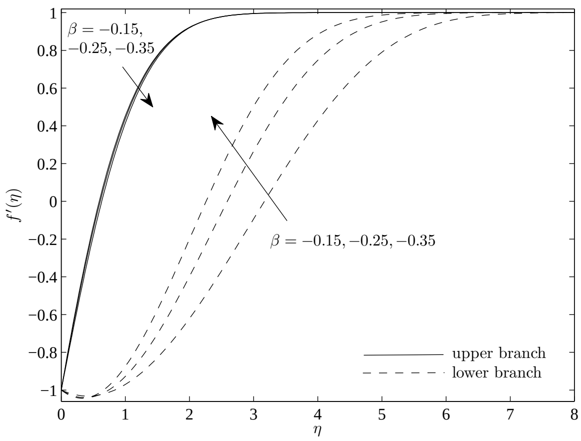

FIGURE 3. Velocity profiles $f'(\eta)$ for different values of β when $s = 0.5$ and $\lambda = -1$

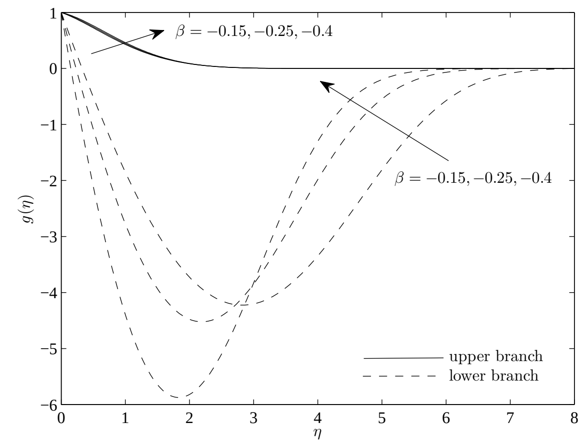

FIGURE 4. Velocity profiles $g(\eta)$ for different values of β when $s = 0.5$ and $\lambda = -1$

The influence of the unsteadiness parameter β on the velocity profiles $f'(\eta)$ and $g(\eta)$ are shown in Figs. 3 and 4, respectively. For the upper branch solution, the increase of $|\beta|$ leads to the increase of the boundary layer thickness and the decrease of the velocity gradient at the wall. The lower branch solution, however, displays the opposite effect from the upper branch solution.

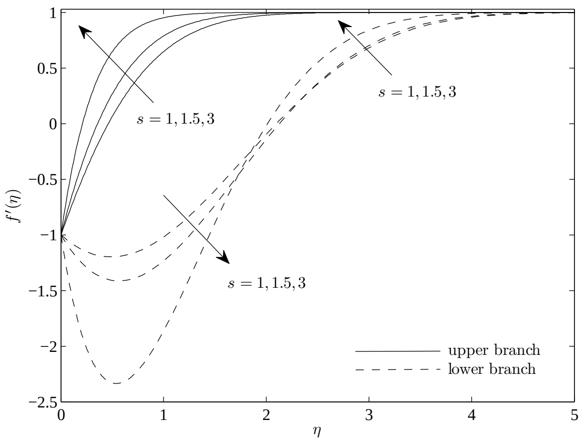

FIGURE 5. Velocity profiles $f'(\eta)$ for different values of s when $\beta = -0.5$ and $\lambda = -1$

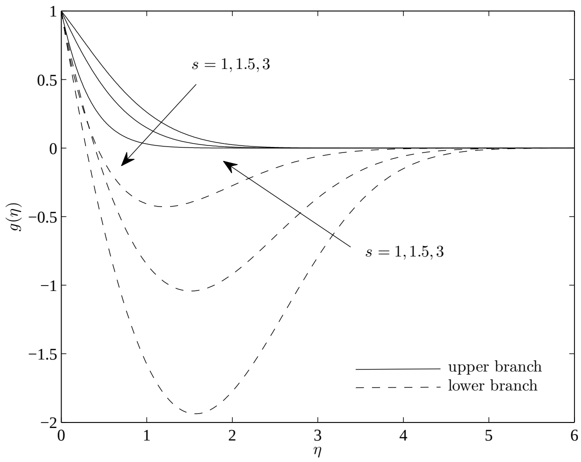

FIGURE 6. Velocity profiles $g(\eta)$ for different values of s when $\beta = -0.5$ and $\lambda = -1$

Meanwhile, the influence of the suction parameter s on the velocity profiles $f'(\eta)$ and $g(\eta)$ are shown in Figs. 5 and 6, respectively. It is seen that the increase of s decreases the boundary layer thickness and increases the velocity gradient at the wall. This happened because suction $(s > 0)$ reduces drag force in order to avoid boundary layer separation.

4. Conclusion

The problem of unsteady three-dimensional stagnation-point flow of a viscous fluid over a permeable stretching/shrinking sheet is solved numerically using the `bvp4c` function in MATLAB. The effects of the unsteadiness parameter β, suction or blowing parameter s and stretching/shrinking parameter λ have been analyzed and presented. Dual solutions are found for a certain range of unsteadiness and stretching/shrinking parameters. Suction or blowing parameter widens the range of stretching/shrinking parameter for which similarity solutions exist. The increase of unsteadiness and stretching/shrinking parameters increase the initial values $f''(0)$ but decrease the initial values $g'(0)$. The boundary layer thickness increases with the increase of unsteadiness and stretching/shrinking parameters, while decreases with the increase of suction or blowing parameter.

ACKNOWLEDGMENT

This work was supported by research grant (GP-IPM/2018/9619000) from Universiti Putra Malaysia, Malaysia.

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