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DIFFERENCE OPERATOR AND DERIVATIVE ON THE DYADIC FIELD

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ABSTRACT. This work gives the form of derivation introduced in [4] in the context of dyadic field. We discuss the relation of this derivative to Fourier transform as well as its appropriate anti derivative.

1. INTRODUCTION

There are many forms of generalized dyadic derivatives. Some of them can be found in [1], [2], [9], [5]. Foundations of Gibbs differential operators and Butzer-Wagner derivative as well as some of their generalizations are explained and discussed in [7] and [8].

The paper of Andreas Klotz [4] provides a new form of dyadic derivative D_γ which is by convolution and with respect to the Hilbert Schmidt norm compatible with the 2-adic difference operator $\triangle_N$.

The derivative D_γ can also be represented by dyadic translations as done in the definition of the Butzer-Wagner dyadic derivative [3]. Like the extension of Butzer and Wagner derivative to the dyadic field F presented in [6, Chapter 9], we extend Klotz's derivative and show some of its properties related to Fourier transform in $L^2(F)$.

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The dyadic field F consists of doubly infinite sequences $x = (x_n, n \in \mathbb{Z})$, where $x_n \in \{0, 1\}$, for every $n \in \mathbb{Z}$, and $x_n \to 0$ as $n \to -\infty$.

The addition is defined componentwise modulo 2, the product is obtained by the relation $\xi_n = \sum\limits_{i+j=n} x_i y_j$ if $\xi = x \cdot y$ and the sum is taken modulo 2.

We define the sequence $(e_i)_{i\in\mathbb{Z}}$ of F by $e_i = (\delta_{i,n}, n \in \mathbb{Z})$.

The norm of any element $x \in G$ is defined by $|x| = \sum\limits_{n\in\mathbb{Z}} x_n 2^{-n-1}$.

The dyadic group G is the additive subgroup of F defined by

$$G = \{x \in F : x_n = 0, \forall n < 0\}.$$

We also introduce the additive subgroups I_N, $N \in \mathbb{Z}$, by

$$I_N = \{x \in F : x_n = 0, \forall n < N\}.$$

Clearly, the dyadic group G is nothing but the subgroup I_0.

If A is a measurable subset of F, then χ_A will denote its characteristic function.

The Walsh-Paley system is the set of functions defined on G as

$$\omega_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k},\ n \in \mathbb{N},\ x \in G,$$

where $n = \sum\limits_{k=0}^{\infty} n_k 2^k$ and the k-th rademacher function is given by $r_k(x) = (-1)^{x_k}$.

For every integrable function f on G we define the sequence of Fourier coefficients $(\hat{f}(n))_n$ by

$$\hat{f}(n) = \int_G f(t)\omega_n(t)dt.$$

On the other hand, additive characters on F are given by the relation

$$\psi_y(x) = \psi_x(y) = (-1)^{(x\cdot y)_{-1}}.$$

Fourier transform for every function from $L^2(F)$ is defined in [6] as

$$\mathbf{F}f(y) = \lim_{N\to+\infty} \int_{I_{-N}} f(x)\psi_y(x)dx,$$

where the limit is taken in $L^2(F)$. It was also proved in [6] that

$$\|\mathbf{F}f\|_2 = \|f\|_2$$

and

$$\mathbf{F}^2 f = \mathbf{F}(\mathbf{F}f) = f.$$

We recall Fine's map $\rho : [0, +\infty) \to F$ by $\rho(x) = (x_n, n \in \mathbb{Z})$, where $(x_n, n \in \mathbb{Z})$ are the elements of the dyadic expansion of x. For positive numbers having two different dyadic expansions we choose the finite one. Fine's map satisfies the relation $\rho(|x|) = x$, for every $x \in F$.

Following A. Klotz in [4] for every nonnegative integer $k = \sum_{i=0}^{\infty} k_i 2^i$, we define the nonnegative integer $\mathfrak{g}_k$ by

$$\mathfrak{g}_k = \sum_{i=0}^{\infty} (k_i + k_{i+1} (\text{mod } 2)) 2^i.$$

Similarly, we define the number $\mathfrak{g}_k^{-1}$ which is such that

$$l = \mathfrak{g}_k^{-1} \Leftrightarrow k = \mathfrak{g}_l.$$

It can be seen that

$$\mathfrak{g}_k^{-1} = \sum_{i=0}^{\infty} (\sum_{s=i}^{\infty} k_s (\text{mod } 2)) 2^i.$$

Moreover, if N and k are nonnegative integers then $2^N \leq k < 2^{N+1}$ if and only if $2^N \leq \mathfrak{g}_k < 2^{N+1}$.

Indeed, assume that for some l and N, $2^N \leq l < 2^{N+1}$, then if $k = \mathfrak{g}_l$, we have

$$k_N = l_N + l_{N+1} = l_N,$$

and

$$k_s = l_s + l_{s+1} = 0, \forall s \geq N + 1.$$

Hence, $2^N \leq k = \mathfrak{g}_l < 2^{N+1}$. Moreover,

$$l_N = k_N = \sum_{s=N}^{\infty} k_s (\text{mod } 2),$$

and if we assume that for some $0 \leq i \leq N - 1$,

$$l_{N-i} = \sum_{s=N-i}^{\infty} k_s (\text{mod } 2),$$

then,

$$\sum_{s=N-i-1}^{\infty} k_s = k_{N-i-1} + l_{N-i} (\text{mod } 2) = l_{N-i-1} + 2 l_{N-i} (\text{mod } 2) = l_{N-i-1} (\text{mod } 2).$$

The generalized dyadic derivative D_γ introduced in [4] is defined for every integrable function f on the group G by

$$D_\gamma f = \lim_{N\to+\infty}\left(f * \sum_{k=0}^{2^N-1} 2(\mathfrak{g}_k^{-1} + (\mathfrak{g}_k^{-1})_0)\omega_k\right),$$

if the limit exists in $L^1(G)$.

2. Main results

Definition 2.1. *Let $f \in L^2(G)$. Define the sequence of operators $(A_N)_{N\in\mathbb{N}}$ and $(D_N)_{N\in\mathbb{N}}$ as follows*

$$A_N f(y) = \sum_{k=0}^{N-1} 2^{k-N} f(y + \sum_{j=k}^{N-1} e_j) + 2^{-N} f(y + \sum_{j=0}^{N-1} e_j),$$

$$D_N f = 2^N (I - A_N) f.$$

Suppose that the sequence $(D_N f)_{N\in\mathbb{N}}$ converges in $L^2(G)$. Then, define the derivative Df by

$$Df = \lim_{N\to\infty} D_N f,$$

where the limit is taken in $L^2(G)$.

Proposition 2.1. *Let $f \in L^2(G)$ be such that $D_\gamma f$ exists in $L^2(G)$. Then, $Df = D_\gamma f$.*

Proof. It suffices to prove that for all $l < 2^N$

$$(D_N f)^\wedge(\mathfrak{g}_l) = 2(l_0 + l)\hat{f}(\mathfrak{g}_l).$$

Indeed,

$$\begin{aligned}
(A_N f)^\wedge(l) &= \int \left(\sum_{k=0}^{N-1} 2^{k-N} f(y + \sum_{j=k}^{N-1} e_j) + 2^{-N} f(y + \sum_{j=0}^{N-1} e_j)\right)\omega_l(y)dy \\
&= 2^{-N}\sum_{k=0}^{N-1} 2^k \int f(y)\omega_l(y + \sum_{j=k}^{N-1} e_j)dy + 2^{-N}\int f(y)\omega_l(y + \sum_{j=0}^{N-1} e_j)dy \\
&= 2^{-N}\sum_{k=0}^{N-1} 2^k \hat{f}(l)\omega_l\left(\sum_{j=k}^{N-1} e_j\right) + 2^{-N}\hat{f}(l)\omega_l\left(\sum_{j=0}^{N-1} e_j\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-N} \hat{f}(l) \left(\sum_{k=0}^{N-1} 2^k \prod_{i=0}^{\infty} l_i \omega_{2^i} \left(\sum_{j=k}^{N-1} e_j \right) + \prod_{i=0}^{\infty} l_i \omega_{2^i} \left(\sum_{j=0}^{N-1} e_j \right) \right) \\
&= 2^{-N} \hat{f}(l) \left(\sum_{k=0}^{N-1} 2^k (-1)^{\sum_{i=k}^{N-1} l_i} + (-1)^{\sum_{i=0}^{N-1} l_i} \right) \\
&= 2^{-N} \hat{f}(l) \left(\sum_{k=0}^{N-1} 2^k (1 - 2 \sum_{i=k}^{N-1} l_i (\text{mod } 2)) + 1 - 2 \sum_{i=0}^{N-1} l_i (\text{mod } 2) \right) \\
&= 2^{-N} \hat{f}(l) \left(2^N - 2\mathfrak{g}_l^{-1} - 2(\mathfrak{g}_l^{-1})_0 \right).
\end{aligned}
$$

The result follows immediately. □

Since this form of derivative on $L^2(G)$ includes addition on the dyadic group as a group of characteristic 2 rather than D_γ which is expressed by means of operations on G as a subset of a 0-characteristic group, this enables us to generate a formula adapted to the dyadic field.

Definition 2.2. *Let $f \in L^2(F)$. Define the sequence of operators A_N and D_N as follows*

$$
A_N f(y) = \sum_{k=-N}^{N-1} 2^{k-1} f(y + \sum_{j=k}^{N-1} e_j)), N \in \mathbb{N},
$$
$$
D_N = (2^{N-1} - 2^{-N-1})I - A_N.
$$

Then, define the derivative Df by

$$
Df = \lim_{N \to \infty} D_N f,
$$

if the limit exists in $L^2(F)$.

Definition 2.3. *We define the functions $(\mathbf{G}_N)_{N \in \mathbb{N}}$ on F in the following way. If $y \in I_{-N}$ and $\xi = (\xi_k, k \in \mathbb{Z}) = \mathbf{G}_N(y)$, then*

$$
\begin{cases} \xi_k = \sum_{j=-N}^{k} y_j (\text{mod } 2), & -N \leq k \leq N-1; \\ \xi_k = 0, & \textit{otherwise.} \end{cases}
$$

If $y \in I_{-N}^c$ then $\mathbf{G}_N(y) = 0$.

It is easily seen that $G_N(y) \neq 0$ if and only if $y \in I_{-N} \setminus I_N$ and that the sequence $(\mathbf{G}_N)_N$ converges pointwise, then define the pointwise limit $\mathbf{G} = \lim_{N \to \infty} \mathbf{G}_N$.

Definition 2.4. *We define the translation operator $T_{2^{-N}}$, where N is a positive integer, by $T_{2^{-N}}f(x) = f(\rho(|x| - 2^{-N}))$. Then, the difference operator $\triangle_N$ is defined as $\triangle_N = 2^N(I - T_{2^{-N}})$, where I is the identity operator.*

For every $N \in \mathbb{N}$ we define the function $M_N : F \setminus I_N \to \mathbb{Z}$ such that $M_N(x)$ is the largest integer from $(-\infty, N-1]$ for which $x_{M_N(x)} = 1$.

Lemma 2.1. *Let $f \in L^2(F)$. Then the sequence $(|\mathbf{G}_N| \cdot f)_N$ converges in $L^2(G)$ if and only if $(|\mathbf{G}| \cdot f) \in L^2(F)$. Moreover, in such a case we have*

$$|\mathbf{G}_N| \cdot f \to |\mathbf{G}| \cdot f, N \to \infty,$$

where convergence holds in $L^2(F)$.

Proof. For every $i \in \{-N, \dots, N-1\}$ and $y \in I_i \setminus I_{i+1}$, we have

$$|\mathbf{G}_N(y)| = \sum_{k=i}^{N-1} 2^{-k-1} \left(\sum_{j=i}^{k} y_j (\mathrm{mod}\ 2) \right),$$
$$|\mathbf{G}(y)| = \sum_{k=i}^{\infty} 2^{-k-1} \left(\sum_{j=i}^{k} y_j (\mathrm{mod}\ 2) \right).$$

Therefore,

$$0 \leq |\mathbf{G}(y)| - |\mathbf{G}_N(y)| \leq 2^{-N}.$$

Hence,

$$(\| |\mathbf{G}_N| \cdot f\|_2 - \|\chi_{I_{-N} \setminus I_N} \cdot |\mathbf{G}| \cdot f\|_2)^2 \leq \|\chi_{I_{-N} \setminus I_N} \cdot (|\mathbf{G}_N| - |\mathbf{G}|) \cdot f\|_2^2 \leq 2^{-N}\|f\|_2^2.$$

The result follows by letting N tend to infinity. □

We prove an analogue of [4, Lemma 6.] concerning the diagonalization of the difference operator by means of additive characters adapted to dyadic field.

Lemma 2.2. *Let N be a positive integer and $y \in I_{-N}$. Then,*

$$\mathbf{F}\left(\triangle_N \psi_y \cdot \chi_{I_{-N} \setminus I_N}\right)(y) = 2^{N+1}|\mathbf{G}_N(y)|.$$

Proof. We can easily verify that

$$\mathbf{F}(T_{2^{-N}}\psi_y \cdot \chi_{I_{-N} \setminus I_N})(y) = 2^N - 2^{-N} - 2|\mathbf{G}_N(y)|.$$

Indeed,

$$\mathbf{F}(T_{2^{-N}}\psi_y \cdot \chi_{I_{-N} \setminus I_N})(y) = \int_{I_{-N} \setminus I_N} \psi_y(\rho(|x| - 2^{-N}))\psi_y(x)dx$$

$$
\begin{aligned}
&= \int_{I_{-N}\setminus I_N} \psi_y \left(\sum_{j=M_N(x)}^{N-1} e_j \right) dx = \sum_{k=-N}^{N-1} \int_{\{x\in I_{-N}:M_N(x)=k\}} \prod_{j=k}^{N-1} (-1)^{y_{-j-1}} dx \\
&= \sum_{k=-N}^{N-1} 2^k (-1)^{\sum_{j=-N}^{-k-1} y_j} = \sum_{k=-N}^{N-1} 2^k \left(1 - 2 \left(\sum_{j=-N}^{-k-1} y_j (\mathrm{mod}\ 2) \right) \right) \\
&= 2^N - 2^{-N} - 2 \sum_{k=-N}^{N-1} 2^k (\sum_{j=-N}^{-k-1} y_j (\mathrm{mod}\ 2)) = 2^N - 2^{-N} - 2|\mathbf{G}_N(y)|.
\end{aligned}
$$

On the other hand, we have

$$
\mathbf{F}\left(\psi_y \cdot \chi_{I_{-N}\setminus I_N}\right)(y) = \int_{I_{-N}\setminus I_N} 1 dx = 2^N - 2^{-N}.
$$

Hence,

$$
\begin{aligned}
2|\mathbf{G}_N(y)| &= 2^N - 2^{-N} - \mathbf{F}(T_{2^{-N}}\psi_y \cdot \chi_{I_{-N}\setminus I_N})(y) \\
&= \mathbf{F}\left(\psi_y \cdot \chi_{I_{-N}\setminus I_N} - T_{2^{-N}}\psi_y \cdot \chi_{I_{-N}\setminus I_N}\right) = 2^{-N}\mathbf{F}\left(\triangle_N \psi_y \cdot \chi_{I_{-N}\setminus I_N}\right)(y).
\end{aligned}
$$

□

Lemma 2.2 enables us to prove an L^2-analogue result of [6, Section 9.2, Theorem 6, p 420] using the function $\mathbf{G}$ on F.

Theorem 2.1. *Let $f \in L^2(F)$. Then $D(\mathbf{F}f)$ exists if and only if $(|\mathbf{G}| \cdot f) \in L^2(F)$. Moreover, in this case we have*

$$
D(\mathbf{F}f) = \mathbf{F}(|\mathbf{G}| \cdot f).
$$

Proof. Let $g \in L^2(F)$. Using Lemma 2.2 we have

$$
\begin{aligned}
2\mathbf{F}(|\mathbf{G}_N| \cdot \mathbf{F}g)(y) &= 2 \int \mathbf{F}g(x) |\mathbf{G}_N(x)| \psi_y(x) dx \\
&= \int \mathbf{F}g(x) \left(2^N - 2^{-N} - \mathbf{F}(T_{2^{-N}}\psi_x \cdot \chi_{I_{-N}\setminus I_N})(x)\right) \psi_y(x) dx \\
&= (2^N - 2^{-N}) \int \mathbf{F}g(x) \psi_y(x) dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int \mathbf{F}g(x) \left(\int T_{2^{-N}} \psi_x(z) \cdot \chi_{I_{-N} \setminus I_N}(z) \psi_x(z) dz \right) \psi_x(y) dx \\
&= (2^N - 2^{-N}) g(y) - \int \chi_{I_{-N} \setminus I_N}(z) (\int \mathbf{F}g(x) \psi_x(y + \sum_{j=M_N(z)}^{N-1} e_j) dx) dz \\
&= (2^N - 2^{-N}) g(y) - \int \chi_{I_{-N} \setminus I_N}(z) g(y + \sum_{j=M_N(z)}^{N-1} e_j) dz \\
&= (2^N - 2^{-N}) g(y) - \sum_{k=-N}^{N-1} 2^k g(y + \sum_{j=k}^{N-1} e_j) = 2 D_N g(y).
\end{aligned}
$$

Hence,

$$D_N(\mathbf{F}f) = \mathbf{F}(|\mathbf{G}_N| \cdot f),$$

which implies the result by our assumptions and by Lemma 2.1. □

Remark 2.1. *From Theorem 2.1 we deduce immediately that for arbitrary $f \in L^2(F)$, Df exists if and only if $|\mathbf{G}| \cdot \mathbf{F}f \in L^2(F)$. In such a case we have*

$$Df = \mathbf{F}(|\mathbf{G}| \cdot \mathbf{F}f).$$

Proceeding as in [6] we suggest the following form of anti-derivative:

Definition 2.5. *Define the sequence of functions in $L^2(F)$:*

$$W_N = \mathbf{F}\left(\chi_{I_{-N} \setminus I_{[\frac{N}{4}]}} \cdot |\mathbf{G}_N|^{-1} \right).$$

Let $f \in L^2(F)$ be such that $(W_N * f)_N$ converges in $L^2(F)$. Then define the anti-derivative of f by

$$If := \lim_{N \to \infty} W_N * f,$$

where the limit is taken in $L^2(F)$.

Lemma 2.3. *Let $f \in L^2(F)$. Then the sequence $(\chi_{I_{-N} \setminus I_{[\frac{N}{4}]}} \cdot |\mathbf{G}_N|^{-1} \cdot f)_N$ converges in $L^2(F)$ if and only if $(|\mathbf{G}|^{-1} \cdot f) \in L^2(F)$. Moreover, in such a case*

$$\chi_{I_{-N} \setminus I_{[\frac{N}{4}]}} \cdot |\mathbf{G}_N|^{-1} \cdot f \to |\mathbf{G}|^{-1} \cdot f, N \to \infty,$$

where convergence holds in $L^2(F)$.

Proof. Let $i \in \{-N, \dots, [\frac{N}{4}] - 1\}$ and $y \in I_i \setminus I_{i+1}$. We have

$$|\mathbf{G}_N(y)| = \sum_{k=i}^{N-1} 2^{-k-1} \left(\sum_{j=i}^{k} y_j (\mathrm{mod}\ 2) \right),$$
$$|\mathbf{G}(y)| = \sum_{k=i}^{\infty} 2^{-k-1} \left(\sum_{j=i}^{k} y_j (\mathrm{mod}\ 2) \right).$$

Therefore,

$$0 \le |\mathbf{G}_N(y)|^{-1} - |\mathbf{G}(y)|^{-1} = \frac{|\mathbf{G}(y)| - |\mathbf{G}_N(y)|}{|\mathbf{G}(y)| \cdot |\mathbf{G}_N(y)|} \le 2^{2i-N+2} \le 2^{-[\frac{N}{2}]+2}.$$

Hence,

$$(\|\chi_{I_{-N} \setminus I_{[\frac{N}{4}]}} \cdot |\mathbf{G}_N|^{-1} \cdot f\|_2 - \|\chi_{I_{-N} \setminus I_{[\frac{N}{4}]}} \cdot |\mathbf{G}|^{-1} \cdot f\|_2)^2$$
$$\le \|\chi_{I_{-N} \setminus I_{[\frac{N}{4}]}} \cdot (|\mathbf{G}_N|^{-1} - |\mathbf{G}|^{-1}) \cdot f\|_2^2 \le 2^{-N+5} \|f\|_2^2.$$

The result follows by letting N tend to infinity. □

Since the Fourier transform preserves the norm in $L^2(F)$, the following result can be immediately obtained from Definition 2.5 and Lemma 2.3.

Lemma 2.4. *Let $f, g \in L^2(F)$. Then, $g = If$ if and only if $\mathbf{F}g = |\mathbf{G}|^{-1} \cdot \mathbf{F}f$.*

Theorem 2.2. *Let $f \in L^2(F)$ be such that If exists. Then, $|\mathbf{G}| \cdot \mathbf{F}(If) \in L^2(F)$ and $f = D(If)$.*

Proof. From Lemma 2.4 we get that

$$\mathbf{F}(If) = |\mathbf{G}|^{-1} \cdot \mathbf{F}f.$$

Hence, $|\mathbf{G}| \cdot \mathbf{F}(If) \in L^2(F)$ and

$$f = \mathbf{F}\left(|\mathbf{G}| \cdot \mathbf{F}(If)\right),$$

which by Theorem 2.1 leads to $f = D(If)$. □

Theorem 2.3. *Let $f \in L^2(F)$ be such that Df exists. Then, $f = I(Df)$.*

Proof. By Remark 2.1 we can see that $\mathbf{F}(Df) = |\mathbf{G}| \cdot \mathbf{F}f$, then from Lemma 2.4 we obtain immediately that $f = I(Df)$. □

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