

## MATHEMATICAL SOLUTION OF IMBIBITION PHENOMENON IN VERTICAL HETEROGENEOUS POROUS MEDIUM

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**ABSTRACT.** The phenomenon of imbibition is of great importance in the petroleum industry. In this analysis, we have studied the Counter-current imbibition through mathematical modelling in a double non-miscible phase in the vertical heterogeneous porous medium, which ensues in the process of oil recovery. The non-linear partial differential equation of second-order is the result of the mathematical formulation. Variational iteration approach is applied to attain approximate analytical solution of the governing equation. The solution's numerical outcomes and graphical representations are identified with the help of MATLAB.

### 1. INTRODUCTION

If the wetting phase (water) enters the porous matrix and displaces the non-wetting phase (oil), imbibition is one of the most significant mechanisms. Imbibition is characterised as the movement of non-wetting phase (oil) through wetting phase (water), where capillary pressure is the driving force. The mechanism of imbibition can be split into two instances: (i) Free or spontaneous imbibition (ii) Forced imbibition. Spontaneous imbibition is the mechanism in which,

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by means of capillary pressures, wetting phase is drawn into a porous media and the curves communicate without any external force among the wetting and non-wetting phases. Spontaneous imbibition is of two kinds: (a) Co-current (b) Counter-current. Both the wetting and non-wetting phases travel in the same direction during the Co-current imbibition, while both travel in the opposite directions during the Counter-current imbibition. In short, Counter-current imbibition is characterised as "When porous medium filled with some fluid is brought in contact with another fluid that ideally moisturises the medium, there is a spontaneous flow of wetting fluid into the medium. This phenomenon occurs as a result of the disparity in wetting potential of fluids circulating in the medium" [3].

With distinct perspectives, several researchers researched this phenomenon. Hossein et. al. described the phenomenon of Counter-current and Co-current through fractured porous medium using method of Homotopy perturbation (HPM) [7]. Parikh et. al investigated mathematical model of the phenomenon for Counter-current imbibition in vertical homogeneous porous media applying Generalized functional separable method (GSM) [3]. Patel and Desai discussed the phenomenon of Counter-current through homogeneous inclined porous medium using Homotopy analysis method (HAM) [12]. Pathak and Singh examined the imbibition phenomenon through homogeneous inclined porous media employing Optimal homotopy analysis method (OHAM) [15]. Patel and Meher studied the mathematical model of imbibition phenomenon in liquid flow across inclined heterogeneous porous media with various porous materials using Adomian decomposition method (ADM) [8]. Choksi and Singh presented the phenomenon of fingering-imbibition with magnetic field effect using Homotopy perturbation sumudu transform method (HPSTM) [5].

This specific study deals with phenomenon of Counter-current imbibition ensuing from the water-oil flow across vertical heterogeneous porous medium with gravitational impact and mean capillary pressure.

A non-linear partial differential equation of second-order is the governing equation for the phenomenon. It is resolved by employing Variational iteration approach [10, 11] with suitable initial and boundary conditions. But, as yet, in vertical downward direction, no researcher had analysed this problem with heterogeneity and gravitational effect. Shah and Parikh analysed mathematical model of the phenomenon for fingering in heterogeneous vertical porous medium using method of Variational iteration (VIM) [6].

Purpose of the research is to measure saturation of the inserted water in the oil recovery process when phenomenon of the imbibition takes place.

## 2. STATEMENT OF THE PROBLEM

In a vertically downward direction, it is presumed that water will be injected uniformly through saturated heterogeneous porous matrix of oil. As water is being injected in an oil-formed porous media in a vertical downward direction into a pipe-shaped part of porous media surrounded by impermeable surfaces, the imbibition happens near the common interface as shown in Figure 1. The fingers are distinct and irregular in size and shape. For the present case, vertical cross-sectional region of pipe-shaped component of the porous media as a rectangular shape is assumed. The fingers are considered to be rectangles and, as shown in Figure 2, the cross-sectional region of the average length of the schematic fingers is considered.

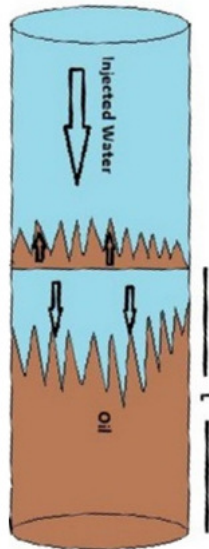


FIGURE 1. Imbibition phenomenon in pipe-shaped part of porous matrix in vertical downward direction

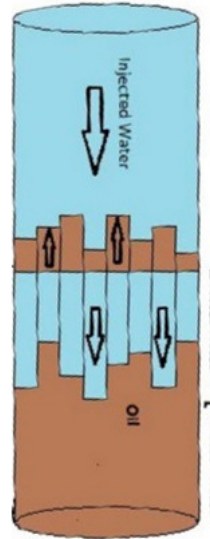


FIGURE 2. Schematic view of Imbibition phenomenon in vertical downward direction

For present study, following assumptions have been made:

- The medium is heterogeneous and the porosity and permeability are therefore chosen as variables.
- The rule of Darcy will be valid to calculate the velocities of water and oil for small Reynold 's number in fluid flow through porous medium.
- The spontaneous flow will take place without any external force in the counter-current direction due to variance in the viscosities of water and oil. In the velocity of injected water, gravitational impact will be introduced while it will be removed from the velocity of oil since fingers of oil occur for small distance and short duration in the upward direction during imbibition.

For specific case, the saturation of inserted water is depicted as a cross-section occupied via schematic fingers of average length at level  $z$  very close to the common interface as shown in the schematic Figure 2. The displacement processes are in  $z$ -direction with respect to time ( $t$ ) . Thus, saturation of inserted water is denoted by  $S_w(z, t)$ . It varies with respect to depth ( $z$ ) and time ( $t$ ).

### 3. MATHEMATICAL FORMULATION OF THE PROBLEM

Assuming the rule of Darcy is valid for the current flow system, it is possible to express the seepage velocities of inserted water as well as native oil as follows [2], [9], [13]:

$$(3.1) \quad V_w = - \left( \frac{K_w}{\mu_w} \right) K \left( \frac{\partial P_w}{\partial z} + \rho_w g \right),$$

$$(3.2) \quad V_o = - \left( \frac{K_o}{\mu_o} \right) K \left( \frac{\partial P_o}{\partial z} - \rho_o g \right).$$

The continuity equation of the water being inserted is given by,

$$(3.3) \quad P \left( \frac{\partial S_w}{\partial t} \right) + \frac{\partial V_w}{\partial z} = 0.$$

Capillary pressure ( $P_c$ ) is the function of phase saturation. It is expressed by pressure variance of the oil and water across their common interface. This is written in the form of

$$(3.4) \quad P_c(S_w) = P_o - P_w.$$

The relation between relative permeability and phase saturation can be noted down as [2],

$$K_w = S_w, K_o = 1 - \alpha S_w \quad (\alpha = 1.11).$$

As a function of saturation of inserted fluid, the capillary pressure is regarded as [14],

$$(3.5) \quad P_c = -\beta S_w.$$

For the investigated flow system in heterogeneous porous medium, porosity and permeability are expressed as functions of variable  $z$  [4],

$$\text{Porosity } P(z) = \frac{1}{a_1 - a_2 z},$$

$$\text{Permeability } K(z) = K_0(1 + bz),$$

where  $a_1, a_2, K_0$  and  $b$  are positive constants. As  $P(z)$  is unable to exceed unity, we presume that  $a_1 - a_2 z \geq 1$ .

For ease,  $K \propto P$  [16]

$$(3.6) \quad \text{Thus, } K = K_c P,$$

where  $K_c$  is constant of proportionality.

Condition for Counter-current imbibition phenomenon at the common interface is given by,

$$V_w + V_o = 0.$$

From (3.1) and (3.2),

$$(3.7) \quad \left(\frac{K_w}{\mu_w}\right) K \left(\frac{\partial P_w}{\partial z} + \rho_w g\right) + \left(\frac{K_o}{\mu_o}\right) K \left(\frac{\partial P_o}{\partial z} - \rho_o g\right) = 0.$$

From (3.7) and (3.4),

$$\left(\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}\right) \frac{\partial P_w}{\partial z} + \left(\frac{K_o}{\mu_o}\right) \frac{\partial P_c}{\partial z} = - \left(\frac{K_o}{\mu_o} \rho_o - \frac{K_w}{\mu_w} \rho_w\right) g.$$

Therefore,

$$(3.8) \quad \frac{\partial P_w}{\partial z} = - \left[ \frac{\left(\frac{K_w}{\mu_w} \rho_w - \frac{K_o}{\mu_o} \rho_o\right) g + \left(\frac{K_o}{\mu_o}\right) \frac{\partial P_c}{\partial z}}{\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}} \right].$$

By using the value of  $\frac{\partial P_w}{\partial z}$  from equation (3.8) into equation (3.1), we get

$$(3.9) \quad V_w = - \left(\frac{K_w}{\mu_w}\right) K \left[ \frac{\left(\frac{K_w}{\mu_w}\right) (\rho_w + \rho_o) g - \left(\frac{K_o}{\mu_o}\right) \frac{\partial P_c}{\partial z}}{\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}} \right].$$

By inserting the equation (3.9) into equation (3.3), we find

$$(3.10) \quad P \left(\frac{\partial S_w}{\partial t}\right) + \frac{\partial}{\partial z} \left[ K \left(\frac{\frac{K_w K_o}{\mu_w \mu_o}}{\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}}\right) \frac{\partial P_c}{\partial S_w} \frac{\partial S_w}{\partial z} \right] - \frac{\partial}{\partial z} \left[ K (\rho_w + \rho_o) g \left(\frac{\frac{K_w K_o}{\mu_w \mu_o}}{\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}}\right) \right] = 0.$$

Water and viscous oil are involved in the present investigation, so we have [1],

$$(3.11) \quad \frac{\frac{K_w K_o}{\mu_w \mu_o}}{\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}} \approx \frac{K_o}{\mu_o} = \frac{1 - \alpha S_w}{\mu_o}.$$

Using values from (3.5), (3.6) and (3.11) into equation (3.10)(15), we obtain

$$(3.12) \quad P \left(\frac{\partial S_w}{\partial t}\right) = \frac{K_c \beta}{\mu_o} \frac{\partial}{\partial z} \left[ P (1 - \alpha S_w) \frac{\partial S_w}{\partial z} \right] + \frac{K_c (\rho_w + \rho_o) g}{\mu_o} \frac{\partial}{\partial z} [P (1 - \alpha S_w)].$$

For further simplification, putting  $S = 1 - \alpha S_w$  in equation (3.12), we get

$$(3.13) \quad \frac{\partial S}{\partial t} = \frac{K_c \beta}{\mu_o} \left[ \frac{\partial}{\partial z} \left( S \frac{\partial S}{\partial z} \right) + S \frac{\partial S}{\partial z} \frac{a_2}{a_1} \right] - \frac{\alpha K_c (\rho_w + \rho_o) g}{\mu_o} \left( \frac{\partial S}{\partial z} + S \frac{a_2}{a_1} \right).$$

$\left( \because \frac{1}{P} \frac{\partial P}{\partial z} = \frac{\partial(\log P)}{\partial z} = \frac{\partial}{\partial z} \left( -\log a_1 + \frac{a_2}{a_1} z \right) = \frac{a_2}{a_1} \right)$  (ignoring terms of  $z$  with higher order)

Employing dimensionless variables,

$$Z = \frac{z}{L} \quad \text{and} \quad T = \frac{K_c \beta t}{\mu_w L^2}.$$

Equation (3.13) reduces to,

$$(3.14) \quad \frac{\partial S}{\partial T} = \frac{\partial}{\partial z} \left( S \frac{\partial S}{\partial z} \right) + BS \frac{\partial S}{\partial z} - A \frac{\partial S}{\partial z} - ABS,$$

where  $A = \frac{\alpha(\rho_w + \rho_o)g}{\beta}$ ,  $B = L \frac{a_2}{a_1}$  and  $S(Z, T) = 1 - \alpha S_w(Z, T)$ . Equation (3.14) expresses a governing non-linear partial differential equation of second-order for the phenomenon of Counter-current imbibition in vertical heterogeneous porous medium.

The suitable initial and boundary conditions for this problem are as follows:

$$\begin{aligned} S(Z, 0) &= S_0(Z), \quad \text{if } Z > 0, \\ S(0, T) &= S_1(T), \quad \text{if } T > 0, \\ S(L, T) &= S_2(T), \quad \text{if } T > 0. \end{aligned}$$

#### 4. SOLUTION WITH VARIATIONAL ITERATION APPROACH

In accordance with the variational iteration approach, correction functional of the equation (3.14) is given by,

$$\begin{aligned} S_{n+1}(Z, T) &= S_n(Z, T) + \int_0^T \lambda(T) \left[ \frac{\partial S_n(Z, \tau)}{\partial \tau} \right. \\ &\quad \left. - \frac{\partial}{\partial z} \left( \tilde{S}_n(Z, \tau) \frac{\partial \tilde{S}_n(Z, \tau)}{\partial z} \right) \right. \\ &\quad \left. - B \tilde{S}_n(Z, \tau) \frac{\partial \tilde{S}_n(Z, \tau)}{\partial z} + A \frac{\partial \tilde{S}_n(Z, \tau)}{\partial z} + AB \tilde{S}_n(Z, \tau) \right] d\tau, \end{aligned}$$

where ' $\lambda$ ' denotes a Lagrange's multiplier which is found as below.

Now,  $\tilde{S}_n(Z, \tau)$  is reflected as restricted variation. That is  $\delta\tilde{S}_n(Z, \tau) = 0$ .

Evaluating variation with respect to  $S_n$ , noting that  $\delta S_n(0) = 0$ , generates

$$\begin{aligned} \delta S_{n+1}(Z, T) = \delta S_n(Z, T) + \delta \int_0^T \lambda(T) & \left[ \frac{\partial S_n(Z, \tau)}{\partial \tau} \right. \\ & - \frac{\partial}{\partial z} \left( \tilde{S}_n(Z, \tau) \frac{\partial \tilde{S}_n(Z, \tau)}{\partial z} \right) \\ & \left. - B\tilde{S}_n(Z, \tau) \frac{\partial \tilde{S}_n(Z, \tau)}{\partial z} + A \frac{\partial \tilde{S}_n(Z, \tau)}{\partial z} + AB\tilde{S}_n(Z, \tau) \right] d\tau. \end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} \delta S_{n+1}(Z, T) = \delta S_n(Z, T) + [\lambda(z) \delta S_n(Z, T)]_{\tau=T} \\ - \int_0^T \lambda'(T) \delta S_n(Z, T) d\tau + \int_0^T \lambda AB \delta S_n(Z, T) d\tau. \end{aligned}$$

Then, its stationary conditions can be ascertained as below,

$$[\lambda(T) + 1]_{\tau=T} = 0,$$

$$[-\lambda'(T) + AB\lambda(T)]_{\tau=T} = 0.$$

Therefore, Lagrange multiplier  $\lambda = -e^{AB(\tau-T)}$ .

Following iteration formula can be found,

$$\begin{aligned} (4.1) \quad S_{n+1}(Z, T) = S_n(Z, T) - \int_0^T e^{AB(\tau-T)} & \left[ \frac{\partial S_n(Z, \tau)}{\partial \tau} \right. \\ & - \frac{\partial}{\partial z} \left( S_n(Z, \tau) \frac{\partial S_n(Z, \tau)}{\partial z} \right) \\ & \left. - BS_n(Z, \tau) \frac{\partial S_n(Z, \tau)}{\partial z} + A \frac{\partial S_n(Z, \tau)}{\partial z} + ABS_n(Z, \tau) \right] d\tau. \end{aligned}$$

Inserting  $n = 0$  in equation (4.1),

$$\begin{aligned} S_1(Z, T) = S_0(Z, T) - \int_0^T e^{AB(\tau-T)} & \left[ \frac{\partial S_0(Z, \tau)}{\partial \tau} \right. \\ & - \frac{\partial}{\partial z} \left( S_0(Z, \tau) \frac{\partial S_0(Z, \tau)}{\partial z} \right) \\ & \left. - BS_0(Z, \tau) \frac{\partial S_0(Z, \tau)}{\partial z} + A \frac{\partial S_0(Z, \tau)}{\partial z} + ABS_0(Z, \tau) \right] d\tau. \end{aligned}$$



We select the initial approximation [3],

$$(4.2) \quad S_0 = e^{-z}.$$

Inserting above approximation (4.2) in iterative formula, we acquire the following approximations

$$(4.3) \quad \begin{aligned} S_1 &= \frac{1}{e^z} - \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{1}{e^{ABT}} - 1 \right) \\ S_2 &= \frac{1}{e^z} - \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{1}{e^{ABT}} - 1 \right) \\ &\quad - \left[ \left( \frac{18}{ABe^{3z}} - \frac{9}{Ae^{3z}} + \frac{4}{Be^{2z}} - \frac{4}{e^{2z}} \right) \right. \\ &\quad \left. + A \left( \frac{4}{ABe^{2z}} - \frac{2}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \right. \\ &\quad \left. - B \left( \frac{6}{ABe^{3z}} - \frac{3}{Ae^{3z}} + \frac{2}{Be^{2z}} - \frac{2}{e^{2z}} \right) \right] \left( \frac{1}{ABe^{ABT}} - \frac{1}{AB} + \frac{T}{e^{ABT}} \right) \\ &\quad - \left[ \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{8}{ABe^{2z}} - \frac{4}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \right. \\ &\quad \left. + \left( \frac{4}{ABe^{2z}} - \frac{2}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right)^2 \right. \\ &\quad \left. - B \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{4}{ABe^{2z}} - \frac{2}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \right] \\ &\quad \left( \frac{1}{ABe^{2ABT}} - \frac{1}{AB} + \frac{2T}{e^{ABT}} \right). \end{aligned}$$

Similarly using iterative formula (4.1) the furthermore iterations can be found. Equation (4.3) is the approximate analytical solution of equation (3.14).

Now,  $S = 1 - \alpha S_w \quad \therefore S_w = \frac{1-S}{\alpha}$ .

Therefore, desired approximate analytical solution of the present problem is attained by,

$$S_{w1} = \frac{1}{\alpha} \left\{ 1 - \left[ \frac{1}{e^z} - \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{1}{e^{ABT}} - 1 \right) \right] \right\}$$

$$\begin{aligned}
S_2 = \frac{1}{\alpha} & \left\{ 1 - \left[ \frac{1}{e^z} - \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{1}{e^{ABT}} - 1 \right) \right. \right. \\
& - \left[ \left( \frac{18}{ABe^{3z}} - \frac{9}{Ae^{3z}} + \frac{4}{Be^{2z}} - \frac{4}{e^{2z}} \right) \right. \\
& + A \left( \frac{4}{ABe^{2z}} - \frac{2}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \\
& - B \left( \frac{6}{ABe^{3z}} - \frac{3}{Ae^{3z}} + \frac{2}{Be^{2z}} - \frac{2}{e^{2z}} \right) \left. \right] \left( \frac{1}{ABe^{ABT}} - \frac{1}{AB} + \frac{T}{e^{ABT}} \right) \\
(4.4) \quad & - \left[ \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{8}{ABe^{2z}} - \frac{4}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \right. \\
& + \left( \frac{4}{ABe^{2z}} - \frac{2}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right)^2 \\
& - B \left( \frac{2}{ABe^{2z}} - \frac{1}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left( \frac{4}{ABe^{2z}} - \frac{2}{Ae^{2z}} + \frac{1}{Be^z} - \frac{1}{e^z} \right) \left. \right] \\
& \left. \left( \frac{1}{ABe^{2ABT}} - \frac{1}{AB} + \frac{2T}{e^{ABT}} \right) \right] \left. \right\}
\end{aligned}$$

## 5. NUMERICAL OUTCOMES AND GRAPHICAL REPRESENTATION

The following values of specific constants are selected from the standard literature.

Density of injected water ( $\rho_w$ ) = 0.1 , Density of native oil ( $\rho_o$ ) = 0.8 , Gravity ( $g$ ) = 9.8 , Length ( $L$ ) = 1 , ( $\beta$ ) = 10.

Therefore,  $A = \frac{\alpha(\rho_w + \rho_o)g}{\beta} \approx 1$ .

The Numerical outcomes and graphical representations of solution (4.4) have been carried out through MATLAB. The numerical outcomes for the saturation of inserted water  $S_w(Z, T)$  at various depth  $Z$  for specified fixed time  $T = 0.01, 0.03, \dots, 0.09$  are expressed in Table 1. The graphs of the saturation of inserted water  $S_w(Z, T)$  at various depth  $Z$  for specified fixed time  $T = 0.01, 0.03, \dots, 0.09$  are exhibited in Figure 3. The graphs of the saturation of inserted water  $S_w(Z, T)$  against  $T$  for specified fixed depth  $Z = 0.1, 0.2, \dots, 1.0$  are presented in Figure 4.

TABLE 1. Numerical outcomes of the saturation of inserted water  $S_w(Z, T)$  at various depth  $Z$  when fixed time  $T > 0$

$T \longrightarrow$	0.01	0.03	0.05	0.07	0.09
$Z \downarrow$	$S_w(Z, T)$				
0.1	0.0781	0.0615	0.0430	0.0227	0.0005
0.2	0.1571	0.1436	0.1287	0.1125	0.0949
0.3	0.2284	0.2175	0.2055	0.1924	0.1784
0.4	0.2929	0.2840	0.2743	0.2638	0.2526
0.5	0.3511	0.3439	0.3360	0.3276	0.3187
0.6	0.4037	0.3978	0.3915	0.3847	0.3775
0.7	0.4513	0.4465	0.4413	0.4359	0.4301
0.8	0.4942	0.4903	0.4862	0.4818	0.4771
0.9	0.5331	0.5299	0.5266	0.5230	0.5192
1.0	0.5682	0.5656	0.5629	0.5600	0.5570

## 6. CONCLUSION

In present study, the phenomenon of Counter-current imbibition in vertical heterogeneous porous medium has been mathematically modelled. Equation (3.14) represents governing non-linear partial differential equation of second-order for the phenomenon of counter-current. The Variational iteration approach was used to find the solution (4.4) of equation (3.14) with suitable initial and boundary conditions. Table 1 represents numerical outcomes of the saturation of inserted water at various depth ( $Z$ ) for various time ( $T$ ). It can be noted from Table 1 that the saturation of inserted water reduces with time as well as rises with depth. So, it will drive oil from oil formatted region towards downward direction. As the bottom is impermeable, it is possible to retain maximum oil at the bottom. In the oil recovery process, it can pass into oil production well via interconnected pipes. Figure3 indicates the graphical solution with depth ( $Z$ ). Figure4 exhibits the graphical solution with time ( $T$ ). It has been observed from the Figure3 and Figure4 that the saturation of inserted water rises with depth and reduces with time which is consistent with the problem's physical nature.

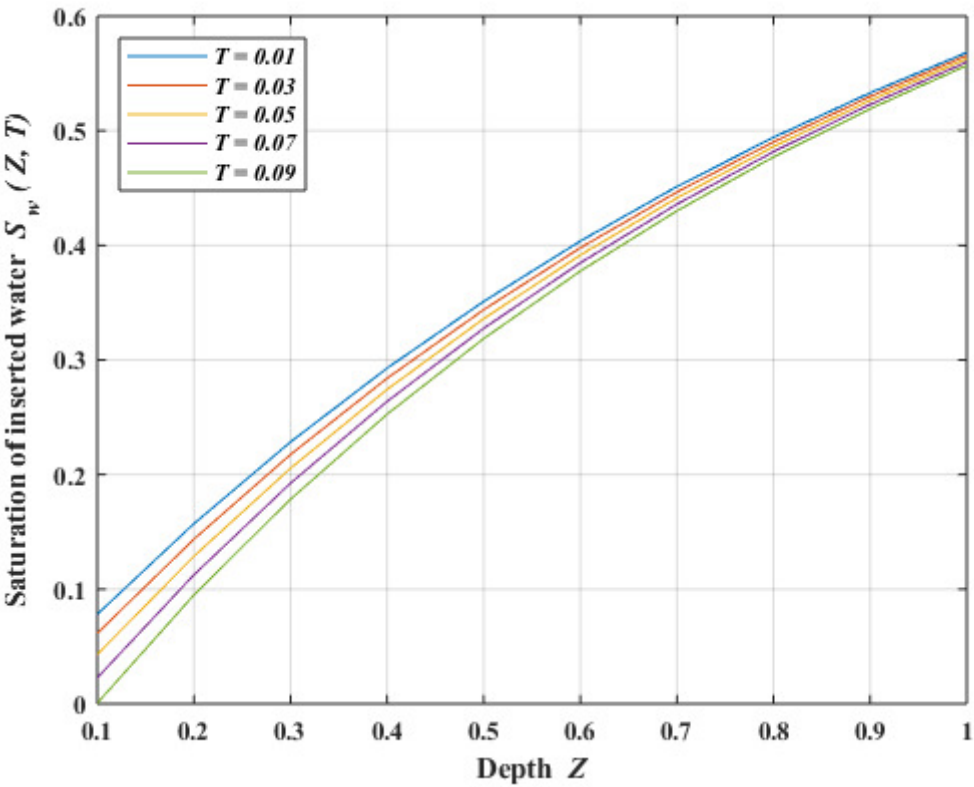


FIGURE 3. 2D graph for the saturation of inserted water  $S_w(Z, T)$  at various depth  $Z$  when fixed time  $T > 0$

NOMENCLATURE

$K$	Permeability of heterogeneous porous medium	$g$	Acceleration due to gravity
$K_w$	Relative permeability of inserted water	$S_w$	Saturation of inserted water
$K_o$	Relative permeability of native oil	$S_o$	Saturation of native oil
$P_w$	Pressure of inserted water	$\beta, K_c$	Constant of proportionality
$P_o$	Pressure of native oil	$a_1, a_2, K_0, b$	Positive constants
$\rho_w$	Density of inserted water	$L$	Length of cylindrical porous matrix
$\rho_o$	Density of native oil	$\tilde{S}_n$	Restricted variation
$\mu_w$	Kinematic viscosity of inserted water	$Z$	Depth
$\mu_o$	Kinematic viscosity of native oil	$T$	Time
$P$	Porosity	$\lambda$	Lagrange's multiplier

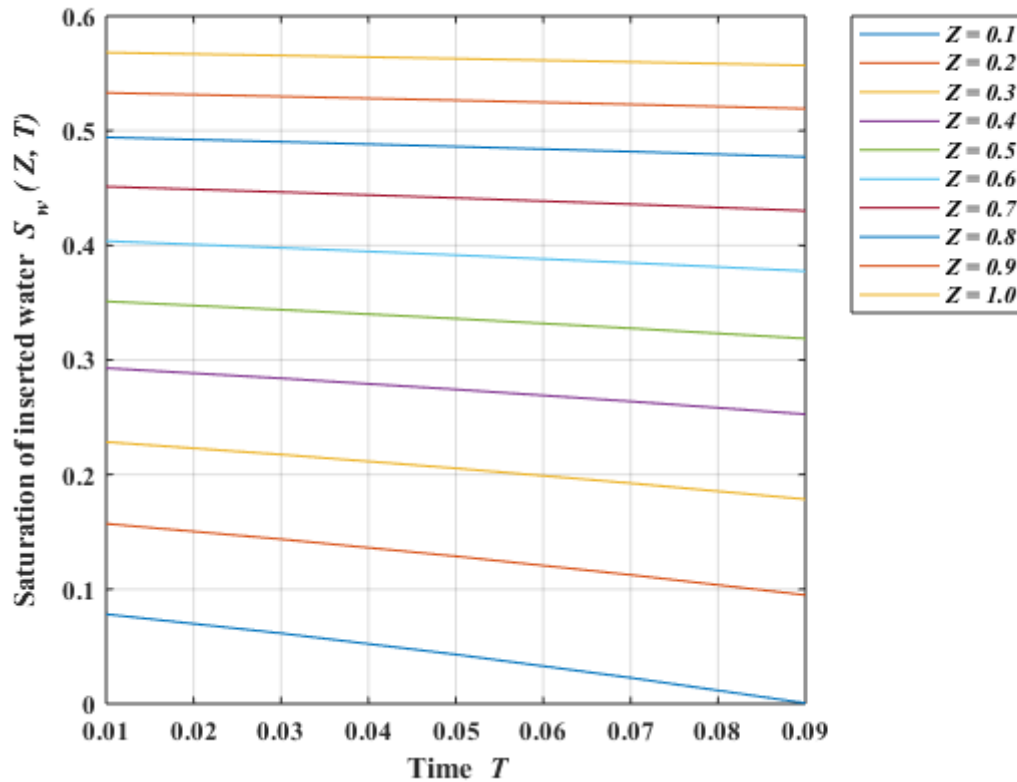


FIGURE 4. 2D graph for the saturation of inserted water  $S_w(Z, T)$  at various time  $T$  when fixed depth  $Z > 0$

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