

WEYL SPACES OF COMPOSITIONS OF SIX BASIC MANIFOLDS

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ABSTRACT. By help of prolonged covariant differentiation, Cartesian compositions of six basic manifolds are studied. Weyl spaces of such compositions are characterized. Eleven-dimensional Riemannian spaces containing compositions of six basic manifolds are also considered.

1. INTRODUCTION

Spaces with a symmetric affine connection, Riemannian spaces and Weyl spaces equipped with additional tensor structures of type (1,1) are extensively studied, cf. [1, 5, 7, 8, 11, 12, 14].

In [18, 19], an apparatus for the study of spaces with a symmetric affine connection and special compositions or nets is introduced. This apparatus is constructed by means of n independent vector fields and their reciprocal covector fields. The same apparatus is applied to the study of triples of compositions in [2], almost contact and almost paracontact structures in [4], and to four-dimensional spaces with additional structures in [3].

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The apparatus of prolonged covariant differentiation in Weyl spaces is constructed in [20, 21]. By help of this apparatus, in [9, 10, 15, 16], the geometry of Weyl spaces is investigated.

In the present work¹, in Weyl spaces, we consider compositions of six basic manifolds whose tangent planes are translated parallelly along any line in the space, i.e. Cartesian compositions. Spaces containing such compositions are characterized by help of the prolonged covariant differentiation. In the parameters of the chosen coordinate net, the fundamental tensor of Weyl spaces with Cartesian compositions is determined. Eleven-dimensional Riemannian spaces of such compositions are also studied.

2. PRELIMINARIES

Let $W_N(g_{\alpha\beta}, \omega_\sigma)$ be a Weyl space with metric tensor $g_{\alpha\beta}$ and additional covector ω_σ . We assume that the metric tensor admits a transformation (renormalization) of the form [6]

$$(2.1) \quad \check{g}_{\alpha\beta} = \lambda^2 g_{\alpha\beta},$$

where λ is a non-zero function of the point. Then, the covector ω_σ is transformed according to the rule [6]

$$(2.2) \quad \check{\omega}_\sigma = \omega_\sigma + \partial_\sigma \ln \lambda,$$

i.e. ω_σ is a normalizer.

The renormalization (2.1) of the metric tensor $g_{\alpha\beta}$ implies the renormalization $\check{g}^{\alpha\beta} = \lambda^{-2} g^{\alpha\beta}$ of its reciprocal tensor $g^{\alpha\beta}$.

Let v^α_{σ} ($\alpha, \sigma = 1, 2, \dots, N$) be independent directional fields. The renormalization of v^α_{σ} follows the rule [20, 21]

$$(2.3) \quad g_{\alpha\beta} v^\alpha_{\sigma} v^\beta_{\sigma} = 1.$$

The reciprocal covector fields $\check{v}^\sigma_{\alpha}$ of v^α_{σ} are defined by

$$(2.4) \quad v^\alpha_{\sigma} \check{v}^\sigma_{\beta} = \delta^\alpha_{\beta} \iff v^\sigma_{\beta} \check{v}^\sigma_{\alpha} = \delta^\alpha_{\beta},$$

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where δ_β^α is the identity tensor. The transformation (2.1) implies

$$\check{v}_\sigma^\alpha = \lambda^{-1} v_\sigma^\alpha, \quad \check{v}_\alpha^\sigma = \lambda v_\alpha^\sigma.$$

According to [20, 21], a pseudo-quantity A in W_N which by the renormalization (2.1) is transformed according to the rule $\check{A} = \lambda^k A$ is called a satellite of $g_{\alpha\beta}$ with weight $\{k\}$. Hence, $g^{\alpha\beta}\{-2\}$, $v_\sigma^\alpha\{-1\}$, $\check{v}_\alpha^\sigma\{1\}$, $\delta_\alpha^\beta\{0\}$.

According to [6], we have

$$(2.5) \quad \nabla_\sigma g_{\alpha\beta} = 2\omega_\sigma g_{\alpha\beta}, \quad \nabla_\sigma g^{\alpha\beta} = -2\omega_\sigma g^{\alpha\beta},$$

where ∇ is the Weyl connection.

The existence of the normalizer ω_σ allows the introduction of the prolonged covariant differentiation of the satellites A of $g_{\alpha\beta}$ with weight $\{k\}$ by the following formula

$$(2.6) \quad \overset{\circ}{\nabla}_\sigma A = \nabla_\sigma A - k\omega_\sigma A.$$

According to [20, 21], the following equalities are valid:

$$(2.7) \quad \overset{\circ}{\nabla}_\sigma g_{\alpha\beta} = \overset{\circ}{\nabla}_\sigma g^{\alpha\beta} = 0, \quad \overset{\circ}{\nabla}_\sigma v_\alpha^\beta = \nabla_\sigma v_\alpha^\beta + \omega_\sigma v_\alpha^\beta, \quad \overset{\circ}{\nabla}_\sigma \check{v}_\beta^\alpha = \nabla_\sigma \check{v}_\beta^\alpha - \omega_\sigma \check{v}_\beta^\alpha.$$

Further in this work we will use the following notation indices:

$$(2.8) \quad \begin{aligned} \alpha, \beta, \gamma, \sigma, \nu, \delta, \tau &= 1, 2, \dots, N; \\ i, j, k, l, s &= 1, 2, \dots, m_1; \quad \bar{i}, \bar{j}, \bar{k}, \bar{l}, \bar{s} = m_1 + 1, m_1 + 2, \dots, m_1 + n_1; \\ a, b, c, d &= m_1 + n_1 + 1, m_1 + n_1 + 2, \dots, m_1 + n_1 + l_1; \\ \bar{a}, \bar{b}, \bar{c}, \bar{d} &= m_1 + n_1 + l_1 + 1, \dots, m_1 + n_1 + l_1 + m_2; \\ p, q, r, t &= m_1 + n_1 + l_1 + m_2 + 1, \dots, m_1 + n_1 + l_1 + m_2 + n_2; \\ \bar{p}, \bar{q}, \bar{r}, \bar{t} &= m_1 + n_1 + l_1 + m_2 + n_2 + 1, \dots, m_1 + n_1 + l_1 + m_2 + n_2 + l_2 = N. \end{aligned}$$

The independent directional fields v_σ^α and the covector fields \check{v}_α^σ satisfy the derivative equations [17, 20]

$$(2.9) \quad \overset{\circ}{\nabla}_\sigma v_\alpha^\beta = \overset{\nu}{T}_\sigma^\alpha v_\nu^\beta, \quad \overset{\circ}{\nabla}_\sigma \check{v}_\beta^\alpha = -\overset{\alpha}{T}_\sigma^\nu \check{v}_\beta^\nu,$$

where $\overset{\beta}{T}_\sigma^\alpha\{0\}$.

We denote by $(v)_\beta$ the lines defined by the directional fields v_β^α and by $\{v\}_\alpha$ the net determined by the independent directional fields v_σ^α ($\sigma = 1, 2, \dots, N$).

Let $\{v_\alpha\}$ be the coordinate net. According to (2.3) and (2.4), we have

$$(2.10) \quad \begin{aligned} & v_1^\alpha \left(\frac{1}{\sqrt{g_{11}}}, 0, \dots, 0 \right), \quad v_2^\alpha \left(0, \frac{1}{\sqrt{g_{22}}}, 0, \dots, 0 \right), \dots, v_N^\alpha \left(0, \dots, 0, \frac{1}{\sqrt{g_{NN}}} \right), \\ & v_\alpha^1(\sqrt{g_{11}}, 0, \dots, 0), \quad v_\alpha^2(0, \sqrt{g_{22}}, 0, \dots, 0), \dots, v_\alpha^N(0, \dots, 0, \sqrt{g_{NN}}). \end{aligned}$$

In addition to the arbitrary coordinates x^α ($\alpha = 1, 2, \dots, N$), in W_N we introduce coordinates with respect to the coordinate net $\{v_\alpha\}$ which we will denote by u^α .

Then for an arbitrary directional field v^α we have $v^\alpha \left(\frac{u^1}{\sqrt{g_{11}}}, \frac{u^2}{\sqrt{g_{22}}}, \dots, \frac{u^N}{\sqrt{g_{NN}}} \right)$.

The Christoffel symbols $\Gamma_{\alpha\beta}^\sigma$ of the Weyl connection ∇ are defined by [6]

$$(2.11) \quad \Gamma_{\alpha\beta}^\sigma = \frac{1}{2} g^{\sigma\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) - (\omega_\alpha \delta_\beta^\nu + \omega_\beta \delta_\alpha^\nu - \omega_\nu g^{\nu\sigma} g_{\alpha\beta}).$$

In the parameters of the coordinate net $\{v_\alpha\}$ we have [4, 18, 19]

$$(2.12) \quad T_{\sigma}^{\beta} = \frac{\sqrt{g_{\beta\beta}}}{\sqrt{g_{\alpha\alpha}}} \Gamma_{\sigma\alpha}^{\beta}, \quad \alpha \neq \beta, \quad T_{\sigma}^{\alpha} = \Gamma_{\sigma\alpha}^{\alpha} - \frac{1}{2} \partial_\sigma \ln g_{\alpha\alpha} + \omega_\sigma.$$

It is known that a directional field v_ν^α is translated parallelly along the lines $(v)_\beta$ if and only if $\nabla_\sigma v_\nu^\alpha v_\beta^\sigma = \mu v_\nu^\alpha$, where μ is a function of the point. According to (2.7), the condition for the parallel translation of the field v_ν^α along the lines $(v)_\beta$ takes the form

$$(2.13) \quad \overset{\circ}{\nabla}_\sigma v_\nu^\alpha v_\beta^\sigma = \lambda v_\nu^\alpha,$$

where λ is a function of the point.

3. WEYL SPACES OF COMPOSITIONS OF SIX BASIC MANIFOLDS

In the Weyl space W_N , we consider a composition $X_{m_1} \times X_{n_1} \times X_{l_1} \times X_{m_2} \times X_{n_2} \times X_{l_2}$ of six basic manifolds denoted by X_{m_1} , X_{n_1} , X_{l_1} , X_{m_2} , X_{n_2} and X_{l_2} , i.e. their topological product.

According to [14], the definition of a composition in W_N is equivalent to the introduction of a tensor a_α^β which satisfies $a_\alpha^\sigma a_\sigma^\beta = \delta_\alpha^\beta$ and the integrability condition (vanishing of the Nijenhuis tensor)

$$(3.1) \quad a_\beta^\sigma \nabla_{[\alpha} a_{\sigma]}^\gamma - a_\alpha^\sigma \nabla_{[\beta} a_{\sigma]}^\gamma = 0.$$

Six positions (tangent spaces to the basic manifolds) pass through each point of W_N which are denoted by $P(X_{m_1})$, $P(X_{n_1})$, $P(X_{l_1})$, $P(X_{m_2})$, $P(X_{n_2})$ and $P(X_{l_2})$, respectively. According to [18, 19], the six projecting tensors are defined by

$$(3.2) \quad \begin{aligned} a_{\alpha}^1 &= v_{\bar{i}}^{\beta} v_{\alpha}^i, & a_{\alpha}^2 &= v_{\bar{i}}^{\beta} v_{\alpha}^{\bar{i}}, & a_{\alpha}^3 &= v_a^{\beta} v_{\alpha}^a, \\ a_{\alpha}^4 &= v_{\bar{a}}^{\beta} v_{\alpha}^{\bar{a}}, & a_{\alpha}^5 &= v_p^{\beta} v_{\alpha}^p, & a_{\alpha}^6 &= v_{\bar{p}}^{\beta} v_{\alpha}^{\bar{p}}. \end{aligned}$$

For an arbitrary directional field v^{α} we have

$$v^{\alpha} = a_{\sigma}^1 v^{\sigma} + a_{\sigma}^2 v^{\sigma} + \dots + a_{\sigma}^6 v^{\sigma} = V_1^{\alpha} + V_2^{\alpha} + \dots + V_6^{\alpha},$$

where $V_1^{\alpha} = a_{\sigma}^1 v^{\sigma} \in P(X_{m_1})$, $V_2^{\alpha} = a_{\sigma}^2 v^{\sigma} \in P(X_{n_1})$, $V_3^{\alpha} = a_{\sigma}^3 v^{\sigma} \in P(X_{l_1})$, $V_4^{\alpha} = a_{\sigma}^4 v^{\sigma} \in P(X_{m_2})$, $V_5^{\alpha} = a_{\sigma}^5 v^{\sigma} \in P(X_{n_2})$ and $V_6^{\alpha} = a_{\sigma}^6 v^{\sigma} \in P(X_{l_2})$. Obviously, $v_{\bar{i}}^{\alpha} \in P(X_{m_1})$, $v_{\bar{i}}^{\alpha} \in P(X_{n_1})$, $v_a^{\alpha} \in P(X_{l_1})$, $v_{\bar{a}}^{\alpha} \in P(X_{m_2})$, $v_p^{\alpha} \in P(X_{n_2})$ and $v_{\bar{p}}^{\alpha} \in P(X_{l_2})$.

Definition 3.1. The composition $X_{m_1} \times X_{n_1} \times X_{l_1} \times X_{m_2} \times X_{n_2} \times X_{l_2}$ is said to be Cartesian if the positions $P(X_{m_1})$, $P(X_{n_1})$, $P(X_{l_1})$, $P(X_{m_2})$, $P(X_{n_2})$ and $P(X_{l_2})$ are translated parallelly along any line in the space W_N .

We prove the following

Theorem 3.1. Equalities

$$(3.3) \quad \overset{\circ}{\nabla}_{\sigma} a_{\alpha}^1 = 0, \quad \overset{\circ}{\nabla}_{\sigma} a_{\alpha}^2 = 0, \quad \overset{\circ}{\nabla}_{\sigma} a_{\alpha}^3 = 0, \quad \overset{\circ}{\nabla}_{\sigma} a_{\alpha}^4 = 0, \quad \overset{\circ}{\nabla}_{\sigma} a_{\alpha}^5 = 0, \quad \overset{\circ}{\nabla}_{\sigma} a_{\alpha}^6 = 0$$

hold if and only if the coefficients of the derivative equations (2.9) satisfy the following conditions

$$(3.4) \quad \begin{aligned} \overset{s}{T}_{\bar{i}}^{\sigma} &= \overset{s}{T}_a^{\sigma} = \overset{s}{T}_{\bar{a}}^{\sigma} = \overset{s}{T}_p^{\sigma} = \overset{s}{T}_{\bar{p}}^{\sigma} = 0, & \overset{\bar{s}}{T}_i^{\sigma} &= \overset{\bar{s}}{T}_a^{\sigma} = \overset{\bar{s}}{T}_{\bar{a}}^{\sigma} = \overset{\bar{s}}{T}_p^{\sigma} = \overset{\bar{s}}{T}_{\bar{p}}^{\sigma} = 0, \\ \overset{a}{T}_i^{\sigma} &= \overset{a}{T}_{\bar{i}}^{\sigma} = \overset{a}{T}_{\bar{b}}^{\sigma} = \overset{a}{T}_p^{\sigma} = \overset{a}{T}_{\bar{p}}^{\sigma} = 0, & \overset{\bar{a}}{T}_i^{\sigma} &= \overset{\bar{a}}{T}_{\bar{i}}^{\sigma} = \overset{\bar{a}}{T}_b^{\sigma} = \overset{\bar{a}}{T}_p^{\sigma} = \overset{\bar{a}}{T}_{\bar{p}}^{\sigma} = 0, \\ \overset{p}{T}_i^{\sigma} &= \overset{p}{T}_{\bar{i}}^{\sigma} = \overset{p}{T}_a^{\sigma} = \overset{p}{T}_{\bar{a}}^{\sigma} = \overset{p}{T}_{\bar{q}}^{\sigma} = 0, & \overset{\bar{p}}{T}_i^{\sigma} &= \overset{\bar{p}}{T}_{\bar{i}}^{\sigma} = \overset{\bar{p}}{T}_a^{\sigma} = \overset{\bar{p}}{T}_{\bar{a}}^{\sigma} = \overset{\bar{p}}{T}_q^{\sigma} = 0. \end{aligned}$$

Proof. According to (2.9), the first equality in (3.3) takes the form

$$(3.5) \quad \overset{\nu}{T}_{\bar{i}}^{\sigma} v_{\nu}^{\beta} v_{\alpha}^i - \overset{i}{T}_{\sigma}^{\nu} v_{\nu}^{\beta} v_{\alpha}^{\nu} = 0.$$

The independence of the covector fields v_β^α implies that (3.5) is equivalent to the following equalities

$$(3.6) \quad \begin{aligned} T_i^\nu v_\nu^\beta - T_i^s v_s^\beta &= 0, & T_i^s v_s^\beta &= 0, & T_a^s v_s^\beta &= 0, \\ T_a^s v_s^\beta &= 0, & T_p^s v_s^\beta &= 0, & T_{\bar{p}}^s v_s^\beta &= 0. \end{aligned}$$

Since the directional fields v_α^β are also independent it follows that equalities (3.6) are equivalent to

$$(3.7) \quad \bar{s} T_\sigma = T_\sigma^a = \bar{a} T_\sigma = T_\sigma^p = \bar{p} T_\sigma = 0, \quad T_\sigma^s = T_\sigma^a = T_\sigma^{\bar{a}} = T_\sigma^p = T_\sigma^{\bar{p}} = 0.$$

Hence, $\overset{\circ}{\nabla}_\sigma a_\alpha^\beta = 0$ if and only if conditions (3.7) hold. Analogously, one can prove that the rest of the equalities in (3.3) hold if and only if conditions (3.4) are satisfied. \square

Keeping in mind (2.12) and (3.4), we get

Corollary 3.1. *In the parameters of the coordinate net $\{v_\alpha\}$ equalities (3.4) have the form*

$$(3.8) \quad \begin{aligned} \Gamma_{\sigma i}^s &= \Gamma_{\sigma a}^s = \Gamma_{\sigma \bar{a}}^s = \Gamma_{\sigma p}^s = \Gamma_{\sigma \bar{p}}^s = 0, & \Gamma_{\sigma i}^{\bar{s}} &= \Gamma_{\sigma a}^{\bar{s}} = \Gamma_{\sigma \bar{a}}^{\bar{s}} = \Gamma_{\sigma p}^{\bar{s}} = \Gamma_{\sigma \bar{p}}^{\bar{s}} = 0, \\ \Gamma_{\sigma i}^a &= \Gamma_{\sigma \bar{i}}^a = \Gamma_{\sigma \bar{b}}^a = \Gamma_{\sigma p}^a = \Gamma_{\sigma \bar{p}}^a = 0, & \Gamma_{\sigma i}^{\bar{a}} &= \Gamma_{\sigma \bar{i}}^{\bar{a}} = \Gamma_{\sigma b}^{\bar{a}} = \Gamma_{\sigma p}^{\bar{a}} = \Gamma_{\sigma \bar{p}}^{\bar{a}} = 0, \\ \Gamma_{\sigma i}^p &= \Gamma_{\sigma \bar{i}}^p = \Gamma_{\sigma a}^p = \Gamma_{\sigma \bar{a}}^p = \Gamma_{\sigma \bar{q}}^p = 0, & \Gamma_{\sigma i}^{\bar{p}} &= \Gamma_{\sigma \bar{i}}^{\bar{p}} = \Gamma_{\sigma a}^{\bar{p}} = \Gamma_{\sigma \bar{a}}^{\bar{p}} = \Gamma_{\sigma q}^{\bar{p}} = 0. \end{aligned}$$

By [8, 19] and (3.8) we obtain

Corollary 3.2. *The Weyl space W_N is a space of the Cartesian composition $X_{m_1} \times X_{n_1} \times X_{l_1} \times X_{m_2} \times X_{n_2} \times X_{l_2}$ if and only if conditions (3.3) hold.*

Now let us consider the case of an orthogonal Cartesian composition $X_{m_1} \times X_{n_1} \times X_{l_1} \times X_{m_2} \times X_{n_2} \times X_{l_2}$. Then the directional fields $v_i^\alpha, v_{\bar{i}}^\alpha, v_a^\alpha, v_{\bar{a}}^\alpha, v_p^\alpha, v_{\bar{p}}^\alpha$ are mutually orthogonal. In the parameters of the coordinate net $\{v_\alpha\}$, the

metric tensor is given by

$$(3.9) \quad (g_{\alpha\beta}) = \begin{pmatrix} g_{ij} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{\bar{i}\bar{j}} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{ab} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{\bar{a}\bar{b}} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{pq} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{\bar{p}\bar{q}} \end{pmatrix}.$$

Following [14], by (3.8), (2.5) and the integrability condition (3.1), in the parameters of the coordinate net $\{v_\alpha\}$, we obtain

$$(3.10) \quad \begin{aligned} g_{ij} &= f_1 \check{g}_{ij}(u^s), & g_{\bar{i}\bar{j}} &= f_2 \check{g}_{\bar{i}\bar{j}}(u^{\bar{s}}), & g_{ab} &= f_3 \check{g}_{ab}(u^c), \\ g_{\bar{a}\bar{b}} &= f_4 \check{g}_{\bar{a}\bar{b}}(u^{\bar{c}}), & g_{pq} &= f_5 \check{g}_{pq}(u^r), & g_{\bar{p}\bar{q}} &= f_6 \check{g}_{\bar{p}\bar{q}}(u^{\bar{r}}), \end{aligned}$$

where the functions $f = f(u^\alpha)$, $f_1 = f(u^\alpha)$, $f_2 = f(u^\alpha)$, $f_3 = f(u^\alpha)$, $f_4 = f(u^\alpha)$, $f_5 = f(u^\alpha)$, $f_6 = f(u^\alpha)$ satisfy the conditions

$$(3.11) \quad \begin{aligned} 2\omega_k &= \partial_k \ln f_2 = \partial_k \ln f_3 = \partial_k \ln f_4 = \partial_k \ln f_5 = \partial_k \ln f_6, \\ 2\omega_{\bar{k}} &= \partial_{\bar{k}} \ln f_1 = \partial_{\bar{k}} \ln f_3 = \partial_{\bar{k}} \ln f_4 = \partial_{\bar{k}} \ln f_5 = \partial_{\bar{k}} \ln f_6, \\ 2\omega_a &= \partial_a \ln f_1 = \partial_a \ln f_2 = \partial_a \ln f_4 = \partial_a \ln f_5 = \partial_a \ln f_6, \\ 2\omega_{\bar{a}} &= \partial_{\bar{a}} \ln f_1 = \partial_{\bar{a}} \ln f_2 = \partial_{\bar{a}} \ln f_3 = \partial_{\bar{a}} \ln f_5 = \partial_{\bar{a}} \ln f_6, \\ 2\omega_p &= \partial_p \ln f_1 = \partial_p \ln f_2 = \partial_p \ln f_3 = \partial_p \ln f_4 = \partial_p \ln f_6, \\ 2\omega_{\bar{p}} &= \partial_{\bar{p}} \ln f_1 = \partial_{\bar{p}} \ln f_2 = \partial_{\bar{p}} \ln f_3 = \partial_{\bar{p}} \ln f_4 = \partial_{\bar{p}} \ln f_5. \end{aligned}$$

According to [6], the integrability condition of equalities (2.5) has the form

$$(3.12) \quad R_{\gamma\delta\alpha\beta} + R_{\gamma\delta\beta\alpha} = -4\nabla_{[\gamma} \omega_{\delta]} g_{\alpha\beta}.$$

Since the Cartesian composition is orthogonal, the following equalities hold

$$(3.13) \quad a_{(\nu}^{i_1\alpha} a_{\sigma)}^{s_1\beta} g_{\alpha\beta} = 0,$$

where $i_1, s_1 = 1, 2, \dots, 6$ and $i_1 \neq s_1$. Equalities (3.12) and (3.13) imply

$$(3.14) \quad R_{\gamma\delta\alpha\beta} a_{(\nu}^{i_1\alpha} a_{\sigma)}^{s_1\beta} = 0.$$

Equalities (3.14) characterize the form of the curvature tensor of Weyl spaces with orthogonal compositions $X_{m_1} \times X_{n_1} \times X_{l_1} \times X_{m_2} \times X_{n_2} \times X_{l_2}$. According to [6], the Weyl space W_N is Riemannian if and only if $\omega_\alpha = \text{grad}$. Obviously, if $\omega_\alpha = \text{grad}$ then $f_1 = f_2 = \dots = f_6$ and vice versa.

4. ELEVEN-DIMENSIONAL RIEMANNIAN SPACES OF COMPOSITIONS OF SIX BASIC MANIFOLDS

Let $N = 11$, $m_1 = n_1 = l_1 = m_2 = n_2 = 2$, $l_2 = 1$ and $\omega_\alpha = \text{grad}$. Then, W_N is an 11-dimensional Riemannian space which we will denote by V_{11} . According to [6], in V_{11} , a normalization of λ can be chosen so that

$$(4.1) \quad \check{\omega}_\alpha = \partial_\alpha \ln \lambda, \quad \nabla_\sigma g_{\alpha\beta} = 0.$$

In the space V_{11} , let us consider Cartesian compositions $X_{m_1} \times X_{n_1} \times X_{l_1} \times X_{m_2} \times X_{n_2} \times X_{l_2}$ of five 2-dimensional and one 1-dimensional basic manifolds and let this composition be orthogonal. Then, in the parameters of the coordinate net $\{v_\alpha\}$, the matrix of the metric tensor $g_{\alpha\beta}$ of V_{11} has the form (3.9) where $g_{\bar{p}\bar{q}} = g_{11,11}$, and conditions (3.10) have the form

$$(4.2) \quad \begin{aligned} g_{ij} &= g_{ij}(u^s), & g_{\bar{i}\bar{j}} &= g_{\bar{i}\bar{j}}(u^{\bar{s}}), & g_{ab} &= g_{ab}(u^c), \\ g_{\bar{a}\bar{b}} &= g_{\bar{a}\bar{b}}(u^{\bar{c}}), & g_{pq} &= g_{pq}(u^r), & g_{11,11} &= g_{11,11}(u^{11}). \end{aligned}$$

Further, let the subnets $\left(\begin{smallmatrix} v \\ s_2 \end{smallmatrix}, \begin{smallmatrix} v \\ s_2+1 \end{smallmatrix}\right)$, $s_2 = 1, 3, 5, 7, 9$, be Chebyshevian which according to [20, 21], means that the directional fields $v_{s_2}^\alpha$ and $v_{s_2+1}^\alpha$ are translated parallelly along the lines $\left(\begin{smallmatrix} v \\ s_2+1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} v \\ s_2 \end{smallmatrix}\right)$, respectively. Then, by (2.9) and (2.13), it follows that the subnet $\left(\begin{smallmatrix} v \\ s_2 \end{smallmatrix}, \begin{smallmatrix} v \\ s_2+1 \end{smallmatrix}\right)$ is Chebyshevian if and only if the coefficients of the derivative equations (2.9) satisfy the conditions

$$(4.3) \quad \overset{\nu}{T}_{s_2+1} = 0, \quad \nu \neq s_2, \quad \overset{\nu}{T}_{s_2} = 0, \quad \nu \neq s_2 + 1.$$

According to (2.12), equalities (4.3) imply

$$(4.4) \quad \Gamma_{s_2+1 \ s_2}^\nu = 0, \quad \nu \neq s_2, \quad \Gamma_{s_2 \ s_2+1}^\nu = 0, \quad \nu \neq s_2 + 1.$$

From the fundamental property $\nabla_\sigma g_{\alpha\beta} = 0$ and equalities (4.4), in the parameters of the coordinate net $\{v_\alpha\}$, we obtain $g_{s_1 s_1} = g_{s_1 s_1}(u^{s_1})$, $s_1 = 1, 2, \dots, 10$.

Since the projecting tensors $\overset{1}{a}_{\alpha}^{\beta}$, $\overset{2}{a}_{\alpha}^{\beta}$, $\overset{3}{a}_{\alpha}^{\beta}$, $\overset{4}{a}_{\alpha}^{\beta}$, $\overset{5}{a}_{\alpha}^{\beta}$ and $\overset{6}{a}_{\alpha}^{\beta}$ have zero weights, then according to (2.6), the prolonged covariant derivative $\bar{\nabla}$ in (3.3) can be replaced by the usual covariant derivative ∇ .

Further, in the space V_{11} of the Cartesian composition $X_{m_1} \times X_{n_1} \times X_{l_1} \times X_{m_2} \times X_{n_2} \times X_{l_2}$, let us consider the tensor defined by

$$(4.5) \quad K_{\alpha\beta}^{\nu\tau} = 2 \overset{1}{a}_{(\alpha}^{\nu} \overset{2}{a}_{\beta)}^{\tau} + \overset{3}{a}_{\alpha}^{\nu} \overset{3}{a}_{\beta}^{\tau} + \overset{4}{a}_{\alpha}^{\nu} \overset{4}{a}_{\beta}^{\tau} - \overset{5}{a}_{\alpha}^{\nu} \overset{5}{a}_{\beta}^{\tau} - \overset{6}{a}_{\alpha}^{\nu} \overset{6}{a}_{\beta}^{\tau}.$$

According to (3.3) and (4.5), we have

$$(4.6) \quad \nabla_{\sigma} K_{\alpha\beta}^{\nu\tau} = 0.$$

Let us consider the following tensor constructed by help of the metric tensor $g_{\nu\tau}$ of V_{11}

$$(4.7) \quad \bar{g}_{\alpha\beta} = K_{\alpha\beta}^{\nu\tau} g_{\nu\tau}.$$

By (4.6) and (4.7) we obtain

$$(4.8) \quad \nabla_{\sigma} \bar{g}_{\alpha\beta} = 0.$$

According to (3.9) and (4.7), in the parameters of the coordinate net $\{v_{\alpha}\}$, the matrix of $\bar{g}_{\alpha\beta}$ is given by

$$(4.9) \quad (\bar{g}_{\alpha\beta}) = \begin{pmatrix} 0 & g_{i\bar{j}} & 0 & 0 & 0 & 0 \\ g_{i\bar{j}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{ab} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{\bar{a}\bar{b}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -g_{pq} & 0 \\ 0 & 0 & 0 & 0 & 0 & -g_{11,11} \end{pmatrix},$$

where $g_{ab} = g_{ab}(u^c)$, $g_{\bar{a}\bar{b}} = g_{\bar{a}\bar{b}}(u^{\bar{c}})$, $g_{pq} = g_{pq}(u^r)$ and $g_{11,11} = g_{11,11}(u^{11})$. According to [7], there exists a function Φ such that $g_{i\bar{j}} = \partial_{i\bar{j}}^2 \Phi = \frac{\partial^2 \Phi}{\partial u^i \partial u^{\bar{j}}}$. The space V_{11} equipped with the metric tensor $\bar{g}_{\alpha\beta}$ is a space of a composition of the four-dimensional manifold $X_{m_1} \cup X_{n_1}$ and the Riemannian manifolds X_{l_1} , X_{m_2} , X_{n_2} and X_{l_2} .

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