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ESTIMATION AND UNIQUENESS OF THE GOMPERTZ FORCE OF MORTALITY IN TERMS OF THE MODAL AGE AT DEATH

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ABSTRACT. The initial level of mortality and the rate at which mortality rises with age are generally expressed in terms of the Gompertz force of mortality (hazard function). In their paper, James W. Vaupel and others define the Gompertz force of mortality as the rate at which mortality rises with age and the modal age at death. In this paper we estimate the Gompertz force of mortality and prove uniqueness theorem.

1. INTRODUCTION

The Gompertz force of mortality as a feature of the mode M (and b) first appears in a short segment of Emil J. Gumbel's Statistics of Extremes citegum, and later in two working papers by John H. Pollard in a demographic sense [7, 8]. Horiuchi et al. [4] recently derived expressions for the hazard in terms of the modal age at death (from senescent causes) in six mortality models: the Gompertz, the Weibull, and the logistic model in the presence or absence of a Makeham word in the Gompertz, Weibull, and logistic models.

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Tests of the dispersion of deaths around the modal age at death have been considered in research on the modal age at death. Rather than analysing the standard deviation around the mean, i.e. around life expectancy, the standard deviation around the mode [1] or the standard deviation above the modal age [4, 5, 9] can be used to determine the dispersion of the death distribution.

As suggested by [5] the standard deviation above the mode pertains to senescent mortality without much distortion from non-senescent mortality beyond the modal age. In Kannisto's study, confirmed by [9], the standard deviation above the mode has declined at a slower pace or stagnated in recent decades and the modal age at death has increased with life expectancy, suggesting that mortality is declining at roughly the same rate at all older ages, leading to a shift in the force of mortality to higher and higher ages [10].

In conclusion, the modal age of death is a valuable metric. In certain cases, it is more useful than the value of the power of death at age zero. As a result, expressing the Gompertz power of mortality in terms of b and M leads to a better interpretation than expressing it in terms of a and b [12]. In section 2, we give an estimation for the Gompertz force of mortality and prove the uniqueness in section 3.

2. Estimation of parameters

The Gompertz force of mortality (or hazard) at age x, $\mu(x)$, has been expressed as

$$\mu(x) = ae^{bx} \tag{2.1}$$

where a denotes the level of mortality at the initial age, i.e., at $x = 0$, and b is the rate of mortality increase over age. Note that $x = 0$ refers to the starting age of analysis and might not correspond to biological age 0.

Alternatively, following Gumbel (1958) [3], the Gompertz force of mortality can be represented as a function of M and b as

$$\mu(x) = be^{b(x-M)} \tag{2.2}$$

where M is the old-age modal age at death, or for short, modal age at death. In other words, M is the age at which the maximum number of deaths occurs after the high number of deaths in the first years of life, assuming stable age groups for populations with senescent mortality. In other words, the power of mortality equals its relative derivative with respect to age at $x = M$.

$$\mu(x) = \frac{d\mu(x)/dx}{\mu(x)} \tag{2.3}$$

In the case of the Gompertz force of mortality given in (2.1), the relationship in (2.3) implies that the mode is

$$M = \frac{1}{b} \ln \frac{b}{a}. \tag{2.4}$$

From (2.4) the parameter a can be expressed in terms of M and b as

$$a = be^{-bM}. \tag{2.5}$$

Substituting (2.5) in (2.1) yields (2.2).

The corresponding survival function can be obtained by integrating the mortality rate function equation (2.2)

$$S(t) = e^{e^{-bM(1-e^{bt})}}. \tag{2.6}$$

In most cases, an experimentalist knows the human lifespans and can estimate the model parameters using standard techniques like MLE or linear regression. When lifespans are not understood precisely or at all for any reason, a problem occurs. Estimating these parameters becomes even more complex in these circumstances.

When it comes to ageing, evolutionary biologists are often left with a survival curve and no data on lifespan. It is crucial in this area to have reliable b figures. If we know M and also know $S(t)$, then equation 2.6 is a transcendental equation in the unknown b that can be solved using normal numerical methods.

In general, agreeing on a specific value of t to use in equation 2.6 is difficult. If we are looking at the evolution of longevity, however, starting with $t = t_m ax$, the known maximum lifespan, is a good place to start.Finally for ease of analysis we may set $S(t_m) = \frac{1}{N*}$ (the populationcontains only one number left from an original population size N^*). Hence we have

$$S(t_m) = e^{e^{-bM(1-e^{btm})}} \approxeq \frac{1}{N*}.$$

Taking natural logarithm on both sides we get

$$-\ln N^* = e^{-bM(1-e^{btm})},$$

which can be rewritten as

$$\ln N^* = e^{-bM(e^{bt_m-1})}$$

Simplifying further we get

$$\begin{aligned} \ln(\ln N^*) &= -bM(e^{bt_m-1}) \\ M &= \frac{1}{b}\left[\frac{\ln(\ln N^*)}{(1-e^{bt_m})}\right]. \end{aligned}$$

The above equation gives an estimation for the modal age with respect to the population size and maximum lifespan.

We obtain the following equation for t_m (the time at which the population has only one number and which approximates the maximum lifespan t_m^*)

$$t_m^* \approx t_m = \frac{1}{b}\ln\left[1+\frac{\ln N^*}{e^{-bM}}\right]. \tag{2.7}$$

The average mortality rate of a steady-state population subject to age-specific mortality rates is (Finch et al [2])

$$A_{avg} = \frac{1}{\int_0^\infty S(t)dt}. \tag{2.8}$$

After a little algebra equation (2.7) leads to

$$e^{-bM} = \frac{\ln N^*}{e^{bt_m}-1} \tag{2.9}$$

and upon substitution of equation (2.6) into equation (2.8) we arrive at

$$\frac{1}{A_{avg}} = \int_0^\infty e^{e^{-bM(1-e^{bt})}}dt. \tag{2.10}$$

A simple substitution in the integral gives

$$b = A_{avg}e^{e^{-bM}}\int_{e^{-bM}}^\infty \frac{e^{-y}}{y}dy \tag{2.11}$$

and using (2.9), we get

$$b = A_{avg}e^{\frac{\ln N^*}{e^{bt_m}-1}}\int_{\frac{\ln N^*}{e^{bt_m}-1}}^\infty \frac{e^{-y}}{y}dy. \tag{2.12}$$

The basic equation (2.12) is transcendental, involving exponential, integral and the age-dependent parameter b is a function of A_{avg}, $\ln N^*$ and t_m.

3. UNIQUENESS OF THE GOMPERTZ FORCE OF MORTALITY

3.1 Uniqueness theorem

Theorem 3.1. *The parameter estimation for the Gompertz force of mortality given in equation (2.12) has a unique solution if* $2A_{avg}t_m < 1$*, for* $b > 0$*.*

Proof. Suppose b_1 and b_2 are two positive distinct solutions of equation (2.12), that is

$$\begin{aligned} b_1 &= A_{avg}e^{\frac{\ln N*}{e^{b_1 t_m}-1}} \int_{\frac{\ln N*}{e^{b_1 t_m}-1}}^{\infty} \frac{e^{-y}}{y}dy, \\ b_2 &= A_{avg}e^{\frac{\ln N*}{e^{b_2 t_m}-1}} \int_{\frac{\ln N*}{e^{b_2 t_m}-1}}^{\infty} \frac{e^{-y}}{y}dy. \end{aligned}$$

Now,

$$b_1 - b_2 = A_{avg}\left[e^{u_1}\int_{u_1}^{\infty}\frac{e^{-y}}{y}dy - e^{u_2}\int_{u_2}^{\infty}\frac{e^{-y}}{y}dy\right],$$

where $u_i = \frac{\ln N*}{e^{b_i t_m}-1}$, for $i = 1, 2$.

A simple substitution in the above equation gives

$$b_1 - b_2 = A_{avg}\int_0^{\infty} e^{-v}\left[\frac{1}{v+u_1} - \frac{1}{v+u_2}\right]dv, \tag{3.1}$$

where $v = y - u_i$, for $i = 1, 2$. Also $e^{-v} \geqslant 1, \forall v \leqslant 0$. Hence we obtain

$$\begin{aligned} |b_1 - b_2| &\leqslant A_{avg}\,|u_1 - u_2| \int_0^{\infty} \frac{dv}{(v+u_1)(v+u_2)} \\ &= A_{avg}\left|\ln[\frac{u_1}{u_2}]\right| \end{aligned}$$

$$\begin{aligned} \therefore\ |b_1 - b_2| &\leqslant A_{avg}\left|\ln\left[\frac{e^{b_2 t_m}-1}{e^{b_1 t_m}-1}\right]\right| \\ &= A_{avg}\left|\ln\left[\frac{e^{b_2 t_m}(1-e^{-b_2 t_m})}{e^{b_1 t_m}(1-e^{-b_1 t_m})}\right]\right| \\ &= A_{avg}\left[\left|\ln\left(\frac{e^{b_2 t_m}}{e^{b_1 t_m}}\right)\right| + \left|\ln\left(\frac{1-e^{-b_2 t_m}}{1-e^{-b_1 t_m}}\right)\right|\right] \\ &= A_{avg}\left[|b_2 t_m - b_1 t_m| + \left|\ln(1-e^{-b_2 t_m}) - \ln(1-e^{-b_1 t_m})\right|\right]. \end{aligned}$$

Applying mean value theorem, we get

$$\begin{aligned} |b_1 - b_2| &\leqslant A_{avg}|b_2 - b_1|t_m + A_v \ln\left|e^{-b_1 t_m} - e^{-b_2 t_m}\right| \\ &= A_{avg}|b_2 - b_1|t_m + A_v|b_2 - b_1|t_m \\ &= 2A_{avg}t_m|b_2 - b_1|. \end{aligned}$$

That is,

$$(2A_{avg}t_m - 1)|b_2 - b_1| \geqslant 0.$$

Since $2A_{avg}t_m < 1$, the last inequality implies that $b_1 = b_2$ for $b > 0$. Hence we conclude that equation (2.12) has a unique solution if $2A_{avg}t_m < 1$, for $b > 0$. □

3.2 Necessary condition for uniqueness

Theorem 3.2. *The necessary condition to have a unique solution of equation (2.12) is that* $\frac{A_{avg}t_m}{\ln N} < 1$*, for* $b > 0$*.*

Proof. Suppose b_1 and b_2 are two positive distinct solutions of equation (2.12), that is from equation (3.1),

$$\begin{aligned} b_1 - b_2 &= A_{avg}(u_2 - u_1)\int_0^\infty \frac{e^{-v}}{(v+u_1)(v+u_2)}dv \\ &= A_{avg}\left(\frac{1}{e^{b_2 t_m} - 1} - \frac{1}{e^{b_1 t_m} - 1}\right)(e^{b_1 t_m} - 1)(e^{b_2 t_m} - 1) \\ &\times \int_0^\infty \frac{e^{-y\ln N*}}{(1 + y(e^{b_1 t_m} - 1))(1 + y(e^{b_2 t_m} - 1))}dy. \end{aligned}$$

Since

$$\frac{e^{-y\ln N*}}{(1 + y(e^{b_1 t_m} - 1))(1 + y(e^{b_2 t_m} - 1))} \leqslant 1,$$

we get

$$\begin{aligned} b_1 - b_2 &\leqslant A_{avg}\left(\frac{1}{e^{b_2 t_m} - 1} - \frac{1}{e^{b_1 t_m} - 1}\right)(e^{b_1 t_m} - 1)(e^{b_2 t_m} - 1)\int_0^\infty e^{-y\ln N*}dy \\ &= A_{svg}\left(\frac{1}{e^{b_2 t_m} - 1} - \frac{1}{e^{b_1 t_m} - 1}\right)(e^{b_1 t_m} - 1)(e^{b_2 t_m} - 1)\frac{1}{\ln N*}. \end{aligned}$$

Hence

$$\begin{aligned}
|b_1 - b_2| &\leqslant \frac{A_{avg}}{\ln N^*}\left|(e^{b_1 t_m}-1)-(e^{b_2 t_m}-1)\right| \\
&= \frac{A_{avg}}{\ln N^*}\left|(b_1 t_m)\left(\frac{e^{b_1 t_m}-1}{b_1 t_m}\right)-(b_2 t_m)\left(\frac{e^{b_2 t_m}-1}{b_2 t_m}\right)\right| \\
&= \frac{A_{avg}}{\ln N^*}\left|\frac{b_1 t_m}{\left(\frac{b_1 t_m}{e^{b_1 t_m}-1}\right)}-\frac{b_2 t_m}{\left(\frac{b_2 t_m}{e^{b_2 t_m}-1}\right)}\right|.
\end{aligned}$$

Thus,

$$|b_1 - b_2| \leqslant \frac{A_{avg} t_m |b_1 - b_2|}{\ln N^* max\left(\frac{b_1 t_m}{e^{b_1 t_m}-1}, \frac{b_2 t_m}{e^{b_2 t_m}-1}\right)}. \tag{3.2}$$

Suppose equation (2.12) has a unique solution, it follows from (3.2) that

$$\frac{A_{avg} t_m}{\ln N^* max\left(\frac{b_1 t_m}{e^{b_1 t_m}-1}, \frac{b_2 t_m}{e^{b_2 t_m}-1}\right)} < 1. \tag{3.3}$$

Since $0 < \frac{t}{e^t - 1} \leqslant 1, \forall t \geqslant 0$, from (3.3) we get

$$\begin{aligned}
\frac{A_{avg} t_m}{\ln N^*} &< min\left(\frac{b_1 t_m}{e^{b_1 t_m}-1}, \frac{b_2 t_m}{e^{b_2 t_m}-1}\right) < max\left(\frac{b_1 t_m}{e^{b_1 t_m}-1}, \frac{b_2 t_m}{e^{b_2 t_m}-1}\right) \\
1 &\leqslant min\left(\frac{b_1 t_m}{e^{b_1 t_m}-1}, \frac{b_2 t_m}{e^{b_2 t_m}-1}\right) \leqslant max\left(\frac{b_1 t_m}{e^{b_1 t_m}-1}, \frac{b_2 t_m}{e^{b_2 t_m}-1}\right).
\end{aligned}$$

Note that $\left(\frac{bt_m}{e^{bt_m}-1}\right)$ attains 1 only if $bt_m = 0$.

Hence the above inequalities implies that $\frac{A_{avg} t_m}{\ln N^*} < 1$, for $b > 0$. Thus we conclude that to have a unique solution of equation (2.12) it is necessary that $\frac{A_{avg} t_m}{lnN^*} < 1$, for $b > 0$ (see Table 1). □

Remark 3.1. *Theorem 3.1 shows that the condition for uniqueness of b is independent of population size N^*, but theorem 2 shows that the necessary condition for uniqueness of b is dependent on population size N^*. As a consequence, Theorem 3.2 outperforms Theorem 3.1.*

TABLE 1. (reprinted from [6])

$A_{avg}/year$	IMR/year	MRD/year	$t_m(years)$
Herring gull			
0.34			
N=(652)	0.18	2.82	11.3
N=10^3			11.5
N=10^9			15.7
Human			
0.015			
N=10^3	0.0002	7.967	105
N=10^5			110
N=10^7			114
N=(80,750,000)			115
N=10^9			117
N=10^{11}			120
Mouse			
0.74			
N=25	0.049	0.27	2.2
N=50			2.3
N=100			2.3
N=(738)			2.5
N=10^3			2.5
N=10^9			2.9
Rat			
0.64			
N=(250)	0.025	0.26	2.6
N=10^3			2.7
N=10^9			3.1

4. CONCLUSION

The aim of this discussion was to look at how to express the Gompertz force of mortality in terms of the rate at which mortality rises with age and the modal age

$A_{avg}/year$	IMR/year	MRD/year	$t_m(years)$
Japanese quail			
0.35			
N=(29)	0.091	1.163	5.8
N=10^3			6.9
N=10^9			8.8

at death. The fact that the modal age parameter M has no impact on estimating and proving the uniqueness of b is quite interesting. It can also be noted that the estimation of the gompertz force of mortality parameter b, is the same as the estimation of the Gompertz parameter.

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