Advances in Mathematics: Scientific Journal **10** (2021), no.10, 3307–3325
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.10.3

# A PARAMETRIZATION OF $\delta$-SPHERICAL FUNCTIONS ON COMMUTATIVE TRIPLES ASSOCIATED WITH NILPOTENT LIE GROUPS

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ABSTRACT. Let $N$ be a connected and simply connected nilpotent Lie group, $K$ be a compact subgroup of $Aut(N)$, the group of automorphisms of $N$ and $\delta$ be a class of unitary irreducible representations of $K$. The triple $(N, K, \delta)$ is a commutative triple if the convolution algebra $\mathfrak{U}^1_\delta(N)$ of $\delta$-radial integrable functions is commutative. In this paper, we obtain first a parametrization of $\delta$ spherical functions by means of the unitary dual $\widehat{N}$ and then an inversion formula for the spherical transform of $F \in \mathfrak{U}^1_\delta(N)$.

## 1. INTRODUCTION

Let $G$ be a locally compact group, $K$ be a compact subgroup of $G$ and $\delta$ be a unitary irreducible representation of $K$. The triple $(G, K, \delta)$ is commutative when the convolution algebra $\mathcal{C}_c(G, \delta, \delta)$ of compactly supported vector-valued functions which are $\delta$-radial, is commutative. It is a generalization of the notion of Gelfand pair which is obtaining when $\delta$ is the one-dimensional trivial representation. The spherical functions associated with commutative triples have been studied by a number of authors such that E. Pedon [12], R. Camporesi [3], A. Samantha and F. Ricci [13]. R. Camporesi has studied the case of semi-simple Lie Group when

2020 *Mathematics Subject Classification.* 43A30, 43A90, 22E25, 22E27.
*Key words and phrases.* Nilpotent Lie groups, Commutative triples, Group representations, Spherical functions, Spherical transform.
*Submitted:* 15.05.2021; *Accepted:* 01.08.2021; *Published:* 12.10.2021.

A. Samantha and F. Ricci have studied the case of nilpotent Lie groups. In this paper, we are interested by the case of nilpotent Lie groups. The first purpose of this paper is to obtain a parametrization of $\delta$-spherical functions on a nilpotent Lie group by means of $(\pi, \xi) \in \widehat{N} \times (\mathcal{H}_\pi \otimes E_\delta)$ where $\widehat{N}$ is the unitary dual of $N$ and $\mathcal{H}_\pi$ (resp. $E_\delta$) is the realization space of $\pi$(resp. $\delta$). This parametrization is an extension of Theorem 8.7 of C. Benson and al. in [1]. Our second purpose is to use this result to obtain an inversion formula for the $\delta$-spherical Fourier transform. The paper is organized as follows. In the next section, we give notations and definitions necessary for the understanding of the paper. In section 3, we first obtain an explicit formula for $\delta$-spherical functions for commutative triples where the first component is a nilpotent Lie group. Our formula permits us to prove that $\delta$-spherical functions for nilpotent groups are definite positive. We end this section by proving that the parametrization obtaining in the first result is unique under the action of $K$ on $\widehat{N}$. In the last section, we show that the $\delta$-spherical Fourier transform defined by means of our formula for $\delta$-spherical functions verifies some classical properties. Thanks to the inversion formula for spherical transform on $\widehat{G}$ where $G = K \propto N$, we obtain an inversion formula for the $\delta$-spherical transform.

## 2. PRELIMINARIES

In this section, we give some notations and definitions for the well understanding of this paper. Let $G$ be a locally compact group and let $K$ be a compact subgroup of $G$. $G$ is equipped with a left Haar measure $dx$ and $K$ is equipped with its normalized Haar measure $dk$. Let $\delta$ be a unitary irreducible representation of $K$ and let us denote by $E_\delta$ the space of the representation $\delta$. We put $F_\delta{:=}Hom(E_\delta, E_\delta)$ the space of endomorphisms of $E_\delta$ and denote by $\mathcal{C}_c(G, F_\delta)$ (resp. $L^1(G, F_\delta)$) the space of compactly supported continuous (resp. integrable) functions of $G$ with values in $F_\delta$. $\mathcal{C}_c(G, F_\delta)$ (resp. $L^1(G, F_\delta)$) is a convolution algebra where the convolution is defined by: for $F, H \in \mathcal{C}_c(G, F_\delta)$ (resp. $L^1(G, F_\delta)$) and $x \in G$,

$$F * H(x) = \int_G F(y^{-1}x)H(y)dy.$$

We set,

$$\mathcal{C}_c(G, F_\delta, \delta, \delta) := \{F \in C_c(G, F_\delta) : F(kxk') = \delta(k'^{-1})F(x)\delta(k^{-1}) \forall k, k' \in K, \forall x \in G\}$$

the space of continuous $\delta-$ radial functions of $G$ with compact support and

$L^1(G, F_\delta, \delta, \delta) := \{F \in L^1(G, F_\delta) : F(kxk') = \delta(k'^{-1})F(x)\delta(k^{-1}) \forall k, k' \in K, \forall x \in G\}$

the space of $\delta-$ radial integrable functions of $G$. $\mathcal{C}_c(G, F_\delta, \delta, \delta)$ (resp. $L^1(G, F_\delta, \delta, \delta)$) is a subalgebra of the convolution algebra $C_c(G, F_\delta)$ (resp. $L^1(G, F_\delta)$). We say that $(G, K, \delta)$ is a commutative triple if the convolution algebra $\mathcal{C}_c(G, F_\delta, \delta, \delta)$ or $L^1(G, F_\delta, \delta, \delta)$ is commutative. If $\delta$ is the one dimensional trivial representation then we obtain the classical notion of Gelfand pair. Also, if $\delta$ and $\delta'$ are unitarily equivalent then the algebras $\mathcal{C}_c(G, F_\delta, \delta, \delta)$ and $\mathcal{C}_c(G, F_{\delta'}, \delta', \delta')$ are isomorphic. In fact, if we designate by $A$ the intertwining operator of $\delta$ and $\delta'$ i.e. $A\delta(k) = \delta'(k)A$, let us consider the map $Q$ from $\mathcal{C}_c(G, F_{\delta'}, \delta', \delta')$ to $\mathcal{C}_c(G, F_\delta, \delta, \delta)$ defined by: $Q(F)(x) = A^{-1}F(x)A$. $Q$ is clearly an isomorphism of algebras. Thus in the following of this paper, $\delta$ will designate a class of unitary irreducible representations of $K$ and $\mu_\delta$ an element of the class $\delta$.

For any $F \in C_c(G, F_\delta)$, the canonical projection of $F$ on $\mathcal{C}_c(G, F_\delta, \delta, \delta)$ is defined by

$$F^\natural(x) = \int_K \int_K \mu_\delta(k_2)F(k_1xk_2)\mu_\delta(k_1)dk_1dk_2.$$

Let us put $\chi_\delta := d(\delta)\xi_\delta$, where $d(\delta)$ is the degree of $\mu_\delta$ and $\xi_\delta$ the character of $\mu_\delta$. Thanks to Schur's orthogonality relations, any $T \in F_\delta$ is written by

$$T = d(\delta) \int_K \mu_\delta(k^{-1})tr(\mu_\delta(k)T)dk$$

where $tr$ designates the trace of an operator. Let us denote by $\widehat{G}$(resp. $\widehat{K}$) the unitary dual of $G$(resp. $K$). For $U \in \widehat{G}$, we denote by $mtp(\delta, U)$ the multiplicity of $\delta$ in $U|K$. We know by [3, 5, 16] that if the triple $(G, K, \delta)$ is commutative then $mtp(\delta, U) \leq 1$. Let $\widehat{G}(\delta)$ be the subset of $\widehat{G}$ consisting of those $U \in \widehat{G}$ that contains $\delta$ upon restriction to $K$. For $U \in \widehat{G}(\delta)$ and $\mathcal{H}$ its realization space, we designate by $\mathcal{H}(\delta)$ the isotypic component of $\delta$ that is the subspace of vectors which transform under $K$ according to $\delta$. The projection $P(\delta)$ from $\mathcal{H}$ onto $\mathcal{H}(\delta)$ is defined by

$$P(\delta) = \int_K \chi_\delta(k^{-1})U(k)dk.$$

If $(G, K, \delta)$ is commutative, a $\delta$-radial continuous function $\phi$ on $G$ with values in $F_\delta$ is said to be a $\delta$-spherical function if the map $F \mapsto \frac{1}{d_\delta}\int_G tr(\phi(x)F(x))dx$ is a unitary character of the commutative algebra $L^1(G, F_\delta, \delta, \delta)$. If $U \in \widehat{G}(\delta)$, the function $\phi_\delta^U(x) = P(\delta)U(x)P(\delta)$ is a positive definite $\delta$-spherical function and any positive definite $\delta$-spherical function is obtained in this manner. We denote by $S_\delta(G)$(resp. $S_\delta^+(G)$) the set of $\delta$-spherical functions (resp. positive definite $\delta$-spherical functions) on $G$. Let $I_c(G)$ denote the set of all $K$-central functions that is, the set of all continuous complex-valued functions $f$ on $G$ and compactly supported such that:

$$f(kxk^{-1}) = f(x), \forall k \in K, \forall x \in G,$$

and $I_\delta(G)$ denotes the set of all continuous complex-valued functions $f$ on $G$ and compactly supported such that: $\chi_\delta * f = f * \chi_\delta = f$ where,

$$\begin{aligned} \chi_\delta * f(x) &:= \int_K \chi_\delta(k^{-1}) f(kx) dk \\ f * \chi_\delta(x) &:= \int_K \chi_\delta(k^{-1}) f(xk) dk. \end{aligned}$$

Let us put $I_{c,\delta}(G) := I_c(G) \cap I_\delta(G)$. $I_{c,\delta}(G)$ is a subalgebra of $\mathcal{C}_c(G)$, where $\mathcal{C}_c(G)$ is the usual convolution algebra of all continuous, complex-valued functions on $G$ and compactly supported. For all $f \in \mathcal{C}_c(G)$, we put

$$f_K(x) = \int_K f(kxk^{-1}) dk.$$

Then the map $f \mapsto \chi_\delta * f_K$ is a continuous projection of $\mathcal{C}_c(G)$ on $I_{c,\delta}(G)$. In [16], it is shown that $I_{c,\delta}(G)$ is isomorphic to $\mathcal{C}_c(G, F_\delta, \delta, \delta)$ thanks to the map

$$\begin{aligned} \psi^\delta : I_{c,\delta}(G) &\to \mathcal{C}_c(G, F_\delta, \delta, \delta) \\ f &\mapsto \psi_f^\delta \,, \end{aligned}$$

where $\psi_f^\delta(x) := \int_K \mu_\delta(k^{-1}) f(kx) dk$. Let $N$ be a connected and simply connected nilpotent Lie group and $K$ be a compact subgroup of $Aut(N)$, the group of automorphisms of $N$. Let us put $G := K \propto N$ the semi-direct product of $K$ by $N$. Writing the action by $k \in K$ on $x \in N$ as $k.x$, the semi-product is defined by: $(k_1, x)(k_2, y) = (k_1 k_2, x k_1.y)$. We set

$$\mathfrak{U}_{c,\delta}(N) := \{F \in C_c(N, F_\delta) : F(k.x) = \mu_\delta(k) F(x) \mu_\delta(k^{-1})\}.$$

$\mathcal{C}_c(G, F_\delta, \delta, \delta)$ and $\mathfrak{U}_{c,\delta}(N)$ are isomorphic as convolution algebras. It suffices to consider the map $F \longmapsto F|_N$ (see [15]). We also set

$$\mathfrak{U}_\delta^1(N) := \{F \in L^1(N, F_\delta) : F(k.x) = \mu_\delta(k) F(x) \mu_\delta(k^{-1})\}.$$

Thus we give the following equivalent definition: $(N, K, \delta)$ is a Gelfand triple if the convolution algebra $\mathfrak{U}_{c,\delta}(N)$(resp. $\mathfrak{U}_\delta^1(N)$) is commutative. For $F \in C_c(N, F_\delta)$, the projection of $F$ on $\mathfrak{U}_{c,\delta}(N)$ is defined by:

$$F^{K,\delta}(x) = \int_K \mu_\delta(k^{-1})F(k.x)\mu_\delta(k)dk.$$

We denote by $\widehat{N}$ the unitary dual of $N$. The subgroup $K$ acts on $\widehat{N}$ by the following way: $k.\pi(x) = \pi^k(x) = \pi(k.x)$ for $x \in N$, $k \in K$ and $\pi \in \widehat{N}$. Let $K_\pi$ denote the stabilizer of $\pi$ by this action,

$$K_\pi = \{k \in K : \pi^k \simeq \pi\}$$

For $k \in K_\pi$, there is an intertwining operator $W_\pi(k)$ with $\pi^k(x) = W_\pi(k)\pi(x)$ $W_\pi(k)^{-1}$. $W_\pi$ is a projective representation of $K_\pi$ on $\mathcal{H}_\pi$ where $\mathcal{H}_\pi$ is the representation space of $\pi$. But, it is known that $W_\pi$ can be chosen to be a unitary representation of $K_\pi$(see [9]).

## 3. $\delta$-SPHERICAL FUNCTIONS

In this section, $N$ is a connected and simply connected nilpotent Lie group and $K$ is a compact subgroup of the group of automorphisms $Aut(N)$. We only assume that $(N, K, \delta)$ is a commutative triple. Therefore $(N, K)$ is a Gelfand pair(see [13], [14]) and $N$ is a two step nilpotent Lie group (see [1]). Let $C(N, F_\delta)$ be the space of continuous function on $N$ with values in $F_\delta$. We recall that $\phi \in C(N, F_\delta)$ is a $\delta$- spherical function on $N$ if it is $\delta$-radial and the map $\chi_\phi : F \mapsto \frac{1}{d_\delta}\int_N tr(F(x)\phi(x))dx$ defines a (necessarily non zero and continuous) character of $\mathfrak{U}_{c,\delta}(N)$. The following result gives an explicit formula of $\delta$-spherical function on $N$. It extends Lemma 8.2 in [1] to $\delta$-spherical function on $N$.

**Theorem 3.1.** *Suppose $\phi$ is a bounded $\delta$-spherical function on $N$. Then there exists $\pi \in \hat{N}$, a unit vector*

$$\xi \text{ in } \mathcal{H}_\pi \otimes E_\delta \text{ such that for all } x \in N, \phi(x)$$
$$= d_\delta^2 \int < \pi(k.x) \otimes \mu_\delta(kk_1)\xi, \xi > \mu_\delta(k^{-1}k_1^{-1})dkdk_1.$$

*Proof.* $N$ being a group with polynomial volume growth (see [11]) then $L^1(N, F_\delta)$ is symmetric (see [10]). So, since $\chi_\phi$ is a one dimensional representation of $\mathfrak{U}^1_\delta(N)$, there exists $(L, E_L)$ (see [6]) an irreducible $*$-representation of $L^1(N, F_\delta)$ and a closed invariant one dimensional subspace $M$ of $E_L$ such that $(L \mid_{\mathfrak{U}^1_\delta(N)}; M)$ is equivalent to $\chi_\phi$. We have $L^1(N,F_\delta) = L^1(N)\otimes F_\delta$. Let $\tilde{\pi}$ be an irreducible $*$-representation of $L^1(N)$. There exists a unitary representation $(\pi,\mathcal{H}_\pi)\in\hat{N}$ such that $\tilde{\pi}(f) = \int_N f(x)\pi(x)dx$ where $f\in L^1(N)$. Hence the representation $L$ of $L^1(N,F_\delta)$ is written by: $L(F) = \int_N \pi(x)\otimes F(x)dx\ \forall\ F\in L^1(N,F_\delta)$. Thus $M$ is a one dimensional subspace of $\mathcal{H}_\pi\otimes E_\delta$. Let us choose $\xi = \sum_i \zeta_i\otimes\eta_i$ in $\mathcal{H}_\pi\otimes E_\delta$ and belonging to $M$ such that $\parallel\xi\parallel = 1$.

$$
\begin{aligned}
\chi_\phi(F) =& < \chi_\phi(F)\xi,\xi >\\
=&\sum_{i,j} < L(F)\zeta_i\otimes\eta_i,\zeta_j\otimes\eta_j >\\
=&\sum_{i,j}\int < \pi(x)\otimes F(x)\zeta_i\otimes\eta_i,\zeta_j\otimes\eta_j > dx\\
=&\sum_{i,j}\int < \pi(x)\zeta_i,\zeta_j >< F(x)\eta_i,\eta_j > dx\\
=&\sum_{i,j}\int < \pi(x)\zeta_i,\zeta_j >< \mu_\delta(k_1^{-1})F(k_1.x)\mu_\delta(k_1)\eta_i,\eta_j > dk_1dx\\
=&\sum_{i,j}\int < \pi(x)\zeta_i,\zeta_j >< F(k_1.x)\mu_\delta(k_1)\eta_i,\mu_\delta(k_1)\eta_j > dk_1dx\\
=&\sum_{i,j}\int < \pi(k_1^{-1}.x)\zeta_i,\zeta_j >< F(x)\mu_\delta(k_1)\eta_i,\mu_\delta(k_1)\eta_j > dk_1dx\\
=&d_\delta\sum_{i,j}\int < \pi(k_1^{-1}.x)\zeta_i,\zeta_j >< \mu_\delta(k)tr(\mu_\delta(k^{-1})F(x))\mu_\delta(k_1)\eta_i,\mu_\delta(k_1)\eta_j >\\
&\times dkdk_1dx\\
=&d_\delta\sum_{i,j}\int < \pi(k_1^{-1}.x)\zeta_i,\zeta_j >< \mu_\delta(kk_1)\eta_i,\mu_\delta(k_1)\eta_j > tr(\mu_\delta(k^{-1})F(x))\\
&\times dkdk_1dx\\
=&d_\delta\sum_{i,j}\int < \pi(k_1^{-1}.x)\zeta_i,\zeta_j >< \mu_\delta(kk_1)\eta_i,\mu_\delta(k_1)\eta_j > tr(\mu_\delta(k^{-1})F(k^{-1}.x))\\
&\times dkdk_1dx\\
=&d_\delta\sum_{i,j}\int < \pi(k_1^{-1}k.x)\zeta_i,\zeta_j >< \mu_\delta(k_1^{-1}kk_1)\eta_i,\eta_j > tr(\mu_\delta(k^{-1})F(x))
\end{aligned}
$$

$$\times\, dkdk_1dx$$

$$=d_\delta \sum_{i,j} \int < \pi(k.x)\zeta_i, \zeta_j >< \mu_\delta(kk_1)\eta_i, \eta_j > tr(\mu_\delta(k^{-1}k_1^{-1})F(x))dkdk_1dx$$

$$=d_\delta \sum_{i,j} \int tr(F(x)(\int < \pi(k.x)\zeta_i, \zeta_j >< \mu_\delta(kk_1)\eta_i, \eta_j > \mu_\delta(k^{-1}k_1^{-1})dkdk_1)dx$$

so we obtain

$$\begin{aligned}\phi(x) =&d_\delta^2 \sum_{i,j} \int < \pi(k.x)\zeta_i, \zeta_j >< \mu_\delta(kk_1)\eta_i, \eta_j > \mu_\delta(k^{-1}k_1^{-1})dkdk_1\\ =&d_\delta^2 \int < \pi(k.x) \otimes \mu_\delta(kk_1)\xi, \xi > \mu_\delta(k^{-1}k_1^{-1})dkdk_1.\end{aligned}$$

□

**Remark 3.1.** *If $\mu_\delta = 1_K$ the one dimensional trivial representation of $K$, we can identify $\mathcal{H}_\pi \otimes E_\delta$ with $\mathcal{H}_\pi$ and so we can identify $\phi(x)$ with the scalar operator $x \mapsto \int < \pi(k.x)\xi, \xi > dk$. Thus we obtain the classical integral formula of $K$-spherical functions on nilpotent Lie groups obtained by C. Benson and al. in [1].*

In the following result, we prove that any bounded $\delta$-spherical function on $N$ is positive definite.

**Corollary 3.1.** *Every bounded $\delta$-spherical function on $N$ is positive definite.*

*Proof.* According to previous theorem, there exist $\pi \in \hat{N}$, $\xi = \sum_i \zeta_i \otimes \eta_i \in \mathcal{H}_\pi \otimes E_\delta$ with $\| \xi \|= 1$ such that $\phi = \phi_{\pi,\xi}$. Let $x_1, x_2, \ldots, x_n \in N$ ;$c_1, c_2, \ldots, c_n \in \mathbb{C}$,

$$\begin{aligned}&\sum_{1\le i,j\le n} c_i\bar{c}_j tr(\phi_{\pi,\xi}(x_i^{-1}x_j))\\ =&d_\delta^2 \sum_{p,q} \sum_{1\le i,j\le n} c_i\bar{c}_j tr(\int < \pi(k.(x_i^{-1}x_j))\zeta_p, \zeta_q > \times\\ &< \mu_\delta(kk_1)\eta_p, \eta_q > \mu_\delta(k^{-1}k_1^{-1})dkdk_1)\\ =&d_\delta^2 \int \sum_{p,q} \sum_{1\le i,j\le n} c_i\bar{c}_j < \pi(k.x_j)\zeta_p, \pi(k.x_i)\zeta_q >\\ &\times < \mu_\delta(kk_1)\eta_p, \eta_q > tr(\mu_\delta(k^{-1}k_1^{-1}))dkdk_1\end{aligned}$$

$$
\begin{aligned}
=& d_\delta^2 \int \sum_{p,q} \sum_{1\le i,j\le n} c_i \bar{c_j} < \pi(k.x_j)\zeta_p, \pi(k.x_i)\zeta_q >< \mu_\delta(kk_1)\eta_p, \eta_q > \\
& \times tr(\mu_\delta(k_1^{-1}k^{-1}))dkdk_1 \\
=& d_\delta^2 \int \sum_{p,q} \sum_{1\le i,j\le n} c_i \bar{c_j} < \pi(k.x_j)\zeta_p, \pi(k.x_i)\zeta_q >< \mu_\delta(k_1)\eta_p, \eta_q > \\
& \times tr(\mu_\delta(k_1^{-1}))dkdk_1 \\
=& d_\delta^2 \int \sum_{p,q} \sum_{1\le i,j\le n} c_i \bar{c_j} < \pi(k.x_j)\zeta_p, \pi(k.x_i)\zeta_q > dk \\
& \times \int < \mu_\delta(k_1)tr(\mu_\delta(k_1^{-1}))\eta_p, \eta_q > dk_1 \\
=& d_\delta \int \sum_{p,q} \sum_{1\le i,j\le n} c_i \bar{c_j} < \pi(k.x_j)\zeta_p, \pi(k.x_i)\zeta_q > dk \\
& \times < d_\delta \int \mu_\delta(k_1)tr(\mu_\delta(k_1^{-1}))dk_1\eta_p, \eta_q > \\
=& d_\delta \int \sum_{p,q} \sum_{1\le i,j\le n} c_i \bar{c_j} < \pi(k.x_j)\zeta_p, \pi(k.x_i)\zeta_q >< \eta_p, \eta_q > dk \\
=& d_\delta \int \sum_{1\le i,j\le n} c_i \bar{c_j} < (\pi(k.x_j) \otimes I_{E_\delta})\xi, (\pi(k.x_i) \otimes I_{E_\delta})\xi > dk \\
=& d_\delta \int \parallel \sum_{1\le i\le n} c_i(\pi(k.x_i) \otimes I_{E_\delta})\xi \parallel^2 dk \ge 0.
\end{aligned}
$$

□

For $\pi \in \widehat{N}$, $\xi \in \mathcal{H}_\pi \otimes E_\delta$ let us put

$$\phi_{\pi,\xi}(x) = d_\delta^2 \int < \pi(k.x) \otimes \mu_\delta(kk_1)\xi, \xi > \mu_\delta(k^{-1}k_1^{-1})dkdk_1.$$

The previous theorem raises two fundamental questions. The first one is: under which conditions on the couple $(\pi, \xi)$, $\phi_{\pi,\xi}$ is $\delta-$spherical? and the second one is: for two couples $(\pi, \xi)$ and $(\pi', \xi')$, under which conditions $\phi_{\pi,\xi}$ and $\phi_{\pi',\xi'}$ coincide? Anwser to these questions is the main goal of the rest of this section and will give an extension of Theorem 8.7 of [1] to $\delta-$spherical functions.

Since $(N, K, \delta)$ is a commutative triple $W_\pi \otimes \mu_\delta|_{K_\pi}$ is multiplicity free(see [13]), where $\mu_\delta|_{K_\pi}$ is the restriction of $\mu_\delta$ to $K_\pi$. We will denote $\mu_\delta|_{K_\pi}$ by $\mu_{\delta_\pi}$. Let $\mathcal{H}_\pi \otimes E_\delta = \sum_\alpha V_\alpha$ be the decomposition of $\mathcal{H}_\pi \otimes E_\delta$ in $K_\pi$-irreducible modules.

**Theorem 3.2.**

(i) $\phi_{\pi,\xi}$ *is* $\delta$*-spherical if and only if* $\xi \in V_\alpha$ *for some* $\alpha$ *and* $\| \xi \|= 1$.
(ii) $\phi_{\pi,\xi} = \phi_{\pi',\xi'}$ *if and only if there exists* $k_0 \in K$ *such that* $\pi' = \pi^{k_0}$*, and* $\xi$ *and* $\mu_\delta(k_0)\xi'$ *belong to the same* $V_\alpha$*.*

*Proof.* (i) For $F \in \mathfrak{U}_{c,\delta}(N)$, $L(F)$ commutes with $W_\pi \otimes \mu_{\delta_\pi}$ (see [13]). In fact,

$$\begin{aligned}
L(F)(W_\pi(k) \otimes \mu_{\delta_\pi}(k)) &= \int_N (\pi(x) \otimes F(x))(W_\pi(k) \otimes \mu_{\delta_\pi}(k))dx \\
&= \int_N W_\pi(k)\pi(k^{-1}.x) \otimes F(x)\mu_{\delta_\pi}(k)dx \\
&= \int_N W_\pi(k)\pi(x) \otimes F(k.x)\mu_{\delta_\pi}(k)dx \\
&=(W_\pi(k) \otimes \mu_{\delta_\pi}(k))(\int_N \pi(x) \otimes F(x)dx) \\
&=(W_\pi(k) \otimes \mu_{\delta_\pi}(k))L(F)
\end{aligned}$$

Since $W_\pi \otimes \mu_{\delta_\pi}$ is multiplicity free then $L(F)$ preserves each $V_\alpha$. Now by Schur's Lemma, the irreducibility of $W_\pi \otimes \mu_{\delta_\pi}$ on $V_\alpha$ implies that $L(F)$ acts as a scalar multiple of the identity on each $V_\alpha$. So $L(F)|_{V_\alpha} = \lambda(F)Id_{V_\alpha}$ and $\lambda(F) = \langle L(F)\xi, \xi\rangle$ for any unit vector $\xi \in V_\alpha$. Let us suppose $\xi \in V_\alpha$ for some $\alpha$ with $\| \xi \|= 1$. Let us first show that $\phi_{\pi,\xi}$ is $\delta$-radial and $\phi_{\pi,\xi}(e) = id_{E_\delta}$. In fact

$$\begin{aligned}
\phi_{\pi,\xi}(\widetilde{k}.x) &=d_\delta^2 \int < \pi(k\widetilde{k}.x) \otimes \mu_\delta(kk_1)\xi, \xi > \mu_\delta(k^{-1}k_1^{-1})dkdk_1 \\
&=d_\delta^2 \int < \pi(k.x) \otimes \mu_\delta(k\widetilde{k}^{-1}k_1)\xi, \xi > \mu_\delta(\widetilde{k}k^{-1}k_1^{-1})dkdk_1 \\
&=d_\delta^2\mu_\delta(\widetilde{k}) \int < \pi(k.x) \otimes \mu_\delta(kk_1)\xi, \xi > \mu_\delta(k^{-1}k_1^{-1}\widetilde{k}^{-1})dkdk_1 \\
&=d_\delta^2\mu_\delta(\widetilde{k}) \int < \pi(k.x) \otimes \mu_\delta(kk_1)\xi, \xi > \mu_\delta(k^{-1}k_1^{-1})dkdk_1\mu_\delta(\widetilde{k}^{-1}) \\
&=\mu_\delta(\widetilde{k})\phi_{\pi,\xi}(x)\mu_\delta(\widetilde{k}^{-1}),
\end{aligned}$$

so $\phi_{\pi,\xi}$ is $\delta$-radial. Taking $x = e$, we have

$$\phi_{\pi,\xi}(e) = d_\delta^2 \int < id_{\mathcal{H}_\pi} \otimes \mu_\delta(kk_1)\xi, \xi > \mu_\delta(k^{-1}k_1^{-1})dkdk_1.$$

Hence

$$\begin{aligned}
tr(\phi_{\pi,\xi}(e)) =& d_\delta^2 \int < id_{\mathcal{H}_\pi} \otimes \mu_\delta(kk_1)\xi, \xi > tr(\mu_\delta(k^{-1}k_1^{-1}))dkdk_1 \\
=& d_\delta^2 \int < id_{\mathcal{H}_\pi} \otimes \mu_\delta(kk_1)\xi, \xi > tr(\mu_\delta(k_1^{-1}k^{-1}))dkdk_1 \\
=& d_\delta^2 \int < id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_1)\xi, \xi > tr(\mu_\delta(k_1^{-1}))dk_1 \\
=& d_\delta < id_{\mathcal{H}_\pi} \otimes (d_\delta \int \mu_\delta(k_1)tr(\mu_\delta(k_1^{-1}))dk_1)\xi, \xi >= d_\delta.
\end{aligned}$$

In other part, since $\phi_{\pi,\xi}$ is $\delta$- radial, $\mu_\delta(k)$ intertwines $\phi_{\pi,\xi}(e)$ for any $k \in K$. Thus, thanks to Schur's Lemma we have $\phi_{\pi,\xi}(e) = cid_{E_\delta}$. It comes that $tr(\phi_{\pi,\xi}(e)) = cd_\delta$ and we deduce that $c = 1$ that is $\phi_{\pi,\xi}(e) = id_{E_\delta}$. It is clear that $\phi_{\pi,\xi}$ is continuous on $N$. Now for $F \in \mathfrak{U}_{c,\delta}(N)$,

$$\begin{aligned}
\chi_{\phi_{\pi,\xi}}(F) =& \frac{1}{d_\delta} \int tr(F(x)\phi_{\pi,\xi}(x))dx \\
=& d_\delta \int < \pi(k.x) \otimes \mu_\delta(kk_1)\xi, \xi > tr(F(x)\mu_\delta(k^{-1}k_1^{-1}))dkdk_1dx \\
=& d_\delta \int < \pi(x) \otimes \mu_\delta(kk_1)\xi, \xi > tr(F(k^{-1}x)\mu_\delta(k^{-1}k_1^{-1}))dkdk_1dx \\
=& d_\delta \int < \pi(x) \otimes \mu_\delta(kk_1)\xi, \xi > tr(F(x)\mu_\delta(k_1^{-1}k^{-1}))dkdk_1dx \\
=& d_\delta \int < \pi(x) \otimes \mu_\delta(k)\xi, \xi > tr(F(x)\mu_\delta(k^{-1}))dkdx \\
=& \int < \pi(x) \otimes (d_\delta \int \mu_\delta(k)tr(\mu_\delta(k^{-1})F(x))\xi dk), \xi > dx \\
=& \int < \pi(x) \otimes F(x)\xi, \xi > dx \\
=& < L(F)\xi, \xi >= \lambda(F).
\end{aligned}$$

Thus if $F, H \in \mathfrak{U}_{c,\delta}(N)$,

$$\begin{aligned}
\chi_{\phi_{\pi,\xi}}(F * H) =& \langle L(F * H)\xi, \xi\rangle = \langle L(F)L(H)\xi, \xi\rangle \\
=& \lambda(H)\langle L(F)\xi, \xi\rangle = \lambda(F)\lambda(H) = \chi_{\phi_{\pi,\xi}}(F)\chi_{\phi_{\pi,\xi}}(H).
\end{aligned}$$

Conversely let us assume that $\phi_{\pi,\xi}$ is $\delta$-spherical with $\xi \in H_\pi \otimes E_\delta$ and $\| \xi \|= 1$. Writing $\xi = \sum_\gamma t_\gamma \xi_\gamma$ with $\xi_\gamma \in V_\alpha$, $\| \xi_\gamma \|= 1$, $t_\gamma \geq 0$ and $\sum_\gamma t_\gamma^2 = 1$, we have

$$\begin{aligned}\chi_{\phi_{\pi,\xi}} = \langle \phi_{\pi,\xi}, F\rangle =&\langle L(F)\xi, \xi\rangle = \sum_{\gamma,\gamma'} t_\gamma t_{\gamma'} \langle L(F)\xi_\gamma, \xi_{\gamma'}\rangle \\ &= \sum_\gamma t_\gamma^2 \langle L(F)\xi_\gamma, \xi_\gamma\rangle = \sum_\gamma t_\gamma^2 \langle \phi_{\pi,\xi_\gamma}, F\rangle.\end{aligned}$$

Thus $\phi_{\pi,\xi} = \sum_\gamma t_\gamma^2 \phi_{\pi,\xi_\gamma}$. So, since $\phi_{\pi,\xi}$ is positive definite $\delta$-spherical function satisfying $\phi_{\pi,\xi}(e) = Id_{E_\delta}$, the map $x \mapsto \langle \phi_{\pi,\xi}(x)v, v\rangle$, for $v \in E_\delta$, is extremal. Therefore $\phi_{\pi,\xi} = \phi_{\pi,\xi_\gamma}$ for some $\gamma$. It comes that $\xi = \zeta_\gamma$

(ii) We suppose that there exists $k_0 \in K$ such that $\pi' = \pi^{k_0}$ and $\xi$ and $id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0)\xi'$ belong to the same $V_\alpha$.

$$\begin{aligned}\langle \phi_{\pi,\xi}, F\rangle =&\langle L(F)\xi, \xi\rangle \\ =&\langle L(F)(id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0))\xi', id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0)\xi'\rangle \\ =&\chi_{\phi_{\pi, id_{\mathcal{H}_\pi}\otimes \mu_\delta(k_0)\xi'}}(F) \\ =&d_\delta \int < \pi(k.x) \otimes \mu_\delta(kk_1)(id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0))\xi', id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0)\xi' > \\ &\times tr(F(x)\mu_\delta(k^{-1}k_1^{-1}))dkdk_1dx \\ =&d_\delta \int < \pi(k_0k.x) \otimes \mu_\delta(k_0kk_1)(id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0))\xi', id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0)\xi' > \\ &\times tr(F(x)\mu_\delta(k^{-1}k_0^{-1}k_1^{-1}))dkdk_1dx \\ =&d_\delta \int < \pi'(k.x) \otimes \mu_\delta(k_0kk_1k_0)\xi', id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0)\xi' > \\ &\times tr(F(x)\mu_\delta(k^{-1}k_0^{-1}k_1^{-1}))dkdk_1dx \\ =&d_\delta \int < \pi'(k.x) \otimes \mu_\delta(k_0kk_1)\xi', id_{\mathcal{H}_\pi} \otimes \mu_\delta(k_0)\xi' > \\ &\times tr(F(x)\mu_\delta(k^{-1}k_1^{-1}))dkdk_1dx \\ =&d_\delta \int < \pi'(k.x) \otimes \mu_\delta(kk_1)\xi', \xi' > tr(F(x)\mu_\delta(k^{-1}k_1^{-1}))dkdk_1dx \\ =&\langle \phi_{\pi',\xi'}, F\rangle\end{aligned}$$

Thus $\phi_{\pi,\xi} = \phi_{\pi',\xi'}$.

For the converse, we will show that the $\delta$-spherical functions $\phi_{\pi,\xi}$ are written as the restrictions of positive definite $\delta$-spherical functions on $K \propto N$. The irreducible representations of $K \propto N$ are well-known. In fact, for $\pi \in \widehat{N}$ and $\sigma \in \widehat{K_\pi}$, the representation $\rho = \overline{\sigma} \otimes \pi W_\pi$ is an irreducible representation of $K_\pi \propto N$ where $\overline{\sigma}$ is the conjugate representation of $\sigma$. Then according Mackey theory, $\tilde{\rho} = Ind_{K_\pi \propto N}^{K \propto N}(\rho) \in \widehat{K \propto N}$. So each irreducible representation of $K \propto N$ is determined by a pair $(\pi, \sigma)$ where $\pi \in \widehat{N}$ and $\sigma \in \widehat{K_\pi}$. Two representations determined by the pair $(\pi, \sigma)$ and $(\pi', \sigma')$ are equivalent if and only if there exists $k_0 \in K$ such that $\pi' = \pi^{k_0}$ and $\sigma' = \sigma o i_0$, where the map $i_0$ is defined from $K_{\pi'}$ to $K_\pi$ by $i_0(k) = k_0 k k_0^{-1}$. We denote by $\widehat{K \propto N}(\delta)$ the subset of $\widehat{K \propto N}$ consisting of representations containing $\delta$ upon restriction to $K$. For $\tilde{\rho} \in \widehat{K \propto N}(\delta)$, let us denote by $\mathcal{H}_{\tilde{\rho}}$ the realization space of $\tilde{\rho}$ and by $\mathcal{H}(\delta)$ the isotypic component of $\delta$. The projection $P(\delta)$ from $\mathcal{H}_{\tilde{\rho}}$ onto $\mathcal{H}(\delta)$ is defined by

$$P(\delta) = \int_K \chi_\delta(k^{-1})\tilde{\rho}(k)dk.$$

The function $\tilde{\phi}(k, n) = P(\delta)\tilde{\rho}(k, n)P(\delta)$ is a positive definite $\delta$-spherical function and each positive definite $\delta$-spherical function is obtained in this manner. Since $(N, K, \delta)$ is commutative and $\tilde{\rho} \in \widehat{K \propto N}(\delta)$ then $mtp(\delta, \tilde{\rho} \mid_K) = 1$ and $\mathcal{H}(\delta) = E_\delta$ . But (see [13])

$$mtp(\delta, \tilde{\rho} \mid_K) = mtp(\delta, Ind_{K_\pi}^K(\overline{\sigma} \otimes W_\pi)) = mtp(\sigma, W_\pi \otimes \delta \mid_{K_\pi}).$$

so that the realization space $\mathcal{H}_\sigma$, of $\sigma$, is isomorphic to an (and only one) irreducible $K_\pi$-module $V_\alpha$ of $\mathcal{H}_\pi \otimes E_\delta$. Let $\{u_i\}_{1 \le i \le n}$ (resp. $\{v_j\}_{1 \le j \le m}$) be an orthonormal system of $\mathcal{H}_\pi$ (resp. $E_\delta$) such that $\{u_i \otimes v_j : u_i \in \mathcal{H}_\pi, v_j \in E_\delta, 1 \le i \le n, 1 \le j \le m\}$ is an orthonormal basis of $V_\alpha$. Let us consider the vector $v = \frac{1}{\sqrt{mn}} \sum_{i,j} u_i \otimes v_j \otimes u_i \in V_\alpha \otimes \mathcal{H}_\pi$. Let $\check{\delta}$ be the contragredient class of $\delta$ and let $\mu_{\check{\delta}}$ be an element of $\check{\delta}$. We have

$$\begin{aligned}
P(\check{\delta})v &= \int_{K_\pi} \chi_{\check{\delta}}(k^{-1})\rho(k)v dk \\
&= \frac{1}{\sqrt{mn}} \sum_{i,j} \int_{K_\pi} \chi_{\check{\delta}}(k^{-1})\rho(k)u_i \otimes v_j \otimes u_i dk \\
&= \frac{1}{\sqrt{mn}} \sum_{i,j} \int_{K_\pi} \chi_{\check{\delta}}(k^{-1})\overline{W_\pi(k)}u_i \otimes \mu_{\check{\delta}}(k)v_j \otimes W_\pi(k)u_i dk
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{mn}} \sum_{i,j,l,p} \int_{K_\pi} \chi_{\check{\delta}}(k^{-1}) \overline{a_{i,l}} u_l \otimes \mu_{\check{\delta}}(k) v_j \otimes a_{i,p} u_p dk \\
&= \frac{1}{\sqrt{mn}} \sum_{l,j} u_l \otimes \int_{K_\pi} \chi_{\check{\delta}}(k^{-1}) \mu_{\check{\delta}}(k) v_j \otimes u_l dk \\
&= \frac{1}{\sqrt{mn}} \sum_{l,j} u_l \otimes v_j \otimes u_l = v,
\end{aligned}
$$

where $A = (a_{i,j})$ is the matrix of the unitary operator of $W_\pi(k)$ restricts to the vector space spanned by $\{u_i : 1 \leq i \leq n\}$ and $\sum \overline{a_{i,l}} a_{i,p} = \delta_{l,p}$, where $\delta_{l,p}$ is the symbol of Kronecker. This calculus shows that $\rho(k)v = \mu_{\check{\delta}}(k)v$ for all $k \in K_\pi$. Now we consider the function $f : K \propto N \to V_\alpha \otimes \mathcal{H}_\pi$ defined by $f(k,n) = 1 \otimes \mu_{\check{\delta}}(k) \otimes \pi(n)v$. We have $f \in \mathcal{H}_{\tilde{\rho}}$. In fact, for $(l,n) \in K_\pi \propto N$ and $(k,e) \in K \propto N$ we have

$$
f((l,n)(k,e)) = f(lk,n) = (1 \otimes \mu_{\check{\delta}}(lk) \otimes \pi(n))v
$$

and

$$
\begin{aligned}
\rho(l,n) f(k,e) &= (\overline{W_\pi(l)} \otimes \mu_{\check{\delta}}(l) \otimes \pi(n) W_\pi(l))(1 \otimes \mu_{\check{\delta}}(k) \otimes 1)v \\
&= (1 \otimes 1 \otimes \pi(n))(\overline{W_\pi(l)} \otimes \mu_{\check{\delta}}(l) \otimes W_\pi(l)) \mu_{\check{\delta}}(k) v \\
&= (1 \otimes 1 \otimes \pi(n)) \rho(l) \mu_{\check{\delta}}(k) v \\
&= (1 \otimes 1 \otimes \pi(n)) \mu_{\check{\delta}}(l) \mu_{\check{\delta}}(k) v \\
&= (1 \otimes \mu_{\check{\delta}}(lk) \otimes \pi(n)) v.
\end{aligned}
$$

So, $\rho(l,n) f(k,e) = f((l,n)(k,e))$. Now for $(k,n') \in K \propto N$ we have

$$
\begin{aligned}
\rho(l,n) f(k,n') &= \rho(l,n) f((e,n')(k,e)) = \rho(l,n) \rho(e,n') f(k,e) \\
&= \rho(l, nl.n') f(k,e) = f(lk, nl.n') = f((l,n)(k,n')),
\end{aligned}
$$

which proves that $f \in \mathcal{H}_{\tilde{\rho}}$. We have also

$$
\begin{aligned}
P(\check{\delta}) f(k',n) &= \int_K \chi_{\check{\delta}}(k^{-1}) \tilde{\rho}(k) f(k',n) dk \\
&= \int_K \chi_{\check{\delta}}(k^{-1}) f(k'k,n) dk \\
&= \int_K \chi_{\check{\delta}}(k^{-1}) (1 \otimes \mu_{\check{\delta}}(k'k) \otimes \pi(n)) v dk \\
&= (1 \otimes \mu_{\check{\delta}}(k') \otimes \pi(n)) v = f(k',n)
\end{aligned}
$$

and it follows that $f \in E_{\check{\delta}}$. Note that $E_{\check{\delta}}$ and $E_\delta$ are isomorphic and $f$ can be consider as belonging to $E_\delta$. Since $\parallel v \parallel = 1$ then it is straightforward to see that $\parallel f \parallel = 1$. The $\delta$-spherical function $\tilde{\phi}$ on $K \propto N$ associated with $\tilde{\rho}$, as mentionned above, is given by $\tilde{\phi}(k,n) = P(\delta)\tilde{\rho}(k,n)P(\delta)$. The restriction to $N$ is given by $\phi(n) = \tilde{\phi}(e,n) = P(\delta)\tilde{\rho}(n)P(\delta)$. Thanks to Schur's orthogonality relations, we have

$$\begin{aligned}\phi(n) =& d_\delta \int \mu_\delta(k^{-1}) tr(\mu_\delta(k)\phi(n))dk\\ =& d_\delta \int \mu_\delta(k^{-1}) tr(\mu_\delta(\tilde{k}k\tilde{k}^{-1})\phi(\tilde{k}.n))d\tilde{k}dk\\ =& d_\delta \int \mu_\delta(\tilde{k}^{-1}k^{-1}) tr(\mu_\delta(\tilde{k}k)P(\delta)\tilde{\rho}(\tilde{k}.n)P(\delta))d\tilde{k}dk.\end{aligned}$$

Since $\mu_\delta$ is irreducible and $E_\delta$ is finite dimensional then there exists $k_1, k_2, \ldots, k_{d_\delta} \in K$ such that $\{\mu_\delta(k_1)f, \mu_\delta(k_2)f, \ldots, \mu_\delta(k_{d_\delta})f\}$ is a basis for $E_\delta$. So we have

$$\begin{aligned}\frac{1}{d_\delta}\phi(n) =& \int_K \mu_\delta(\tilde{k}^{-1}k^{-1}) tr(\mu_\delta(\tilde{k}k)P(\delta)\tilde{\rho}(\tilde{k}.n)P(\delta))d\tilde{k}dk\\ =& \int_K \mu_\delta(\tilde{k}^{-1}k^{-1}) \sum_{j=1}^{d_\delta}(\mu_\delta(\tilde{k}k)P(\delta)\tilde{\rho}(\tilde{k}.n)P(\delta)\mu_\delta(k_j)f, \mu_\delta(k_j)f)d\tilde{k}dk\\ =& \int_K \mu_\delta(\tilde{k}^{-1}k^{-1}) \sum_{j=1}^{d_\delta}(\mu_\delta(\tilde{k}k)\tilde{\rho}(\tilde{k}.n)\mu_\delta(k_j)f, \mu_\delta(k_j)f)d\tilde{k}dk\\ =& \int_K \int_{K/K_\pi} \mu_\delta(\tilde{k}^{-1}k^{-1}) \sum_{j=1}^{d_\delta}(\mu_\delta(\tilde{k}k)\tilde{\rho}(\tilde{k}.n)\mu_\delta(k_j)f(k'), \mu_\delta(k_j)f(k'))dk'd\tilde{k}dk\\ =& d_\delta \int_K \int_{K/K_\pi} \mu_\delta(\tilde{k}^{-1}k^{-1})(\mu_\delta(\tilde{k}k)\tilde{\rho}(\tilde{k}.n)f(k'), f(k'))dk'd\tilde{k}dk\\ =& d_\delta \int_K \int_{K/K_\pi} \mu_\delta(\tilde{k}^{-1}k^{-1})(f(k'\tilde{k}k, k'\tilde{k}.n), f(k'))dk'd\tilde{k}dk\\ =& d_\delta \int_K \int_{K/K_\pi} \mu_\delta(\tilde{k}^{-1}k^{-1})(1 \otimes \mu_{\check{\delta}}(k'\tilde{k}k) \otimes \pi(k'\tilde{k}.n)v, 1 \otimes \mu_{\check{\delta}}(k') \otimes 1v)dk'd\tilde{k}dk.\end{aligned}$$

So it follows that

$$\phi(n) = d_\delta^2 \int_K \int_{K/K_\pi} \mu_\delta(\tilde{k}^{-1}k^{-1})(1 \otimes \mu_{\check{\delta}}(k'\tilde{k}k) \otimes \pi(k'\tilde{k}.n)v, 1 \otimes \mu_{\check{\delta}}(k') \otimes 1v)dk'd\tilde{k}dk.$$

For $k' \in K$, we have

$$\begin{aligned}(1 \otimes \mu_{\check{\delta}}(k'\tilde{k}k) \otimes \pi(k'\tilde{k}.n)v, 1 \otimes \mu_{\check{\delta}}(k') \otimes 1v) =& \frac{1}{mn} \sum_{i,j,l,p} (u_i \otimes \mu_{\check{\delta}}(k'\tilde{k}k)v_j \otimes \pi(k'\tilde{k}.n)u_i, \\ & u_l \otimes \mu_{\check{\delta}}(k')v_p \otimes u_l) \\ =& \frac{1}{mn} \sum_{i,j,p} (\mu_{\check{\delta}}(\tilde{k}k)v_j, v_p)(\pi(k'\tilde{k}.n)u_i, u_i).\end{aligned}$$

For $k' \in K_\pi$,

$$\begin{aligned}\sum_i (\pi(k'\tilde{k}.n)u_i, u_i) &= \sum_i (\pi(k'\tilde{k}.n)u_i, u_i) \\ &= \sum_i (W_\pi(k')\pi(\tilde{k}.n)W_\pi(k'^{-1})u_i, u_i) \\ &= \sum_i (\pi(\tilde{k}.n)u_i, u_i).\end{aligned}$$

Thus

$$\begin{aligned}\phi(n) =& \frac{d_\delta^2}{mn} \sum_{i,j,p} \int_K \int_K \mu_\delta(\tilde{k}^{-1}k^{-1})(\mu_{\check{\delta}}(\tilde{k}k)v_j, v_p)(\pi(k'\tilde{k}.n)u_i, u_i)dk'd\tilde{k}dk \\ =& \frac{d_\delta^2}{mn} \sum_{i,j,p} \int_K \int_K \mu_\delta(\tilde{k}^{-1}k^{-1})(\mu_\delta(k'^{-1}\tilde{k}kk')v_j, v_p)(\pi(\tilde{k}.n)u_i, u_i)dk'd\tilde{k}dk \\ =& \frac{d_\delta^2}{mn} \sum_{i,j,p} \int_K \int_K \mu_\delta(\tilde{k}^{-1}k^{-1})(\mu_\delta(\tilde{k}k)v_j, v_p)(\pi(\tilde{k}.n)u_i, u_i)d\tilde{k}dk \\ =& \frac{d_\delta^2}{mn} \sum_i \int_K \int_K (\pi(\tilde{k}.n) \otimes \mu_\delta(\tilde{k}k)u_i \otimes \sum_j v_j, u_i \otimes \sum_p v_p)\mu_\delta(\tilde{k}^{-1}k^{-1})d\tilde{k}dk \\ =& \sum_i \phi_{\pi, \frac{1}{\sqrt{mn}} u_i \otimes \sum_j v_j}(n) \\ =& \phi_{\pi, \frac{1}{\sqrt{mn}} \sum_{i,j} u_i \otimes v_j}(n).\end{aligned}$$

So $\phi = \phi_{\pi,\xi}$, for some $\xi \in V_\alpha$. Now if $\phi_{\pi,\xi} = \phi_{\pi',\xi'}$ then the spherical functions $\tilde{\phi}$ and $\tilde{\phi}'$ on $K \propto N$ such that $\tilde{\phi}|N = \phi_{\pi,\xi}$ and $\tilde{\phi}'|N = \phi_{\pi',\xi'}$ are equal. In fact,

$$\begin{aligned}\tilde{\phi}(k,n) =& \tilde{\phi}((e,n)(k,e)) = \mu_\delta(k)\tilde{\phi}((e,n)) = \mu_\delta(k)\phi_{\pi,\xi}(n) = \mu_\delta(k)\phi_{\pi',\xi'}(n) \\ =& \mu_\delta(k)\tilde{\phi}'((e,n)) = \tilde{\phi}'(k,n)\end{aligned}$$

Thus the representations $\tilde{\rho}$ and $\tilde{\rho'}$, associated to $\tilde{\phi}$ and $\tilde{\phi'}$, determined respectively by $(\pi, \sigma)$ and $\pi', \sigma')$, are equivalent. So according to Mackey theory recalled above, we have done. $\square$

## 4. $\delta$ SPHERICAL TRANSFORM AND ITS INVERSION FORMULA

In this section, we assume that $(N, K, \delta)$ is commutative. This implies that $(G, K, \delta)$ is commutative that is $\mathcal{C}_c(G, \delta, \delta)$ is commutative. We give an inversion formula for the spherical transform on $S_\delta(N)$ from the Fourier inversion formula of $G = K \propto N$.

**Definition 4.1.** *The spherical transform for $F \in \mathcal{U}_{c,\delta}(N)$ is the function $\widehat{F}$ on $S_\delta(N)$ defined by*

$$\widehat{F}(\phi) = \frac{1}{d_\delta} \int_N Tr(F(x)\phi(x))dx, \phi \in S_\delta(N).$$

According theorem 3.1, we can write

$$\widehat{F}(\phi_{\pi,\xi}) = \frac{1}{d_\delta} \int_N Tr(F(x)\phi_{\pi,\xi}(x))dx, \pi \in \widehat{N}, \xi \in V_\alpha.$$

Given a function $F \in \mathcal{C}_c(N, F_\delta)$, we define $F^*$ by $F^*(x) = F(x^{-1})^*$, where $\star$ designates the adjoint operator. The following result gives some properties of the spherical transform.

**Theorem 4.1.** *For $F, H \in \mathcal{U}_{c,\delta}(N)$, we have*
*(i) $\widehat{F * G}(\phi_{\pi,\xi}) = \widehat{F}(\phi_{\pi,\xi})\widehat{H}(\phi_{\pi,\xi})$,*
*(ii) $\widehat{H^*}(\phi_{\pi,\xi}) = \overline{\widehat{H}(\phi^*_{\pi,\xi})}$.*

*Proof.*
(i) We can notice that $\widehat{F}(\phi_{\pi,\xi}) = \chi_{\phi_{\pi,\xi}}(F)$ and since $\chi_{\phi_{\pi,\xi}}$ is an homomorphism (see the proof of theorem 3.4) then we have the result.
(ii) For $H \in \mathcal{U}_{c,\delta}(N)$, we have

$$\begin{aligned}\widehat{H^*}(\phi_{\pi,\xi}) =& \frac{1}{d_\delta} \int_N tr(H(x^{-1})^*\phi_{\pi,\xi}(x))dx \\ =& \overline{\int_N tr(H(x)\phi_{\pi,\xi}(x^{-1})^*)dx} = \overline{\widehat{H}(\phi^*_{\pi,\xi})}.\end{aligned}$$

$\square$

We recall now the definition of the spherical transform on $G$.

**Definition 4.2.** *The spherical transform of $F \in \mathcal{C}_c(G,\delta,\delta)$ is the function $\widehat{F}$ on $\widehat{G}(\delta)$ defined by:*

$$\widehat{F}(U) = \frac{1}{d_\delta}\int_G Tr(F(x)\phi_\delta^U(x))dx, U \in \widehat{G}(\delta).$$

So thanks to the bijective correspondence between $S_\delta(G)^+$ and $\widehat{G}(\delta)$, we have the following definition.

**Definition 4.3.** *The spherical transform of $F \in \mathcal{C}_c(G,\delta,\delta)$ is the function $\widehat{F}$ on $S_\delta(G)^+$ defined by:*

$$\widehat{F}(\phi) = \frac{1}{d_\delta}\int_G Tr(F(x)\phi(x))dx, \phi \in S_\delta(G).$$

**Remark 4.1.** *If $\phi_\delta^U$ is the spherical function associated to $U$, then we have $\widehat{F}(\phi_\delta^U) = \widehat{F}(U)$.*

**Theorem 4.2.** *The spherical transform is inverted by*

$$F(x) = \frac{1}{d_\delta}\int_{S_\delta(N)} \phi(x^{-1})\widehat{F}(\phi)d\kappa(\phi), F \in \mathcal{U}_{c,\delta}(N).$$

*Proof.* The spherical transform for $F \in \mathcal{C}_c(G,\delta,\delta)$ is inverted by (see [3])

$$F(g) = \int_{\widehat{G}(\delta)} \phi_\delta^U(g^{-1})\widehat{F}(U)d\mu(U),$$

where $d\mu$ is the Plancherel measure on $\widehat{G}(\delta)$. The map $\theta : U \to P(\delta)UP(\delta)$ is a bijection between $\widehat{G}(\delta)$ and $S_\delta^+(G)$, the space of positive definite elements of $S_\delta(G)$. Therefore $S_\delta^+(G)$ can make into a measure space by transporting the measure space structure of $\widehat{G}(\delta)$. In fact, $A \subset S_\delta^+(G)$ is a measurable set if and only if $\theta^{-1}(A)$ is measurable in $\widehat{G}(\delta)$ and the map $\nu$ defined by $\nu(A) = \mu(\theta^{-1}(A))$ is a measure on $S_\delta^+(G)$. So we obtain an inversion formula for the spherical transform that is for $F \in \mathcal{C}(G,\delta,\delta)$ we have

$$F(g) = \int_{S_\delta^+(G)} \phi(g^{-1})\widehat{F}(\phi)d\nu(\phi).$$

We know [15] that the space of $\delta$-radial functions on $N$, $\mathcal{U}_{c,\delta}(N)$, is isomorphic to $\mathcal{C}_c(G,\delta,\delta)$. This isomorphism is defined by restriction. Also from the content of the last part of Theorem 3.4 proof , it is clear that that the restriction map $\theta'$ from

$S_\delta^+(G)$ to $S_\delta(N)$ is surjective. It is also injective. In fact, if $\psi|N = \psi'|N$ then we have for all $(k,n) \in G = k \propto N$,

$$\psi(k,n) = \psi(e,n)\mu_\delta(k) = \psi'(e,n)\mu_\delta(k) = \psi'(k,n).$$

So using the map $\theta'$, we can equip $S_\delta(N)$ with a measure space structure and a measure $\kappa$ following the method just describe above. Therefore the inversion is written by $F \in \mathcal{U}_{c,\delta}(N)$

$$F(x) = \frac{1}{d_\delta}\int_{S_\delta(N)} \phi(x^{-1})\widehat{F}(\phi)d\kappa(\phi).$$

□

**Theorem 4.3.** *Let* $F, H \in \mathcal{U}_{c,\delta}(N)$. *Then*

$$\int_N tr(F(x)H(x)^*)dx = \int_{S_\delta(N)} \widehat{F}(\phi)\overline{\widehat{H}(\phi^*)}d\phi.$$

*Proof.* For $F, H \in \mathcal{U}_{c,\delta}(N)$, we have

$$tr(H^* * F(e)) = \int_G tr(H^*(x^{-1})F(x))dx = \int_G tr(H(x)^*F(x))dx.$$

Since $H^* * F \in \mathcal{U}_{c,\delta}(N)$ then by inversion formula and theorem 4.2 (ii)

$$\begin{aligned} tr(H^* * F(e)) =& \frac{1}{d_\delta}\int_{S_\delta(N)} tr(\phi(e)\widehat{H^* * F}(\phi))d\phi = \frac{1}{d_\delta}\int_{S_\delta(N)} tr(\phi(e)\widehat{H^*}(\phi)\widehat{F}(\phi))d\phi \\ =& \frac{1}{d_\delta}\int_{S_\delta(N)} tr(\phi(e))\widehat{H^*}(\phi)\widehat{F}(\phi)d\phi = \int_{S_\delta(N)} \widehat{F}(\phi)\overline{\widehat{H}(\phi^*)}d\phi \end{aligned}$$

and we have done. □

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