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RECURSIVE FORMULAE FOR DROPLETS TRANSIENT HEATING AND EVAPORATION MODELS VIA A COMBINED METHOD OF INTEGRAL TRANSFORMS

Kwassi Anani[1]

ABSTRACT. The transient heating of a spherical droplet at rest in a hot gas environment, is analysed when the temperature distribution is initially assumed to be non uniform inside the droplet. A combined method of integral transforms, namely the classical Fourier cosine transform together with the unilateral Laplace transform, is used in solving the resulting initial-boundary value problem, stated in the dimensionless form. Explicit solutions of the problem are first obtained in the Laplace domain, and then analytical approximations in short time limits (timessteps) are derived for the droplet internal and surface temperature fields. The analytical approximation for the droplet internal temperature during the time step is proven to be highly accurate, while the innovative recursive formula obtained for the droplet surface temperature may lead to computationally efficient droplets and sprays vaporization models.

1. INTRODUCTION

In this paper, a combined method of two classical integral transforms is used in solving the spherically symmetric droplet transient heating equation. The related

[1]*corresponding author*

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initial-boundary value problem is stated in the dimensionless form as:

$$r\frac{\partial T}{\partial t} - \frac{\partial^2 (rT)}{\partial r^2} = 0, \qquad 0 < r < 1, \qquad t > 0, \tag{1.1}$$

with the initial and boundary conditions:

$$T(r, t = 0) = T_0(r); \tag{1.2}$$

$$\frac{\partial T}{\partial r}(r = 0, t) = 0; \tag{1.3}$$

$$\frac{\partial T}{\partial r}(r = 1, t) = K(T_g(t) - T_s(t)) = q_s(t). \tag{1.4}$$

Equation (1.1) can be recast as:

$$\frac{\partial T}{\partial t} - \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) = 0, \tag{1.5}$$

where $T \equiv T(r, t)$ represents the temperature function to be determined at time t and distance r from the unity-radius droplet centre. A non-uniform temperature distribution $T(r, t = 0) = T_0(r)$, is initially assumed inside the droplet or at the beginning of the time step Δt. The zero temperature gradient at the centre assures the spherical symmetry of the droplet which is surrounded by a gas phase at time-evolving temperature $T_g(t)$. The coefficient $K = k_g/k_l$, $0 < K < 1$, denotes the ratio of liquid and gas thermal conductivity. These latter are assumed to be constant with time as well as the specific heat capacities and the densities of both phases. Thus, in the dimensionless form of the problem, the thermal diffusivity of the droplet is scaled to unity. The unknown temperature gradient $q_s(t)$ and surface temperature $T_s(t)$ are related by the boundary condition of the third kind (1.4), and must be determined as part of the solution of equation (1.1).

In Computational Fluid Dynamics (CFD) codes for sprays, the droplet transient heating and evaporation processes are treated by a time step analysis, where the location of the droplet surface is assumed fixed during the time step, but varies from one time step to another. Nevertheless, the time-evolving temperature of the surrounding gas-phase mixture at the immediate vicinity of the droplet can be estimated at each time step (confer [1] and [2]). Due to the important number of droplets involved in liquid fuel combustion mechanisms, the combination of model simplicity and capability to accurately predict droplet surface temperature (which solely affects evaporation rate) is crucial for efficient use in a CFD spray model.

In the latter case, the use of high-precision numerical as in [3] or exact analytical solutions of the heat transfer equation when considering a single droplet, seems to be impractical. Thus, many approximate analytical approaches as the power law, the polynomial approximations and the heat balance integral methods, have been formulated through physical considerations [4]. On the other hand, the method of the Laplace integral transform and its inversion formula as well as that of the separation of variables, have been used for deriving exact series solutions for the spherical solid body heating/cooling problem with prescribed expressions of the ambient gas temperature as in [5], [6], [7] and [8]. But, the classical Fourier sine or cosine integral applied to space coordinates, has being validated only for infinite and semi-infinite solids, while the Laplace transform method is found to be not appropriate for solving boundary value problems with a non-uniform initial space function [5].

In the present paper, efficient analytical solutions in short time limits are obtained for the spherically symmetric droplet heating problem (1.1)-(1.4), by using the classical Fourier cosine integral transform (FCIT) in combination with the Laplace integral transform (LIT). Similarly to a former combined method of the Fourier sine and the Laplace transforms as performed in [9], the present combined method is introduced in section 2, and an integral form of the temperature distribution inside the droplet is obtained. In section 3, explicit solutions in the Laplace domain are derived for the droplet internal and surface temperature fields. In section 4, analytical solutions in short time limits are obtained for the droplet internal and surface temperatures. The recursive formula derived for the droplet surface temperature is proved to be sufficiently simple for implementing in spray CFD codes. Finally, section 5 outlines the conclusion.

2. The Fourier cosine integral transform method

The classical Fourier cosine integral transform (see [11]) is applied to the problem (1.1)-(1.4). The surrounding gas temperature $T_g(t)$ is assumed to be bounded and continuous with time.

Lemma 2.1. *Assuming that $T = T(r,t)$ is a solution of the problem (1.1)-(1.4), then, the FCIT or Fourier Cosine Integral Transform $V_c(\lambda, t)$ of the function $T =$*

$T(r,t)$, *defined as:*

$$V_c(\lambda,t) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} T\cos(\lambda r)dr = \sqrt{\frac{2}{\pi}} \int_0^1 T\cos(\lambda r)dr$$

is solution of the following differential equation:

$$\begin{aligned} &\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial t}\frac{\partial V_c(\lambda,t)}{\partial\lambda} + \lambda^2\frac{\partial V_c(\lambda,t)}{\partial\lambda}\right) \\ &= -\sqrt{\frac{2}{\pi}}(q_s(t)\cos\lambda + \lambda T_s(t)\sin\lambda), \qquad \lambda \geq 0, \qquad t > 0. \end{aligned} \tag{2.1}$$

Proof. Since we are here concerned only with the temperature within the droplet, the function $T(r,t)$ can be taken null outside the interval $[0,1]$ without lost of generality. According to equation (1.5), the solution $T = T(r,t)$ should be continuously differentiable for $0 \leq r \leq 1$ and for $t > 0$. Therefore, T is absolutely integrable in respect to the variable r on $[0,1] \subset [0,+\infty[$ and the FCIT of the function $T \equiv T(r,t)$ is reduced to:

$$V_c(\lambda,t) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} T\cos(\lambda r)dr = \sqrt{\frac{2}{\pi}} \int_0^1 T\cos(\lambda r)dr.$$

Likewise, at any time $t > 0$, the terms $r\frac{\partial T}{\partial r}$, $\frac{\partial^2(rT)}{\partial r^2}$ mentioned in equation (1.1) are absolutely integrable in respect to the radial variable r on $[0,+\infty[$. The FCIT can now be applied to these terms. For convenience, equation (1.1) is first multiplied by r and reads:

$$\frac{r^2\partial T}{\partial t} - r\frac{\partial^2(rT)}{\partial r^2} = 0. \tag{2.2}$$

Applying the FCIT (denoted by F_c) to the first term of equation (2.2), we have:

$$A(\lambda) = F_c\left[\frac{r^2\partial T}{\partial t}\right] = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial t}\int_0^1 Tr^2\cos(\lambda r)dr,$$

which is equivalent to:

$$A(\lambda) = -\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial t}\frac{\partial}{\partial\lambda}\int_0^1 rT\sin(\lambda r)dr = -\frac{\partial}{\partial t}\frac{\partial^2}{\partial\lambda^2}V_c(\lambda,t), \tag{2.3}$$

since, $\frac{\partial^2}{\partial\lambda^2}\cos(\lambda r) = -r^2\cos(\lambda r)$. The application of the FCIT to the diffusion term of equation (2.2) leads to:

$$C(\lambda) = F_c\left[-r\frac{\partial^2(rT)}{\partial r^2}\right] = -\sqrt{\frac{2}{\pi}}\int_0^1 r\frac{\partial^2(rT)}{\partial r^2}\cos(\lambda r)dr. \tag{2.4}$$

The integral (2.4) can be transformed using the technique of integration by parts. The first integration by parts is performed by writing:

$$u_1(r) = r\cos(\lambda r) \Rightarrow u_1'(r) = \cos(\lambda r) - r\lambda\sin(\lambda r),$$

and

$$v_1'(r) = \frac{\partial^2(rT)}{\partial r^2} \Rightarrow v_1(r) = \frac{\partial(rT)}{\partial r}.$$

Equation (2.4) is then transformed into:

$$\begin{aligned}
&-\sqrt{\frac{\pi}{2}}C(\lambda) = \int_0^1 r\frac{\partial^2(rT)}{\partial r^2}\cos(\lambda r)dr = [u_1v_1]_0^1 - \int_0^1 u_1'v_1 dr \\
&= \left[r\cos(\lambda r)\frac{\partial(rT)}{\partial r}\right]_0^1 - \int_0^1 \frac{\partial(rT)}{\partial r}\cos(\lambda r)dr + \int_0^1 \frac{\partial(rT)}{\partial r} r\lambda\sin(\lambda r)dr \\
&= C_1 - C_2 + C_3,
\end{aligned} \tag{2.5}$$

where, the term C_1 is calculated as:

$$\begin{aligned}
C_1 &= \left[r\cos(\lambda r)\frac{\partial(rT)}{\partial r}\right]_0^1 = \left[r\cos(\lambda r)\left(T + r\frac{\partial T}{\partial r}\right)\right]_0^1 \\
&= \cos(\lambda)T_s(t) + q_s(t)\cos(\lambda).
\end{aligned} \tag{2.6}$$

The term C_2 of equation (2.5) can be equally transformed by using an integration by parts with:

$$u_2' = \frac{\partial(rT)}{\partial r} \Rightarrow u_2 = rT; \quad v_2 = \cos(\lambda r) \Rightarrow v_2' = -\lambda\sin(\lambda r).$$

This leads to:

$$\begin{aligned}
C_2 &= \int_0^1 \frac{\partial(rT)}{\partial r}\cos(\lambda r)dr = [u_2v_2]_0^1 - \int_0^1 u_2v_2' dr \\
&= [rT\cos(\lambda r)]_0^1 + \int_0^1 rT\lambda\sin(\lambda r)dr \\
&= \cos(\lambda)T_s(t) + \int_0^1 rT\lambda\sin(\lambda r)dr \\
&= \cos(\lambda)T_s(t) - \sqrt{\frac{\pi}{2}}\lambda\frac{\partial V_c(\lambda,t)}{\partial\lambda}.
\end{aligned} \tag{2.7}$$

The third term C_3 in equation (2.5) can also be calculated by integration by parts with:

$$u_3 = r\sin(\lambda r) \quad \Rightarrow \quad u_3' = \sin(\lambda r) + \lambda r\cos(\lambda r);$$

$$v_3' = \frac{\partial(rT)}{\partial r} \Rightarrow v_3 = rT,$$

and gives:

$$
\begin{aligned}
C_3 &= \int_0^1 \frac{\partial(rT)}{\partial r} r\lambda \sin(\lambda r) dr = \lambda \int_0^1 \frac{\partial(rT)}{\partial r} r \sin(\lambda r) dr \\
&= \lambda \left([u_3 v_3]_0^1 - \int_0^1 u_3' v_3 dr \right) = \lambda [r^2 T \sin(\lambda r)]_0^1 \\
&\quad + \lambda \left(- \int_0^1 rT \sin(\lambda r) dr - \lambda \int_0^1 r^2 T \cos(\lambda r) dr \right) \\
&= \lambda \left(\sin(\lambda) T_s + \sqrt{\frac{\pi}{2}} \frac{\partial V_c(\lambda, t)}{\partial \lambda} + \sqrt{\frac{\pi}{2}} \lambda \frac{\partial^2 V_c(\lambda, t)}{\partial \lambda^2} \right).
\end{aligned} \tag{2.8}
$$

So, by using successive integration by parts over the unity radius of the droplet with consideration to the boundary conditions (1.3)-(1.4), equation (2.5) is transformed, by combination of equations (2.6)-(2.8), into:

$$
C(\lambda) = -\sqrt{\frac{2}{\pi}} (q_s \cos \lambda + \lambda T_s \sin \lambda) - \frac{\partial}{\partial \lambda} \left(\lambda^2 \frac{\partial V_c(\lambda, t)}{\partial \lambda} \right). \tag{2.9}
$$

By combining equations (2.2)-(2.4) and (2.9), the system (1.1)-(1.4) is finally transformed into equation (2.1) as in the Lemma 2.1, the proof of which is completed. □

Proposition 2.1. *The temperature function $T(r,t)$, solution of the initial-boundary value problem (1.1)-(1.4) can be written as:*

$$
\begin{aligned}
T(r,t) &= \frac{1}{r\sqrt{\pi}} \int_0^t T_s(t-\eta) \frac{\left(-\frac{(r+1)}{2} e^{-\frac{(r+1)^2}{4\eta}} + \frac{(1-r)}{2} e^{-\frac{(1-r)^2}{4\eta}}\right)}{2\eta^{\frac{3}{2}}} d\eta \\
&\quad - \frac{1}{r\sqrt{\pi}} \int_0^t P(t-\eta) \frac{\left(\frac{1}{2} e^{-\frac{(r+1)^2}{4\eta}} - \frac{1}{2} e^{-\frac{(r-1)^2}{4\eta}}\right)}{\sqrt{\eta}} d\eta \\
&\quad - \frac{1}{r\sqrt{\pi}} \int_0^1 x T_0(x) \frac{\left(\frac{1}{2} e^{-\frac{(r+x)^2}{4t}} - \frac{1}{2} e^{-\frac{(r-x)^2}{4t}}\right)}{\sqrt{t}} dx,
\end{aligned} \tag{2.10}
$$

where $P = T_s + q_s$, T_s and q_s being respectively the temperature and its gradient at the droplet surface.

Proof. Let us consider the partial differential equation (2.1). A first integration with respect to λ gives:

$$
\begin{aligned}
&\frac{\partial^2 V_c(\lambda, t)}{\partial t \partial \lambda} + \lambda^2 \frac{\partial V_c(\lambda, t)}{\partial \lambda} \\
&= -\sqrt{\frac{2}{\pi}} [q_s(t) \sin \lambda + (-\lambda \cos \lambda + \sin \lambda) T_s(t)] + a_1(t),
\end{aligned} \tag{2.11}
$$

with $a_1(t)$ a function to be determined. By tending λ to 0 in equation (2.11), we find:

$$\left.\frac{\partial^2 V_c(\lambda,t)}{\partial t\partial\lambda}\right|_{\lambda=0} = a_1(t). \tag{2.12}$$

Now the quantity $\left.\frac{\partial^2 V_c(\lambda,t)}{\partial t\partial\lambda}\right|_{\lambda=0} = -\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial t}\int_0^1 rTsin(\lambda r)dr\bigg|_{\lambda=0} = 0$. Thus, from equation (2.12), one has

$$a_1(t) = 0. \tag{2.13}$$

Equation (2.11) is then reduced into:

$$\begin{aligned}\frac{\partial^2 V_c(\lambda,t)}{\partial t\partial\lambda} + \lambda^2\frac{\partial V_c(\lambda,t)}{\partial\lambda}\\ = -\sqrt{\frac{2}{\pi}}[q_s(t)\sin\lambda + (-\lambda\cos\lambda + \sin\lambda)T_s(t)].\end{aligned} \tag{2.14}$$

Writing $W_c(\lambda,t) = \frac{\partial V_c(\lambda,t)}{\partial\lambda}$ and multiplying the equation (2.14) by $\mathrm{e}^{\lambda^2 t}$, we obtain:

$$\begin{aligned}\mathrm{e}^{\lambda^2 t}\frac{\partial W_c(\lambda,t)}{\partial t} + \lambda^2\mathrm{e}^{\lambda^2 t}W_c(\lambda,t)\\ = -\sqrt{\frac{2}{\pi}}[q_s(t)\sin\lambda + (-\lambda\cos\lambda + \sin\lambda)T_s(t)]\mathrm{e}^{\lambda^2 t},\end{aligned}$$

which can be reduced into:

$$\begin{aligned}\frac{\partial(\mathrm{e}^{\lambda^2 t}W_c(\lambda,t))}{\partial t}\\ = \sqrt{\frac{2}{\pi}}[-q_s(t)\sin\lambda + (\lambda\cos\lambda - \sin\lambda)T_s(t)]\mathrm{e}^{\lambda^2 t}.\end{aligned} \tag{2.15}$$

By integrating equation (2.15) with respect to the time variable, η going from 0 to t, it can be written that:

$$\begin{aligned}\mathrm{e}^{\lambda^2 t}W_c(\lambda,t) - W_c(\lambda,t=0)\\ = \sqrt{\frac{2}{\pi}}\int_0^t[-q_s(\eta)\sin\lambda + (\lambda\cos\lambda - \sin\lambda)T_s(\eta)]\mathrm{e}^{\lambda^2\eta}d\eta.\end{aligned}$$

Now,

$$\begin{aligned}W_c(\lambda,t=0) &= \frac{\partial V_c(\lambda,t=0)}{\partial\lambda} = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial\lambda}\int_0^1 T_0(r)\cos(\lambda r)dr\\ &= -\sqrt{\frac{2}{\pi}}\int_0^1 rT_0(r)\sin(\lambda r)dr,\end{aligned}$$

as the initial condition $T(r, t = 0) \equiv T_0(r)$. Equation (2.15) is then integrated as:

$$(2.16)\quad \begin{aligned} W_c(\lambda, t) &= \frac{\partial V_c(\lambda, t)}{\partial \lambda} \\ &= \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\lambda^2 t} \int_0^t [-q_s(\eta) \sin \lambda + (\lambda \cos \lambda - \sin \lambda) T_s(\eta)] \mathrm{e}^{\lambda^2 \eta} d\eta \\ &\quad - \sqrt{\frac{2}{\pi}} \mathrm{e}^{-\lambda^2 t} \int_0^1 r T_0(r) \sin(\lambda r) dr. \end{aligned}$$

Since $V_c(\lambda, t)$ is a FCIT and therefore cancels when λ tends to $+\infty$ according to the Fourier transform properties, $V_c(\lambda, t)$ can be written as:

$$V_c(\lambda, t) = -\int_\lambda^\infty \frac{\partial V_c(x, t)}{\partial x} dx,$$

and equation (2.16) can be integrated with respect to λ, the dummy variable x going from λ to $+\infty$. By using such integration of equation (2.16) and then reversing the order of integration (that is allowed due the uniform convergence of $V_s(\lambda, t)$ and of its derivative relatively to λ), it can be deduced for $t > 0$:

$$(2.17)\quad \begin{aligned} V_c(\lambda, t) &= \sqrt{\frac{2}{\pi}} \int_0^t d\eta \left[T_s(\eta) \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta - t)} x \cos(x) dx \right) \right] \\ &\quad + \sqrt{\frac{2}{\pi}} \int_0^t d\eta \left[(-q_s(\eta) - T_s(\eta)) \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta - t)} \sin(x) dx \right) \right] \\ &\quad - \sqrt{\frac{2}{\pi}} \int_0^1 R T_0(R) dR \left(-\int_\lambda^\infty \sin(xR) \mathrm{e}^{-x^2 t} dx \right). \end{aligned}$$

To obtain $T(r, t)$ from the FCIT $V_c(\lambda, t)$, the inversion formula reads:

$$(2.18)\quad T(r, t) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} V_c(\lambda, t) \cos(r\lambda) d\lambda,$$

and can be applied to equation (2.17). We first calculate the following integrals:

$$I_a = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta - t)} x \cos(x) dx \right) \cos(\lambda r) d\lambda,$$

$$I_b = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-\int_\lambda^\infty \mathrm{e}^{x^2(\eta - t)} \sin(x) dx \right) \cos(\lambda r) d\lambda,$$

and

$$I_i = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(-\int_\lambda^\infty \mathrm{e}^{x^2 t} \sin(xR) dx \right) \cos(\lambda r) d\lambda.$$

An integration by parts is performed on I_a as follows:

$$u_4 = -\int_\lambda^\infty e^{x^2(\eta-t)} x\cos(x)dx \Rightarrow u_4' = e^{\lambda^2(\eta-t)}\lambda\cos(\lambda);$$

$$v_4' = \cos(\lambda r) \Rightarrow v_4 = \frac{\sin(\lambda r)}{r},$$

and then

$$\begin{aligned} I_a &= \sqrt{\frac{2}{\pi}}\left[-\frac{\sin(\lambda r)}{r}\int_\lambda^\infty e^{x^2(\eta-t)}x\cos(x)dx\right]_0^{+\infty} \\ &-\frac{\sqrt{2}}{r\sqrt{\pi}}\int_0^\infty e^{\lambda^2(\eta-t)}\lambda\cos(\lambda)\sin(\lambda r)d\lambda \\ &= -\frac{\sqrt{2}}{4r(t-\eta)^{\frac{3}{2}}}\left(\frac{(1+r)}{2}e^{-\frac{(1+r)^2}{4(t-\eta)}} - \frac{(1-r)}{2}e^{-\frac{(1-r)^2}{4(t-\eta)}}\right). \end{aligned}$$

Likewise, for I_b, the integration by parts:

$$u_5 = -\int_\lambda^\infty e^{x^2(\eta-t)}\sin(x)dx \Rightarrow u_5' = e^{\lambda^2(\eta-t)}\sin(\lambda);$$

$$v_5' = \cos(\lambda r) \Rightarrow v_5 = \frac{\sin(\lambda r)}{r},$$

leads to:

$$\begin{aligned} I_b &= \sqrt{\frac{2}{\pi}}\left[-\frac{\sin(\lambda r)}{r}\int_\lambda^\infty e^{x^2(\eta-t)}\sin(x)dx\right]_0^{+\infty} \\ &-\frac{\sqrt{2}}{r\sqrt{\pi}}\int_0^\infty e^{\lambda^2(\eta-t)}\sin(\lambda r)\sin(\lambda)d\lambda \\ &= -\frac{1}{r\sqrt{2(t-\eta)}}\left(-\frac{1}{2}e^{-\frac{(1+r)^2}{4(t-\eta)}} + \frac{1}{2}e^{-\frac{(r-1)^2}{4(t-\eta)}}\right). \end{aligned}$$

And by an alike technique of integration by parts:

$$u_6 = -\int_\lambda^\infty e^{x^2 t}\sin(xR)dx \Rightarrow u_6' = e^{\lambda^2 t}\sin(\lambda R);$$

$$v_6' = \cos(\lambda r) \Rightarrow v_6 = \frac{\sin(\lambda r)}{r},$$

I_i is obtained for $t > 0$ as:

$$\begin{aligned} I_i &= \sqrt{\frac{2}{\pi}}\left[-\frac{\sin(\lambda r)}{r}\int_\lambda^\infty e^{-x^2 t}\sin(xR)dx\right]_0^{+\infty} \\ &-\frac{\sqrt{2}}{r\sqrt{\pi}}\int_0^\infty e^{-\lambda^2 t}\sin(\lambda R)\sin(\lambda r)d\lambda \\ &= -\frac{1}{r\sqrt{2t}}\left(-\frac{1}{2}e^{-\frac{(r+R)^2}{4t}} + \frac{1}{2}e^{-\frac{(r-R)^2}{4t}}\right). \end{aligned}$$

Now, equation (2.17) is reversed by using the formula (2.18) and the temperature function $T(r,t)$ is derived through the above expressions of I_a, I_b and I_i. Finally, by changing integration variables from η and R to $\eta' = t-\eta$ and $x = R$, the temperature function $T(r,t)$ is determined by equation (2.10) as in the proposition 2.1. This completes the proof of the proposition. □

Corollary 2.1. *The initial condition (1.2) of the problem (1.1)-(1.4) is satisfied by the integral form of the temperature function $T(r,t)$ given by equation (2.10).*

Proof. In order to verify if the initial condition (1.2) is satisfied by the above expression of $T(r,t)$, we consider the following function $G(t,r,x)$, $0 \le r, x \le 1$, $t > 0$:

$$G(t,r,x) = -\frac{1}{\sqrt{\pi t}}\left(\frac{1}{2}\mathrm{e}^{-\frac{(x+r)^2}{4t}} - \frac{1}{2}\mathrm{e}^{-\frac{(r-x)^2}{4t}}\right), \tag{2.19}$$

such as the last integral in equation (2.10) can be equally written:

$$-\frac{1}{r}\frac{1}{\sqrt{\pi t}}\int_0^1 xT_0(x)\left[\frac{1}{2}\mathrm{e}^{-\frac{(x+r)^2}{4t}} - \frac{1}{2}\mathrm{e}^{-\frac{(r-x)^2}{4t}}\right]dx = \frac{1}{r}\int_0^1 xT_0(x)G(t,r,x)dx. \tag{2.20}$$

It is seen from the definition (2.19) that $G(t,r,x)$ tends to zero when $t \to 0$ at all points of the unity square $0 \le r, x \le 1$ with the exception of the diagonal $x = r$ where it becomes infinitely large. One may admit that $G(t,r,x)$ is an analogue of Green's function and:

$$\lim_{t\to 0} G(t,r,x) = \delta(x-r),$$

where $\delta(x-r)$ is the Dirac delta function. In consequence, the initial condition (1.2) is satisfied by the integral expression (2.10). Indeed, when $t \to 0$, equation (2.10) reduces to:

$$rT(r,t=0) = \int_0^1 xT_0(x)\delta(x-r)dx = \int_0^1 rT_0(r)\delta(r-x)dr = rT_0(r). \tag{2.21}$$

The verification of the corollary 2.1 is completed. □

Remark 2.1. *The temperature distribution $T(r,t)$ expressed in equation (2.10) may be separated into two parts and we can set:*

$$T(r,t) = T_1(r,t) + T_2(r,t), \tag{2.22}$$

where

$$
\begin{aligned}
T_1(r,t) &= \frac{1}{r\sqrt{\pi}}\int_0^t T_s(t-\eta)\frac{(-\frac{(r+1)}{2}\mathrm{e}^{-\frac{(r+1)^2}{4\eta}}+\frac{(1-r)}{2}\mathrm{e}^{-\frac{(1-r)^2}{4\eta}})}{2\eta^{\frac{3}{2}}}d\eta \\
&-\frac{1}{r\sqrt{\pi}}\int_0^t P(t-\eta)\frac{(\frac{1}{2}\mathrm{e}^{-\frac{(r+1)^2}{4\eta}}-\frac{1}{2}\mathrm{e}^{-\frac{(r-1)^2}{4\eta}})}{\sqrt{\eta}}d\eta,
\end{aligned} \tag{2.23}
$$

and

$$
\begin{aligned}
T_2(r,t) &= -\frac{1}{r\sqrt{\pi}}\int_0^1 xT_0(x)\frac{(\frac{1}{2}\mathrm{e}^{-\frac{(r+x)^2}{4t}}-\frac{1}{2}\mathrm{e}^{-\frac{(r-x)^2}{4t}})}{\sqrt{t}}dx \\
&= \frac{1}{r}\int_0^1 xT_0(x)G(t,r,x)dx.
\end{aligned} \tag{2.24}
$$

It's easily verifiable that the function $T_2(r,t)$ is an exact solution of the equation (1.1) taken into consideration in this study.

Remark 2.2. *The FCIT method has permitted us to determine an integral expression of the solution of the problem (1.1)-(1.4). But, the unknown droplet surface temperature $T_s(t)$ and temperature gradient $q_s(t)$ are not easily derivable from the internal temperature function expressed by equation (2.10). The Laplace integral transform (LIT) will be introduced in order to express these time-varying functions in the Laplace domain.*

3. Laplace domain solutions

Proposition 3.1. *The droplet surface temperature $T_s(t)$, the surface temperature gradient $q_s(t)$ and the temperature field inside the droplet $T(r,t)$ are respectively expressed in the Laplace domain by $\mathcal{L}T_s(p)$, $\mathcal{L}q_s(p)$ and $\mathcal{L}T(r,p)$ as follow:*

$$
\mathcal{L}T_s(p) = -\frac{K\,(\mathrm{e}^{-2\sqrt{p}}-1)\mathcal{L}T_g(p)+\mathrm{U}(T_0(x),p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p}-K\,\mathrm{e}^{-2\sqrt{p}}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}+K-1}, \tag{3.1}
$$

$$
\mathcal{L}q_s(p) = \frac{K\,(\mathrm{e}^{-2\sqrt{p}}\sqrt{p}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}-1)\mathcal{L}T_g(p)+K\,\mathrm{U}(T_0(x),p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p}-K\,\mathrm{e}^{-2\sqrt{p}}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}+K-1}, \tag{3.2}
$$

and

$$\mathcal{L}T(r,p) = \frac{1}{2}\frac{\mathrm{e}^{-\sqrt{p}}(\mathrm{e}^{r\sqrt{p}}-\mathrm{e}^{-r\sqrt{p}})[2K\sqrt{p}\mathcal{L}T_g(p)+(K-\sqrt{p}-1)\mathrm{U}(T_0(x),p)]}{r(\mathrm{e}^{-2\sqrt{p}}\sqrt{p}-K\,\mathrm{e}^{-2\sqrt{p}}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}+K-1)\sqrt{p}} - \frac{1}{2r\sqrt{p}}\int_0^1 xT_0(x)\left(\mathrm{e}^{-(x+r)\sqrt{p}}-\mathrm{e}^{-|r-x|\sqrt{p}}\right)dx, \tag{3.3}$$

where the p-dependent operator U *is applied to the initial temperature function* $T_0(x)$ *as:*

$$\mathrm{U}(T_0(x),p) = \mathrm{e}^{-\sqrt{p}}\int_0^1 xT_0(x)\mathrm{e}^{-x\sqrt{p}}dx - \mathrm{e}^{-\sqrt{p}}\int_0^1 xT_0(x)\mathrm{e}^{x\sqrt{p}}dx, \tag{3.4}$$

and $\mathcal{L}T_g(p)$ *is the Laplace transform of the gas temperature* $T_g(t)$*.*

Proof. Since the LIT of the droplet internal temperature $T(r,t)$ is denoted by $\mathcal{L}T(r,p)$, the initial-boundary-value problem (1.1)-(1.4) can be reformulated in the Laplace domain as:

$$p\,r\mathcal{L}T(r,p) - \frac{d^2(r\mathcal{L}T(r,p))}{dr^2} = 0, \tag{3.5}$$

subject to the initial condition:

$$p\mathcal{L}T(r,p)|_{r,p=\infty} = T_0(r), \tag{3.6}$$

and to the boundary conditions:

$$\begin{cases} \left.\dfrac{d\mathcal{L}T(r,p)}{dr}\right|_{r=0,p} = 0 \\ \left.\dfrac{d\mathcal{L}T(r,p)}{dr}\right|_{r=1,p} = \mathcal{L}q_s(p) = K\left(\mathcal{L}T_g(p) - \mathcal{L}T_s(p)\right) \end{cases}. \tag{3.7}$$

The equation (3.6), which expresses the initial condition in the Laplace domain, results from the Initial Value Theorem (see Debnath [12]).

According to the remark 2.1, the LIT of equation (2.10) is obtained as the sum of the LIT of functions $T_1(r,t)$ and $T_2(r,t)$:

$$\mathcal{L}T(r,p) = \mathcal{L}T_1(r,p) + \mathcal{L}T_2(r,p). \tag{3.8}$$

The Convolution Theorem (see again Debnath [12]) can be applied to the equation (2.23) and the LIT of $T_1(r,t)$ (see remark 2.1) reads:

$$\mathcal{L}T_1(r,p) = \frac{1}{r\sqrt{p}}\mathrm{e}^{-\sqrt{p}}\left(\frac{1}{2}\mathrm{e}^{r\sqrt{p}} - \frac{1}{2}\mathrm{e}^{-r\sqrt{p}}\right)[\mathcal{L}q_s(p) + \mathcal{L}T_s(p) + \sqrt{p}\mathcal{L}T_s(p)]. \tag{3.9}$$

Next, the LIT of the time-varying function $T_2(r,t)$ as mentioned in the remark 2.1, is obtained by using Maple software or tables of transforms as in [13]. Since $0 \le r,\ x \le 1$, we obtain:

$$\mathcal{L}T_2(r,p) = -\frac{1}{2r\sqrt{p}}\int_0^1 xT_0(x)\left(\mathrm{e}^{-(x+r)\sqrt{p}} - \mathrm{e}^{-|r-x|\sqrt{p}}\right)dx, \tag{3.10}$$

where $|r-x|$ is the absolute value of $r-x$. Formula (3.8) now reads:

$$\begin{aligned}\mathcal{L}T(r,p) &= \frac{1}{r\sqrt{p}}\mathrm{e}^{-\sqrt{p}}\left(\frac{1}{2}\mathrm{e}^{r\sqrt{p}} - \frac{1}{2}\mathrm{e}^{-r\sqrt{p}}\right)[\mathcal{L}q_s(p) + \mathcal{L}T_s(p) + \sqrt{p}\mathcal{L}T_s(p)] \\ &\quad - \frac{1}{2r\sqrt{p}}\int_0^1 xT_0(x)\left(\mathrm{e}^{-(x+r)\sqrt{p}} - \mathrm{e}^{-|r-x|\sqrt{p}}\right)dx.\end{aligned} \tag{3.11}$$

Now, according to the above equation (3.9), $r\mathcal{L}T_1(r,p)$ can be written in the form of:

$$r\mathcal{L}T_1(r,p) = C(p)\frac{\left(\mathrm{e}^{r\sqrt{p}} - \mathrm{e}^{-r\sqrt{p}}\right)}{2} = \sinh(r\sqrt{p}),$$

where the p-dependent coefficient $C(p) = \frac{1}{\sqrt{p}}\mathrm{e}^{-\sqrt{p}}[\mathcal{L}q_s(p) + \mathcal{L}T_s(p) + \sqrt{p}\mathcal{L}T_s(p)]$ doesn't depend on the variable r. Under this form, it is easily verifiable that $\mathcal{L}T_1(r,p)$ is an exact solution of the Laplace domain equation (3.5). Consequently, $T_1(r,p)$ is an exact solution of the time domain equation (1.1) considered in this study, whatever is the value of $C(p)$. As already mentioned in the remark 2.1, the function $T_2(r,t)$ is also an exact solution of the equation (1.1). Hence, the integral form $T(r,t) = T_1(r,t) + T_2(r,t)$, as reported in equation (2.10), is an exact solution of the time domain equation (1.1) and satisfies the initial condition (1.2). This implies that $\mathcal{L}T(r,p) = \mathcal{L}T_1(r,p) + \mathcal{L}T_2(r,p)$ is a so-called operational solution to the initial value problem (3.5)-(3.6). It remains to show that $\mathcal{L}T(r,p)$ satisfies also the boundary conditions (3.7). By substituting $r=1$ in the above expression of $\mathcal{L}T(r,p)$, the droplet surface temperature is derived in the Laplace domain as:

$$\mathcal{L}T_s(p) = -\frac{(\mathrm{e}^{-2\sqrt{p}} - 1)\mathcal{L}q_s(p) + \mathrm{U}(T_0(x),p)}{\mathrm{e}^{-2\sqrt{p}}\sqrt{p} + \mathrm{e}^{-2\sqrt{p}} + \sqrt{p} - 1}, \tag{3.12}$$

where $\mathrm{U}(T_0(x),p)$ is the operator defined by the equation (3.4) of the proposition 3.1. The LIT of the temperature gradient at the droplet surface, written in conditions (3.7), is recalled as:

$$\mathcal{L}q_s(p) = K\left(\mathcal{L}T_g(p) - \mathcal{L}T_s(p)\right), \tag{3.13}$$

where $K = k_g/k_l$ is a constant ratio such as $0 < K < 1$, and $\mathcal{L}T_g(p)$ is the Laplace transform of the surrounding gas-phase temperature $T_g(t)$. Combining the above equations (3.12) and (3.13) that are expressions of $\mathcal{L}T_s(p)$ and $\mathcal{L}q_s(p)$, the surface temperature and its gradient are respectively obtained in the Laplace domain by equations (3.1) and (3.2) as in the proposition 3.1. Combining now equations (3.1), (3.2) and (3.11), an explicit solution in the Laplace domain $\mathcal{L}T(r,p)$ of the temperature field inside the droplet is obtained as in proposition 3.1 by the equation (3.3). As expected, the boundary conditions (3.7) are satisfied by the Laplace domain solution $\mathcal{L}T(r,p)$ given by the equation (3.3) when considering the transform of the droplet surface temperature (3.1) and that of the temperature gradient (3.2). Finally, the proof of the proposition 3.1 is completed. □

Remark 3.1. *In all cases, inverse Laplace transforms can be accomplished numerically regardless of the complexity of the involved solutions (see for examples [14] and [15]). However, analytical solutions from the Laplace domain into the time domain may be also sought by means of Laplace inversion theorems.*

4. Analytical solutions in short time limits

We now seek for a general approximate solution for the droplet internal temperature, which is valid during any small value of the time step Δt, ($t \in [0, \Delta t]$). The order of magnitude of the time step Δt is typically $10^{-6}\,\mathrm{s}$ for internal combustion engines as mentioned in [10] and we recall that the dimensionless coefficient $K = k_g/k_l$ verifies $0 < K < 1$.

Proposition 4.1. *A truncated expansion of the droplet internal temperature $rT(r,t)$ during a short time step Δt ($t \in [0,\ \Delta t]$) can be expressed as:*

$$\begin{aligned} rT(r,t) \; &= K\int_0^t T_g(t-\eta)\mathrm{e}^{\frac{-(1-r)^2}{4\eta}}\left(\frac{1}{\sqrt{\pi\eta}} + 2(1-K)^2\sqrt{\frac{\eta}{\pi}}\right)d\eta \\ &+K(1-K)(r+K-Kr)\int_0^t T_g(t-\eta)\mathrm{erfc}\left(\frac{1-r}{2\sqrt{\eta}}\right)d\eta \\ &+T_0(1)\int_0^t \mathrm{e}^{\frac{-(1-r)^2}{4\eta}}\left(\frac{(1-r)}{2\sqrt{\pi}\eta^{3/2}} + \frac{2(1-K)}{\sqrt{\pi\eta}} + 4(1-K)^3\sqrt{\frac{\eta}{\pi}}\right)d\eta \end{aligned}$$

$$\begin{aligned}&+2(1-K)^2(r+K-Kr)T_0(1)\int_0^t \operatorname{erfc}\left(\frac{1-r}{2\sqrt{\eta}}\right)d\eta\\&+rT_0(r)+O(E(r,t)),\end{aligned} \tag{4.1}$$

for $0 \le r < 1$*, while at the droplet surface* $r = 1$*, the following first order truncated expansion is obtained for the temperature* $T_s(t)$*:*

$$\begin{aligned}&T_s(t)\\&=K\int_0^t T_g(t-\eta)\left(\frac{1}{\sqrt{\pi\eta}}+(1-K)+2(1-K)^2\sqrt{\frac{\eta}{\pi}}\right)d\eta\\&+2(1-K)T_0(1)\int_0^t\left(\frac{1}{\sqrt{\pi\eta}}+(1-K)+2(1-K)^2\sqrt{\frac{\eta}{\pi}}\right)d\eta\\&+T_0(1)+O(t).\end{aligned} \tag{4.2}$$

In expressions (4.1) and (4.2), $T_0(r)$ *and* $T_0(1)$ *are respectively the initial temperature distribution inside the droplet* $0 \le r < 1$ *and at its surface* $r = 1$*, the big* $O()$ *is the asymptotic notation,* erfc *is the complementary error function defined as* $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$*, and*

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{z^2}dz,$$

and finally

$$E(r,t)=\sqrt{\frac{t}{\pi}}(-1+r)\mathrm{e}^{\frac{-(1-r)^2}{4t}}+\frac{[(r-1)^2+2t]}{2}\operatorname{erfc}\left(\frac{1-r}{2\sqrt{t}}\right),$$

is the inverse Laplace transform of $\mathrm{e}^{-(1-r)\sqrt{p}}/p^2$ *as computed with Maple software.*

Proof. The Laplace domain solution (3.3) can be recast as the sum of three terms:

$$\begin{aligned}&\mathcal{L}T(r,p)\\&=\frac{K\mathrm{e}^{-\sqrt{p}}(\mathrm{e}^{r\sqrt{p}}-\mathrm{e}^{-r\sqrt{p}})}{r(\mathrm{e}^{-2\sqrt{p}}\sqrt{p}-K\,\mathrm{e}^{-2\sqrt{p}}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}+K-1)}\mathcal{L}T_g(p)\\&+\frac{(K-\sqrt{p}-1)\mathrm{e}^{-\sqrt{p}}(\mathrm{e}^{r\sqrt{p}}-\mathrm{e}^{-r\sqrt{p}})}{r(\mathrm{e}^{-2\sqrt{p}}\sqrt{p}-K\,\mathrm{e}^{-2\sqrt{p}}+\mathrm{e}^{-2\sqrt{p}}+\sqrt{p}+K-1)}\frac{[\mathrm{U}(T_0(x),p)]}{2\sqrt{p}}\\&-\frac{1}{2r\sqrt{p}}\int_0^1 xT_0(x)\left(\mathrm{e}^{-(x+r)\sqrt{p}}-\mathrm{e}^{-|r-x|\sqrt{p}}\right)dx,\end{aligned} \tag{4.3}$$

where $\mathrm{U}(T_0(x),p)$ is given by the relation (3.4). The limiting case of short time duration (t tending to 0) corresponds to a very large Laplace domain variable (p tending to $+\infty$). According to the LIT properties, the transforms $\mathcal{L}T_g(p)$ and $\mathrm{U}(T_0(x),p)$ tend to zero as p tends to $+\infty$. Then, the asymptotic expansion of second order for the droplet internal temperature $\mathcal{L}T(r,p)$, can be derived from the formula (4.3) by computing the asymptotic expansions of the corresponding terms (the last term remains unchanged):

(4.4)

$$\begin{aligned} r\mathcal{L}T(r,p) &= K\mathcal{L}T_g(p)\left(\frac{1}{\sqrt{p}}+\frac{(1-K)}{p}+\frac{(1-K)^2}{p^{3/2}}\right)\mathrm{e}^{-(1-r)\sqrt{p}} \\ &-\frac{\mathrm{U}(T_0(x),p)}{2\sqrt{p}}\left(1+\frac{2(1-K)}{\sqrt{p}}+\frac{2(1-K)^2}{p}+\frac{2(1-K)^3}{p^{3/2}}\right)\mathrm{e}^{-(1-r)\sqrt{p}} \\ &+\mathcal{L}T_2(r,p)+O\left(\frac{\mathrm{e}^{-(1-r)\sqrt{p}}}{p^2}\right), \end{aligned}$$

as the last integral term is identical to $\mathcal{L}T_2(r,p)$, according to the equation (3.10). By using the corollary 2.1, the inverse Laplace transform of $\mathcal{L}T_2(r,p)$, which is $T_2(r,t)$, tends to $rT_0(r)$ when p tends to $+\infty$ or equivalently when t tends to 0. It is also remarkable that the factor $-\mathrm{U}(T_0(x),p)/(2\sqrt{p})$ in the expression (4.4) can be identified to $\mathcal{L}T_2(r=1,p)$. Consequently, this factor goes in the time domain to $T_0(1)$ as p tends to $+\infty$. Concerning the first term, the inverse Laplace transform of $\mathcal{L}T_g(p)$ is evidently the time-varying temperature of the gas phase $T_g(t)$. Due to all these considerations, the convolution theorem (see [12]) and the converse to Watson's Lemma (see [16]), can be used in order to derive from equation (4.4), analytical approximations of the droplet internal and surface temperatures at the earliest time of the process or after any short time step Δt ($t \in [0, \Delta t]$). This leads to the formulae (4.1) and (4.2) as specified in the proposition 4.1. Since for $0 \leq r < 1$, it can be written that the limit when t tends to 0 of $E(r,t)/t^n = 0$ for all $n \geq 1$, the analytical approximation (4.1) is valid for an arbitrary order $n \geq 1$ of the related truncated expansion, if yet the surrounding gas temperature $T_g(t)$ at the immediate vicinity of the droplet is assumed to be known during the time step. This implies that, the absolute error committed by this approximation of $rT(r,t)$, is at the same order of magnitude as $(\Delta t)^n$ for all $n \geq 1$. This proves the accuracy of the formula (4.1) expressing the droplet internal temperature during a short time step Δt. However, for $r = 1$ i.e. at the droplet surface, $E(r = 1, t) = t$ and the formula (4.2) is valid only for the first order $n = 1$. Consequently, the absolute error committed in the evaluation of the droplet surface temperature $T_s(t)$ by using formula (4.2) is at the same order of magnitude as Δt. Now, the formula (4.2) is recursive since the droplet surface temperature $T_s(t)$ during the time step Δt is given as a function of $T_0(1)$, which is the surface temperature at the end of the previous time step. In CFD codes for droplets and sprays, as mentioned in the introduction, the surface temperature evaluation is sufficient, instead of the whole droplet internal temperature field, to determine the estimation of the heat and mass transferred by an evaporating droplet during a short time step. □

Corollary 4.1. *In the case of constant temperature $T_g(t) = \overline{T}_g$ of the surrounding gas phase during the small time step Δt, the analytical approximation of the droplet surface temperature $T_s(t)$ is obtained during $t \in [0, \Delta t]$ as:*

$$T_s(t) = T_0(1) + 2K\frac{[\overline{T}_g - 2(1-K)T_0(1)]}{\sqrt{\pi}}\sqrt{t} + O(t), \tag{4.5}$$

where $T_0(1)$ is recalled as the initial temperature of the droplet surface at the beginning of the time step Δt.

Proof. If the surrounding gas phase is assumed to be at constant temperature $T_g(t) = \overline{T}_g$ during the time step Δt, then $\mathcal{L}T_g(q) = \overline{T}_g/q$ and the approximation formula (4.2) can be explicitly evaluated. Retaining only the terms of order less than t in this evaluation, the analytical approximation of the droplet surface temperature $T_s(t)$ is derived from relation (4.2) by the approximation (4.5) as in the corollary. This completes the proof of the corollary. □

5. Conclusion

In this study, the combination of the classical Fourier cosine and Laplace integral transforms has permitted to obtain explicit solutions in the Laplace and time domains, for the spherically symmetric heat diffusion equation inside a motionless droplet suspended in a hot gas environment. Numerical inverse transforms of the droplet surface and internal temperatures from the Laplace domain into the time domain will be possible, regardless of the complexity of the involved solutions. However, the analytical approximations in short time limits obtained in the study can be applied to time-step models of vaporizing droplets and sprays, since they are recursive, and depend only on the appreciable time-evolving temperature of the gas-mixture at the immediate vicinity of the droplet. The new analytical approximations are proved to be computationally efficient, and may be generally useful for applications in droplets and spray modelling, as practised in CFD codes. Moreover, the combined method of integral transforms, as presented in the above study, is promising for divers moving-boundary-value problems, which often arise from engineering models that involve one-dimensional transient heat or mass transfer.

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DEPARTMENT OF MATHEMATICS,UNIVERSITY OF LOMÉ, 02 BP 1515, LOMÉ,TOGO.
Email address: ananikwassi@yahoo.fr, kanani@univ-lome.tg