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A FOURTH ORDER ENERGY STABLE FINITE DIFFERENCE SCHEME FOR A TIMOSHENKO BEAM EQUATIONS WITH LOCALLY DISTRIBUTED FEEDBACK

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ABSTRACT. This work deals with the numerical solution of a control problem governed by the Timoshenko beam equations with locally distributed feedback. We apply a fourth-order Compact Finite Difference (CFD) approximation for the discretizing spatial derivatives and a Forward second order method for the resulting linear system of ordinary differential equations. Using the energy method, we derive energy relation for the continuous model, and design numerical scheme that preserve a discrete analogue of the energy relation. Numerical results show that the CFD approximation of fourth order give an efficient method for solving the Timoshenko beam equations.

1. INTRODUCTION

The boundary and internal damping of Timoshenko beam are interesting problems and have recently attracted much attention with the rapid development of high technology such as space science and flexible robots. A number of authors [3], [2] and [4] have considered control problems associated with the Timoshenko beam and obtained many interesting results. At the same time, the finite

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element method and the finite difference method are effectively applied to the Timoshenko beam for solving the boundary stabilization and the computation of the numerical solution [1]. If the cross section dimension of the beam is negligible in comparison with its length, the transversal vibration of an elastic beam is described by the Euler-Bernoulli beam equations. Many studies performed in this cas and various analytical and numerical method in frequency as well as time domaine have been developed for modeling and numerical solutions of this equations (see [11], [12] and [13]). If the cross-section dimension is not negligible, then it is necessary to consider the effect of the rotatory inertia. If the deflection due to shear is not negligible either, then the transversal vibration is described by the so-called Timoshenko beam equations. A basic linear model, developed in [6], for describing the transverse vibration of beams is given by two coupled partial differential equations

$$\begin{cases} \rho w_{tt} - K(w_x - \varphi)_x = 0 \\ I_\rho \varphi_{tt} - EI\varphi_{xx} - K(w_x - \varphi) = 0 \end{cases} \text{on } (0, L) \times \mathbb{R}^+, \tag{1.1}$$

where, t is the time variable, x the space coordinate and L is the length of the beam in its equilibrium position. The function w is the transverse displacement and φ is the rotation angle of a filament of the beam. The coefficients ρ, I_ρ, EI and K are respectively mass density, moment of mass inertia, rigidity coefficient and the shear modulus of elasticity.

In this paper, we develop and analyse a compact finite difference scheme for the following Timoshenko beam equations with unique locally distributed feedback, given by

$$\begin{cases} \rho w_{tt} - K(w_x - \varphi)_x = f(x,t) \\ I_\rho \varphi_{tt} - EI\varphi_{xx} - K(w_x - \varphi) - \mu\varphi_t = g(x,t) \end{cases} \text{on } (0, L) \times \mathbb{R}^+, \tag{1.2}$$

with initial conditions

$$w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x) \tag{1.3}$$

and the boundary conditions

$$w(0,t) = 0,\ \varphi(0,t) = 0,\ w(L,t) = 0,\ \varphi(L,t) = 0, \tag{1.4}$$

where the functions $w_0,\ w_1,\ \varphi_0$ and φ_1 belongs to a suitable functional space, and μ is a positive constant. The boundary conditions in (1.4) mean that the beam is clamped at the two ends of $(0, L)$.

A. Soufyane and al. in [7] studied the exponential stability of system (1.2)-(1.4) by using energy multiplier method. Brandts in [10] studied the discretization by the mixed finite element and obtained the superconvergence of the mixed finite element solutions to projections of the real solutions on the approximating spaces in the global H^1-norm uniform in d. Li in [2] studied a finite difference of the Timoshenko beam equations with boundary feedback scheme. It is proved by the discrete energy method that the scheme is uniquely solvable, unconditionally stable and second order convergent in L^∞ norm. In this paper we propose a numerical scheme based on the finite difference method for solving system (1.2)-(1.4), which is energy stable and has the fourth-order accuracy in space and two-order in time components. We use the CFD approximation of fourth-order for discretizing spatial derivatives of Timoshenko beam equations and a Forward second order method for the resulting linear system of ordinary differential equations.

The remainder of the rest of the article is arranged as follows. In section 2, we introduce the fourth order finite difference space discretization for the Timoshenko system then we establishe the stability of the semi-discretized problem. In section 3, we present the fully discretization and we investigate the energy stability of the proposed method. Some numerical experiments are presented in section 4, to illustrate the efficiency of our scheme. Finally, a conclusion is drawn in the last section.

2. THE SEMI-DISCRETIZED PROBLEM

In this section, we proceed as in [8], [9] to construct a fourth order finite difference scheme for the Timoshenko system (1.2) then we establishe the energy stability.

For positive integer N, let $h := \frac{L}{N}$ be a uniform mesh step size, and define for $i \in \{0, 1, \ldots, N\}$, $x_i = ih$. For $i \in \{0, 1, \ldots, N-1\}$, define $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h$. The primal mesh, $\mathcal{M}_p$ is defined as: $\mathcal{M}_p = (x_i),\ i \in \{0, 1, \ldots, N\}$ and the dual mesh, $\mathcal{M}_d$ is defined as $\mathcal{M}_d = (x_{i+\frac{1}{2}}),\ i \in \{0, 1, \ldots, N-1\}$. On the primal mesh $\mathcal{M}_p$, we define the discrete space:

$$V_h^0 := \left\{ v_h := (v_i)_i,\ i = 0, \ldots, N : \|v_h\|_{0,h} := h \sum_{i=0}^{N} v_i^2 < \infty \right\} + \text{ periodic BC.}$$

On the dual mesh $\mathcal{M}_d$, we define the discrete space:

$$V_h^{\frac{1}{2}} := \left\{ u_h := (u_{i+\frac{1}{2}})_i,\ i = 0, \dots, N-1 : \|v_h\|_{\frac{1}{2},h} := h \sum_{i=0}^{N} v_{i+\frac{1}{2}}^2 < \infty \right\}$$
$$+ \text{ periodic BC}.$$

We denote by $\|\ \|_{0,h}$ and $\|\ \|_{\frac{1}{2},h}$ respectively the l^2 norms on V_h^0 and $V_h^{\frac{1}{2}}$, are derived from corresponding l^2 scalar products $< .,. >_{0,h}$ and $< .,. >_{\frac{1}{2},h}$. We define the fourth order difference operators $\delta_x : V_h^0 \to V_h^{\frac{1}{2}}$ and $\delta_x^* : V_h^{\frac{1}{2}} \to V_h^0$ such that

$$\forall v_h \in V_h^0,\ (\delta_x v_h)_{i+\frac{1}{2}} := \frac{9}{8}\frac{v_{i+1} - v_i}{h} - \frac{1}{24}\frac{v_{i+2} - v_{i-1}}{h}, \tag{2.1}$$

$$\forall u_h \in V_h^{\frac{1}{2}},\ (\delta_x^* u_h)_i := \frac{9}{8}\frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{h} - \frac{1}{24}\frac{u_{i+\frac{3}{2}} - u_{i-\frac{3}{2}}}{h}, \tag{2.2}$$

Operators (2.1)-(2.2) are the fourth-order discrete approximations of the operator ∂_x with mesh size h. We note that under the assumption of periodic boundary conditions, the opperator δ_x^* is the adjoint of the discrete operator δ_x with respect to the l^2 scalar product.

The semi-discrete problem of the Timoshenko system (1.2) reads: Find $w_h \in V_h^0$ and $\varphi_h \in V_h^{\frac{1}{2}}$ such that

$$\begin{cases} \dfrac{\rho}{K}\partial_{tt} w_h - \delta_x^*\delta_x w_h + \delta_x^*\varphi_h = f_h \\ \dfrac{I_\rho}{K}\partial_{tt}\varphi_h - \dfrac{EI}{K}\delta_x\delta_x^*\varphi_h - \delta_x w_h + \varphi_h - \dfrac{\mu}{K}\partial_t\varphi_h = g_h \end{cases} \text{on } (0,T), \tag{2.3}$$

The following theorem gives an estimation of the energy at time $t \in (0,T]$. For simplicity, we will consider problem (2.3) with homogenuous right hand sides (except for section 4).

Theorem 2.1. *Assume $w_h \in C^1((0,T), V_h^0)$ and $\varphi_h \in C^1((0,T), V_h^{\frac{1}{2}})$. There exist a positive constant C independent of h and the time t such that:*

$$\mathcal{E}_{w\varphi}^h(t) \le C\,\mathcal{E}_{w\varphi}^h(0),\ \forall t \in (0,T).$$

where the energy $\mathcal{E}_{w\varphi}^h(t)$ is defined as

$$\mathcal{E}_{w\varphi}^h(t) = \frac{1}{2}\big(\frac{\rho}{K}\|\partial_t w_h\|_{0,h}^2 + \frac{I_\rho}{K}\|\partial_t\varphi_h\|_{\frac{1}{2},h}^2 + \|\delta_x w_h - \varphi_h\|_{\frac{1}{2},h}^2 + \frac{EI}{K}\|\delta_x^*\varphi_h\|_{0,h}^2\big).$$

Proof. Multiplying in (2.3) the first equation by $\partial_t w_h$ and the second equation by $\partial_t \varphi_h$ then taking the scalar product, we get

$$\frac{d}{dt}\mathcal{E}^h_{w\varphi}(t) = \frac{\mu}{2K}\|\partial_t \varphi_h\|_{\frac{1}{2},h}, \tag{2.4}$$

where we have used the fact that δ_x is self adjoint and $\mathcal{E}^h_{w\varphi}(t)$ is given by

$$\mathcal{E}^h_{w\varphi}(t) := \frac{1}{2}\left(\frac{\rho}{K}\|\partial_t w_h\|^2_{0,h} + \frac{I_\rho}{K}\|\partial_t \varphi_h\|^2_{\frac{1}{2},h} + \|\delta_x w_h - \varphi_h\|^2_{\frac{1}{2},h} + \frac{EI}{K}\|\delta^*_x \varphi_h\|^2_{0,h}\right).$$

Integrating the equation (2.4) in time between 0 and t, we obtain:

$$\mathcal{E}^h_{w\varphi}(t) = \mathcal{E}^h_{w\varphi}(0) + \frac{\mu}{2K}\int_0^t \|\partial_s \varphi_h\|_{\frac{1}{2},h} ds,$$

then

$$\mathcal{E}^h_{w\varphi}(t) \le \mathcal{E}^h_{w\varphi}(0) + \frac{\mu}{I_\rho}\int_0^t \mathcal{E}^h_{w\varphi}(s) ds.$$

Using Gronwall inequality, we have, for $t \ge 0$

$$\mathcal{E}^h_{w\varphi}(t) \le \mathcal{E}^h_{w\varphi}(0)\exp\left(\frac{\mu}{I_\rho}\int_0^t ds\right) \le C\,\mathcal{E}^h_{w\varphi}(0),$$

with $C := \exp\left(\frac{\mu T}{I_\rho}\right)$. □

Remark 2.1. *In the case without damping $\mu = 0$, the spatially discrete system (2.3) satisfies the energy conservation result:*

$$\frac{d}{dt}\mathcal{E}^h_{w\varphi}(t) = 0,$$

the energy is conserved in time.

3. Fully discretization

In this section, we present the space-time discretization for the Timoshenko system (1.2), we apply a fourth order finite difference scheme for spatial discretization and a second order Forward scheme for the time variable. For positive integer M, let $\Delta t := \frac{T}{M}$ be a uniform time step, and define for $n \in \{0, 1, \dots, M\}$, $t_n = n\Delta t$. For any function v and $v^n_i = v(x_i, t_n)$, we define the difference operators δ_t and δ_{tt} as

$$\delta_t v_i^{n+\frac{1}{2}} := \frac{v_i^{n+1} - v_i^n}{\Delta t} \text{ and } \delta_{tt} v_i^n := \frac{v_i^{n+1} - 2v_i^n + v_i^{n-1}}{\Delta t^2},$$

The fully discrete problem can be written in vector form as

$$\mathcal{M}\delta_{tt}\mathbf{U}_i^n + \mathcal{A}\mathbf{U}_i^n = (f^n, g^n + \frac{\mu}{K}\delta_t\varphi_{i+\frac{1}{2}}^{n+\frac{1}{2}})^t, \tag{3.1}$$

where $\mathbf{U}_i^n = (w_i^n, \varphi_{i+\frac{1}{2}}^n)^t$, the matrix operators $\mathcal{M}$ and $\mathcal{A}$ are defined by

$$\mathcal{M} = \begin{pmatrix} \frac{\rho}{K} & 0 \\ 0 & \frac{I_\rho}{K} \end{pmatrix} \text{ and } \mathcal{A} = \begin{pmatrix} -\delta_x^*\delta_x & \delta_x^* \\ -\delta_x & -\frac{EI}{K}\delta_x\delta_x^* + id \end{pmatrix}.$$

For any $\tilde{\mathbf{U}}_i^n = (\tilde{w}_i^n, \tilde{\varphi}_{i+\frac{1}{2}}^n)^t$ and $\hat{\mathbf{U}}_i^n = (\hat{w}_i^n, \hat{\varphi}_{i+\frac{1}{2}}^n)^t$, we define

$$< \tilde{\mathbf{U}}^n, \hat{\mathbf{U}}^n >_h := h\sum_{i=0}^{M-1} < \tilde{\mathbf{U}}_i^n, \hat{\mathbf{U}}_i^n > .$$

The fully discrete system for (w, φ) in one dimension satisfies the following energy result:

Theorem 3.1. *(Fully discrete energy estimate) Assuming periodic boundary conditions. If $\Delta t < \frac{I_\rho}{\mu}$, then there exist a positive constant C depends only on μ and T such that:*

$$\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} \leq C\,\mathcal{E}_{w\varphi}^{h,\frac{1}{2}},\ \forall n = 1, 2, \cdot, N.$$

where the energy $\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}}$ is defined as

$$\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} = \frac{1}{2}\left(\frac{\rho}{K}\|\delta_t w_h^{n+\frac{1}{2}}\|_{0,h}^2 + \frac{I_\rho}{K}\|\delta_t\varphi_h^{n+\frac{1}{2}}\|_{\frac{1}{2},h}^2 + < \mathcal{A}\mathbf{U}^{n+1}, \mathbf{U}^n >_h\right),$$

with $\mathbf{U}^n = (w^n, \varphi^n)^t$. Under the Courant-Friedrichs-Lewy (CFL) condition $\frac{K\,\Delta t}{h} < 1$, we have that $\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}}$, and is then called a discrete energy, which is conserved in time.

Proof. Multiplying the equation (3.1) by $\tilde{\mathbf{U}}_i^n = \frac{1}{2\Delta t}(\mathbf{U}_i^{n+1} - \mathbf{U}_i^{n-1})$ and summing over $i = 0, ..., M-1$, we get

$$< \mathcal{M}\delta_{tt}\mathbf{U}^n + \mathcal{A}\mathbf{U}^n, \tilde{\mathbf{U}}^n >_h = < \mathcal{B}\delta_t\mathbf{U}^{n+\frac{1}{2}}, \tilde{\mathbf{U}}^n >_h, \tag{3.2}$$

where $\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\mu}{K} \end{pmatrix}$.

We can easily show that

$$< \mathcal{M}\delta_{tt}\mathbf{U}^n, \tilde{\mathbf{U}}^n >_h = \frac{1}{2\Delta t}\left(\|\mathcal{M}\delta_t\mathbf{U}^{n+\frac{1}{2}}\|_h^2 - \|\mathcal{M}\delta_t\mathbf{U}^{n-\frac{1}{2}}\|_h^2\right)$$

$$= \frac{1}{2\Delta t}\left(\|\delta_t w_h^{n+\frac{1}{2}}\|_{0,h}^2 + \frac{I_\rho}{K}\|\delta_t\varphi_h^{n+\frac{1}{2}}\|_{\frac{1}{2},h}^2 - \|\delta_t w_h^{n-\frac{1}{2}}\|_{0,h}^2 - \frac{I_\rho}{K}\|\delta_t\varphi_h^{n-\frac{1}{2}}\|_{\frac{1}{2},h}^2\right).$$

As the operator $\mathcal{A}$ is self-adjoint, the following identity holds:

$$< \mathcal{A}\mathbf{U}^n, \tilde{\mathbf{U}}^n >_h = \frac{1}{2\Delta t}\left(< \mathcal{A}\mathbf{U}^{n+1}, \mathbf{U}^n >_h - < \mathcal{A}\mathbf{U}^{n-1}, \mathbf{U}^n >_h\right).$$

We obtain

$$\frac{1}{\Delta t}\left(\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} - \mathcal{E}_{w\varphi}^{h,n-\frac{1}{2}}\right) = \frac{1}{2} < \mathcal{B}\delta_t\mathbf{U}^{n+\frac{1}{2}}, \delta_t\mathbf{U}^{n+\frac{1}{2}} >_h + \frac{1}{2} < \mathcal{B}\delta_t\mathbf{U}^{n+\frac{1}{2}}, \delta_t\mathbf{U}^{n-\frac{1}{2}} >_h .$$

We get

$$\frac{1}{\Delta t}\left(\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} - \mathcal{E}_{w\varphi}^{h,n-\frac{1}{2}}\right) = \frac{\mu}{2K}\|\delta_t\varphi^{n+\frac{1}{2}}\|^2_{\frac{1}{2},h} + \frac{\mu}{2K} < \delta_t\varphi^{n+\frac{1}{2}}, \delta_t\varphi^{n-\frac{1}{2}} >_{\frac{1}{2},h},$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\frac{1}{\Delta t}\left(\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} - \mathcal{E}_{w\varphi}^{h,n-\frac{1}{2}}\right) &\leq \frac{3\mu}{4K}\|\delta_t\varphi^{n+\frac{1}{2}}\|^2_{\frac{1}{2},h} + \frac{\mu}{4K}\|\delta_t\varphi^{n-\frac{1}{2}}\|^2_{\frac{1}{2},h},\\ &\leq \frac{3\mu}{2\,I_\rho}\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} + \frac{\mu}{2\,I_\rho}\mathcal{E}_{w\varphi}^{h,n-\frac{1}{2}}.\end{aligned}$$

Equivalently

$$\left(1 - \frac{3\mu}{2\,I_\rho}\Delta t\right)\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} \leq \left(1 + \frac{\mu}{2\,I_\rho}\Delta t\right)\mathcal{E}_{w\varphi}^{h,n-\frac{1}{2}}.$$

For $\Delta t < \frac{\mu}{I_\rho}$, the following inequality holds:

$$\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} \leq \left(\frac{1 + \frac{\mu}{2\,I_\rho}\Delta t}{1 - \frac{3\mu}{2\,I_\rho}\Delta t}\right)\mathcal{E}_{w\varphi}^{h,n-\frac{1}{2}}.$$

The result is concluded using discrete Gronwall inequality. □

Remark 3.1. *In the case $\mu = 0$ (without damping), The fully discrete system (3.1) satisfies the following energy conservation result:*

$$\delta_t\mathcal{E}_{w\varphi}^{h,n} := \frac{1}{\Delta t}\left(\mathcal{E}_{w\varphi}^{h,n+\frac{1}{2}} - \mathcal{E}_{w\varphi}^{h,n-\frac{1}{2}}\right) = 0,$$

the energy is conserved in time.

4. Numerical experiments

To test our compact difference scheme (3.1), we consider a simple and boundary value problem. We choose the coefficients to be $K = \rho = I_\rho = EI = 1$. In this section we present the numerical results of the method on three test problems. We performed our computations using Matlab 2010 software on a PC with Intel Core i3 CPU, 2.27 GHZ and 3 GB RAM. We test the accuracy and convergence of the method presented in this paper by performing the mentioned method for different values of Δt and h. The L_∞ and Root-Mean Square (RMS) errors obtained by the method are shown. Also we calculate the computational orders of the method presented in this study (denoted by C-Order) with the following formula:

$$\frac{\log(\frac{E_1}{E_2})}{\log(\frac{h_1}{h_2})} \tag{4.1}$$

in which E_1 and E_2 are errors correspond to grids with mesh size h_1 and h_2 respectively.

4.1. **Problem 1.**

In this exemple, we choose $K = \rho = I_\rho = EI = \mu = 1$ and we consider the following Timoshenko beam equations

$$\begin{aligned} w_{tt} &= (w_x - \varphi)_x + e^{-t}(2\sin(x) + cos(x)), \\ \varphi_{tt} &= \varphi_{xx} + (w_x - \varphi) - \varphi_t + e^{-t}(2\sin(x) - cos(x)), \end{aligned} \quad \text{on } (0, L) \times \mathbb{R}^+ \tag{4.2}$$

The exact solution is given by

$$\begin{aligned} w(x,t) &= e^{-t}\sin(x),\ x \in [0,\pi], t > 0, \\ \varphi(x,t) &= e^{-t}\sin(x),\ x \in [0,\pi], t > 0. \end{aligned} \tag{4.3}$$

The boundary, initial conditions and the right hand functions $h(x,t)$ and $k(x,t)$ can be obtained from exact solution. We compare the numerical results of the method presented in this paper. Table 1 (resp. Table 2) shows the L_∞ and Root-Mean-square (RMS) errors, C-Order and CPU time of the numerical solution w (resp. φ) for solving Test problem 1 with the time step $\Delta t = h^2$ (satisfying the CFL condition) and the final time $T = 1$.

Table 1 and 2 give also the errors of the numerical solutions, in which the maximal errors are defined as follows:

$$\begin{aligned} \|w - w_{h,k}\| &= \max_{0\le k\le n}\{\max_{0\le t\le M} \mid w(x_i,t_k) - w_i^k \mid\}, \\ \|\varphi - \varphi_{h,k}\| &= \max_{0\le k\le n}\{\max_{0\le t\le M} \mid \varphi(x_i,t_k) - \varphi_i^k \mid\}. \end{aligned} \tag{4.4}$$

We test the accuracy of the method by solving this problem with several values of h for $L = \pi$ at the final time $T = 1$. Numerical results in Table 1 and Table 2 confirm that the order of the method presented in this paper has fourth-order of accuracy in space.

TABLE 1. Error as a function of mesh size, w

h	RMS	L_∞	C-Order	CPU time(s)
$\frac{\pi}{5}$	3.755434×10^{-3}	3.540239×10^{-3}	$----$	0.03120020
$\frac{\pi}{10}$	2.311524×10^{-4}	2.200251×10^{-4}	4.0081	0.04680030
$\frac{\pi}{20}$	1.419201×10^{-5}	1.387278×10^{-5}	3.9873	0.1404009
$\frac{\pi}{40}$	8.881627×10^{-7}	8.670589×10^{-7}	4.0000	0.6240040
$\frac{\pi}{80}$	5.545431×10^{-8}	5.419213×10^{-8}	4.0000	6.208840
$\frac{\pi}{160}$	3.465521×10^{-9}	3.386924×10^{-9}	4.0000	81.99413

TABLE 2. Error as a function of mesh size, φ

h	RMS	L_∞	C-Order	CPU time(s)
$\frac{\pi}{5}$	3.488629×10^{-3}	3.197449×10^{-3}	$----$	$0.03.120020$
$\frac{\pi}{10}$	2.156233×10^{-4}	2.113690×10^{-4}	3.9191	0.03120020
$\frac{\pi}{20}$	1.332199×10^{-5}	1.334139×10^{-5}	3.9858	0.1404009
$\frac{\pi}{40}$	8.334153×10^{-7}	8.339090×10^{-7}	3.9999	0.6552042
$\frac{\pi}{80}$	5.205373×10^{-8}	5.212084×10^{-8}	4.0000	6.255640
$\frac{\pi}{160}$	3.253204×10^{-9}	3.257596×10^{-9}	4.0000	81.77572

TABLE 3. Error as a function of time step, $h = \frac{\pi}{100}$, w

Δt	RMS	L_∞	C-Order	CPU time(s)
$1/5$	9.674348×10^{-4}	9.616796×10^{-4}	$----$	0.1560010
$1/10$	2.375954×10^{-4}	2.388695×10^{-4}	2.0257	0.2652017
$1/20$	5.887851×10^{-5}	5.981516×10^{-5}	2.0127	0.4056026
$1/40$	1.465425×10^{-5}	1.494601×10^{-5}	2.0064	0.6708043
$1/80$	3.654409×10^{-6}	3.735319×10^{-6}	2.0036	1.263608
$1/160$	9.114517×10^{-7}	9.329793×10^{-7}	2.0034	2.402415

Table 3 presents the numerical results in solving Test problem 1 for several values of Δt, mesh size h is choosed such that the CFL condition is satisfied. As we see with the above parameters, the method achieve an accuracy of order 10^{-7} in under 3 seconds and confirm that the order of the method has order two in time. Figure 1 shows the surface plot of approximate solution of Test problem 1.

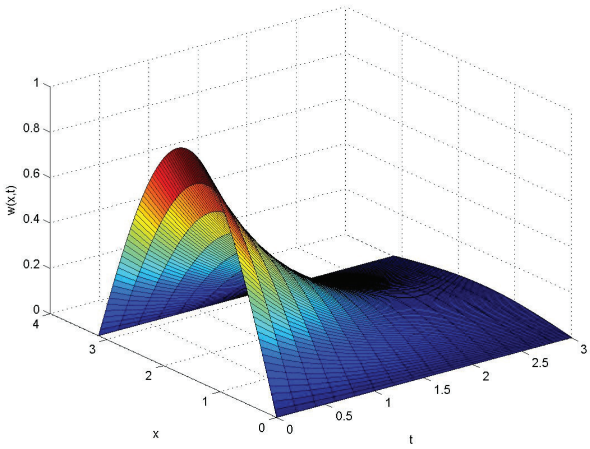

FIGURE 1. Surface plots of approximate solutions w of problem 1

4.2. **Problem 2.** In order to verify the asymptotic behavior of the solution of the Timoshenko system when one locally distributed damping is applied, we consider the following data: $K = \rho = I_\rho = EI = 1,\ h(x,t) = k(x,t) = 0$, then the system (1.2) becomes

$$\begin{aligned} w_{tt} &= (w_x - \varphi)_x, \\ \varphi_{tt} &= \varphi_{xx} + (w_x - \varphi) - \mu\varphi_t, \end{aligned} \qquad \text{on } (0,L)\times\mathbb{R}^+ \tag{4.5}$$

with boundary condition

$$w(0,t) = 0,\ \varphi(0,t) = 0,\ w(2\pi,t) = 0,\ \varphi(2\pi,t) = 0, \tag{4.6}$$

and initial condition

$$w(x,0) = 0,\ w_t(x,0) = sin(x),\ \varphi(x,0) = 0,\ \varphi_t(x,0) = cos(x). \tag{4.7}$$

The natural energy of the beam is given by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^l \rho|\partial_t w|^2 + I_\rho|\partial_t\varphi|^2 + EI|\partial_x\varphi|^2 + K|\partial_x w - \varphi|^2)dx. \tag{4.8}$$

In [7], the well-posedness and asymptotic behavior solution of the Timoshenko system (4.5)-(4.6)-(4.7) are demonstrated. In Figure 2, the asymptotic behavior of the solution was calculated by taking the maximum value of the function w, in $x \in [0, 2\pi]$, for $\mu = 1,\ \Delta t = \frac{1}{5}$ throughout time. Thus, it also was obtained numerically the asymptotic behavior of the solution confirming the theory developed.

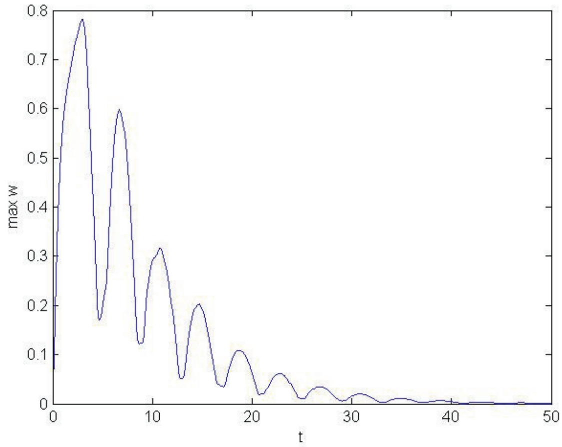

FIGURE 2. Plots of max of w throughout time for $x \in [0, 2\pi]$

Figure 3 represents the case without damping $\mu = 0, \quad \Delta t = \frac{1}{5},\ t \in [0, 500]$, the presence of damping is very evident, obtaining the asymptotic behavior of the solution immediately. and as was expected, there is no convergence of the solution of the system (4.5)-(4.6)-4.7).

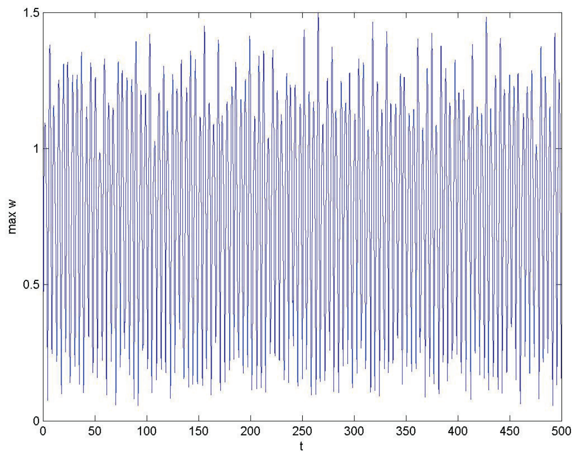

FIGURE 3. plots of max of w throughout time for $x \in [0, 2\pi]$

In Figure 4, we show the graph of function $w(x,t)$ where $x \in [0, 2\pi]$, $t \in [0, 30]$, $\mu = 1$ and we choose only 100 iterations along time a.e. $\Delta t = \frac{1}{100}$. With respect to rotation angle φ, we observe that it exhibits the same behavior that the function w.

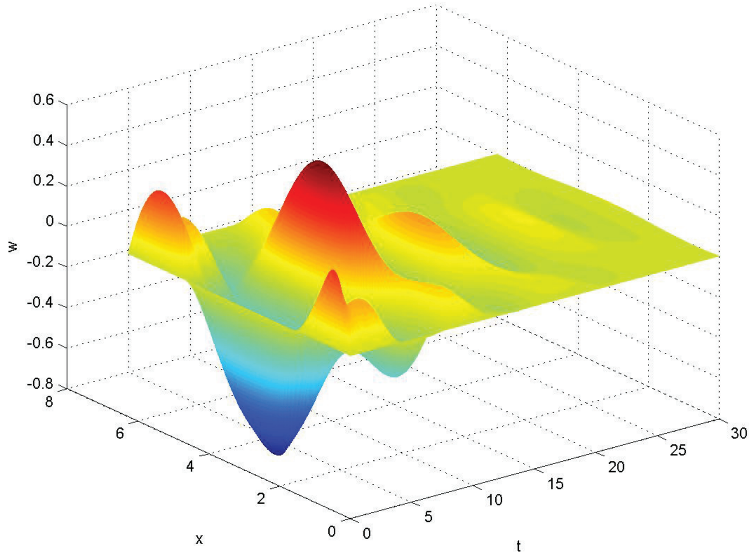

FIGURE 4. Surface plots of approximate solutions w

4.3. **Problem 3.** In this exemple, we choose $K = \rho = I_\rho = EI = \mu = 1.$ and we consider the Timoshenko beam equations

(4.9)
$$\left.\begin{aligned} w_{tt} &= (w_x - \varphi)_x + e^{-t}(\arctan(\alpha(x - \frac{1}{\pi})) + \frac{2\alpha^3(x-\frac{1}{\pi})}{(1+\alpha^2(x-\frac{1}{\pi})^2)^2}, \\ \varphi_{tt} &= \varphi_{xx} + (w_x - \varphi) - \varphi_t + e^{-t}\sin(\pi x)(1 + \pi^2 - \frac{\alpha}{1+\alpha^2(x-\frac{1}{\pi})^2}), \end{aligned}\right. \quad \text{on } (0,1)\times\mathbb{R}^+.$$

The exact solution is given by

$$\begin{aligned} w(x,t) &= e^{-t}\arctan(\alpha(x - \frac{1}{\pi}), \;\; x \in [0,1], t > 0, \\ \varphi(x,t) &= e^{-t}\sin(\pi x), \;\; x \in [0,1], t > 0. \end{aligned} \tag{4.10}$$

For higher α the function w is not smooth, there is a high gradient near the point $x = \frac{1}{\pi}$. The figure bellow represent the L^2 error as a function of the mesh size h.

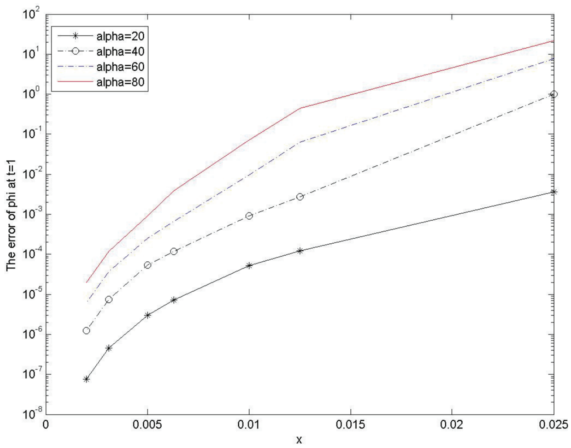

FIGURE 5. Curves of the errors of φ between the exact solution and the numerical solution.

The figure shows the convergence of the numerical scheme and the convergence becomes slow for higher α.

5. CONCLUSION

In this study, we have proposed and analysed a fourth order compact scheme for spatial discretization and a second order forward method for time variable for solving 1D Timoshenko beam equations. We have established stability of the continuous model, by proving energy estimates for the semi and fully-discrete schemes. Computational experiments demonstrate the theoretical results, confirme hight accuracy of the method and presente that this method can achieve good accuracy in short CPU time.

When two locally distributed damping are applied to the Timoshenko beam equation (2.3), the following Timoshenko beam [14]

$$\begin{aligned} \rho w_{tt}(x,t) &= K(w_x - \varphi')_x(x,t) - \mu_1 w_t(x,t), \\ I_\rho \varphi_{tt}(x,t) &= EI\varphi_{xx}(x,t) + K(w_x - \varphi)(x,t) - \mu_2 \varphi_t(x,t), \end{aligned} \quad \text{on } (0,L)\times\mathbb{R}^+, \tag{5.1}$$

where $\mu_1 > 0,\ \mu_2 > 0$ are positive constants can be considered similarly. In [14], the exponential decay of the solution is shown by using a method developed by Z. Liu and S. Zheng [15].

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