Advances in Mathematics: Scientific Journal **10** (2021), no.11, 3381–3393
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.11.2

ON SOME ASPECTS OF COHERENT STATES QUANTIZATION WITH RELATED EXAMPLES

Isiaka Aremua[1], Komi Sodoga[2], and Paalamwé Komi Tchakpélé[3]

ABSTRACT. This work addresses the general procedure of quantization also known as the Berezin-Klauder-Toeplitz quantization, or as coherent state (CS) (anti- Wick) quantization. The method is first illustrated by the motion of a particle on the circle. Then, we take as second example, a set of generalized photon-added coherent states related to associated hypergeometric functions. The nonclassical behaviour of this set of coherent states is also investigated.

1. INTRODUCTION

Coherent states (CSs), known as an overcomplete family of vectors, represent one of the most fundamental framework for the analysis, or decomposition, of states in the Hilbert spaces, which are the underlying mathematical structures of several physical phenomena. CSs were introduced for the first time by Schrödinger in 1926 [1] in his study of quantum states that restore the classical behavior of

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 81R30, 81S08, 81V80, 81P15, 81P40.

Key words and phrases. Coherent states, Quantization, Mandel Q-parameter, Poissonian statistics.

Submitted: 25.10.2021; *Accepted:* 08.11.2021; *Published:* 21.11.2021.

a quantum observable. Klauder, Glauber, and Sudarshan reconsidered the definition of CSs introduced by Schrödinger at the beginning of the 1960s. In addition, Klauder formulated the so-called “Klauder’s minimal prescriptions (or conditions)”that any CS must meet. For various generalizations, approaches and their properties one may consult [2]- [5] and references therein.

Quantization is commonly understood as the transition from classical to quantum mechanics, relating to a larger discipline than specific domains of physics [4–6]. The coherent states (CSs) quantization (the quantization of measure spaces through CSs) related to integral quantization in the complex quantum mechanics has attracted much investigations [5]. For e.g., taking the phase space $X = \mathbb{C} = \left\{z = \frac{q+ip}{\sqrt{2}}\right\}$ endowed with the measure d^2z/π or $X \equiv S^1 \times \mathbb{R}$, CSs have been demonstrated to be useful in the quantization procedure, see for e.g., [7, 8] and references therein. CSs can be defined over complex domains with $\mathfrak{H} = span\{\phi_m, m \in \mathbb{N}\}$, the separable Hilbert space of the physical phenomena described by a quantum Hamiltonian, as linear superposition of the states or eigenfunctions ϕ_m:

$$|z\rangle = (\mathcal{N}(|z|))^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho(m)}} |\phi_m\rangle, \qquad z = re^{i\theta}, \tag{1.1}$$

where $0 \le r \le \infty, 0 \le \theta \le 2\pi$, $\{\rho(m)\}_{m=0}^{\infty}$ is a sequence of non-zero positive numbers, determining the internal structure of the CSs, chosen to ensure the convergence of the sum in a non-empty open subset $\mathfrak{D}$ of the complex plane, $\mathcal{N}(|z|)$ is the normalization factor ensuring that $\langle z|z\rangle = 1$. Our work deals with some aspects of the general procedure of CS quantization [4, 5] illustrated by some quantum models.

The paper is organized as follows. In section 2, we revisit the CS quantization procedure, and provide some notions on optical behaviour of a nonclassical field containing photons. Section 3 revisits the quantization problem of the motion of a particle on the circle studied in a previous work [7]. Let us note that the quantization procedure has been also achieved in the complex plane in [8]. Here, we focus on the construction of the building method of the operators acting on a Hilbert space from their associated classical observables describing the physical system by giving some details of calculations and proofs. As second illustration, we study in section 4, a set of generalized photon-added associated hypergeometric CSs (GPAH-CSs) related to associated hypergeometric functions [9]. These latter are obtained from a given analytical function not ever studied in the literature. The resolution of the identity obtained from a positive weight function, provided in terms of Meijer's G-function, allows the CS quatization. The field of GPAH-CSs are also characterized by the calculation of their corresponding Mandel's Q-parameter depicted through graphics ensuring their nonclassical behaviour. Concluding remarks are given in section 5.

2. Some preliminaries

2.1. **Coherent State Quantization: General Scheme.** Let X be a set of parameters equipped with a measure μ and let $L^2(X,\mu)$ be its associated Hilbert space of complex-valued square integrable functions with respect to μ. Let us choose in $L^2(X,\mu)$ a finite or countable orthonormal set $\mathcal{O} = \{\phi_n\,,\, n = 0,1,2,\dots\}$ satisfying:

$$\begin{aligned} \langle \phi_m | \phi_n \rangle &= \int_X \overline{\phi_m(x)}\,\phi_n(x)\,\mu(dx) = \delta_{mn}, \\ 0 < \sum_n |\phi_n(x)|^2 &:= \mathcal{N}(x) < \infty \quad \text{a.e.}\,. \end{aligned} \tag{2.1}$$

Let $\mathcal{H} := \overline{\mathrm{span}(\mathcal{O})}$ in $L^2(X,\mu)$ be a separable complex Hilbert space with orthonormal basis $\{|e_n\rangle\,, n = 0,1,2,\dots\}$, in one-to-one correspondence with the elements of $\mathcal{O} = \{\phi_n\,,\, n = 0,1,2,\dots\}$. One defines the family of states $\mathcal{F}_{\mathcal{H}} = \{|x\rangle\,,\, x \in X\}$ in $\mathfrak{H}$ as:

$$|x\rangle = \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n(x)}\,|e_n\rangle \ \in \mathcal{H}.$$

From conditions (2.1) these CSs are normalized, $\langle x|x\rangle = 1$ and resolve the identity in $\mathfrak{H}$:

$$\int_X \mathcal{N}(x)\,|x\rangle\langle x|\;\mu(dx) = \mathbb{I}_{\mathcal{H}}\,. \tag{2.2}$$

The relation (2.2) allows us to implement a *coherent state quantization* of the set of parameters X by associating to a function $X \ni x \mapsto f(x)$ that satisfies appropriate conditions the operator A_f in $\mathcal{H}$ as:

$$f(x) \mapsto A_f := \int_X \mathcal{N}(x)\,f(x)\,|x\rangle\langle x|\;\mu(dx).$$

The matrix elements of A_f with respect to the basis $|e_n\rangle$ are given by:

$$\left(A_f\right)_{nm} = \langle e_n|A_f|e_m\rangle = \int_X f(x)\,\overline{\phi_n(x)}\,\phi_m(x)\;\mu(dx).$$

The operator A_f is (i) symmetric if $f(x)$ is real valued, (ii) bounded if $f(x)$ is bounded (iii) self-adjoint if $f(x)$ is real semi-bounded (through Friedrich's extension, self ajoint extension). The "lower symbol" of A_f, the mean value of A_f is defined as:

$$\check{f}(x) := \langle x|A_f|x\rangle = \int_X \mathcal{N}(x')\, f(x')\, |\langle x|x'\rangle|^2\, \mu(dx'),$$

with f being the "upper symbol" of A_f.

2.2. **Mandel parameter.** Several parameters can be introduced to characterize the statistical properties, and the most popular one is the Mandel parameter [10], denoted here by $\mathcal{Q}$, known as a convenient noise-indicator of a non-classical field, which is frequently used to measure the deviation from Poisson distribution, and thus to distinguish quantum process from classical ones [11]- [13].

The Mandel parameter $\mathcal{Q}$ is defined as [10]

$$\begin{aligned} \mathcal{Q} &\equiv \frac{(\Delta N)^2 - \langle N\rangle}{\langle N\rangle} \\ &= \frac{2\langle I\rangle}{T}\int_0^T dt_2 \int_0^{t_2} dt_1 [1+\lambda(t_1)] - \langle I\rangle T, \end{aligned}$$

where:

- $\langle N\rangle$ is the average counting number;
- $(\Delta N)^2$ is the corresponding square variance;
- $\langle I\rangle = \langle N\rangle/T$ is the steady-state photon-counting rate expressed in units of cps;
- $\lambda(\tau) = \langle \Delta I(t)\Delta I(t+\tau)\rangle/\langle I(t)\rangle\langle I(t+\tau)\rangle$ is the normalized two-time correlation of intensity fluctuations $(\Delta I(t) = I(t) - \langle I(t)\rangle)$ of time difference equal to the time τ [11].

Moreover, the Mandel parameter $\mathcal{Q}$:

$$\mathcal{Q} = \frac{(\Delta N)^2}{\langle N\rangle} - 1 \equiv \mathcal{F} - 1,$$

is closely related to the normalized variance, also called the quantum Fano factor $\mathcal{F}$ [12], given by $\mathcal{F} = (\Delta N)^2/\langle N\rangle$, of the photon distribution. For $\mathcal{F} < 1 (\mathcal{Q} \leq 0)$, the emitted light is referred to as sub-Poissonian (corresponding to nonclassical states); $\mathcal{F} = 1, \mathcal{Q} = 0$ corresponds to the Poisson distribution (case of standard CS), whereas for $\mathcal{F} > 1, (\mathcal{Q} > 0)$ the light is called super-Poissonian (corresponding to classical states) [10]- [14].

Let us take as direct application of the above scheme the motion of a particle on the circle.

3. Quantization of the motion of a particle on the circle

The CS quantization of this problem, is achieved by taking the observation set X as the phase space of a particle moving on the circle, i.e., the cylinder:

$$X \equiv S^1 \times \mathbb{R} = \{x \equiv (\varphi, J), |0 \leq \varphi < 2\pi, J \in \mathbb{R}\},$$

equipped with the measure $\mu(dx) = \frac{1}{2\pi} dJ d\varphi$. The variables $J \in \mathbb{R}$ and $\varphi\, (0 \leq \varphi < 2\pi)$ are the action and angle variables satisfying:

$$\{J, \varphi\} = \frac{\partial J}{\partial J}\frac{\partial \varphi}{\partial \varphi} - \frac{\partial J}{\partial \varphi}\frac{\partial \varphi}{\partial J} = 1. \tag{3.1}$$

Introduce a probability distribution on the range of the variable J:

$$\mathbb{R} \ni J \mapsto \varpi^\sigma(J)\,, \quad \varpi^\sigma(J) = \varpi^\sigma(-J)\,, \quad \int_{-\infty}^{+\infty} \varpi^\sigma(J)\, dJ = 1\,,$$

being a non-negative, even, well-localized and normalized integrable function. $\sigma > 0$ is a kind of width parameter with the function ϖ^σ satisfying some required conditions [7].

Introducing the weighted Fourier exponentials $\phi_n(x) = \sqrt{\varpi^\sigma_n(J)}\, e^{in\varphi}\,, n \in \mathbb{Z}\,,$ the correspondent family of CSs on the circle reads as [7]:

$$|J, \varphi\rangle = \frac{1}{\sqrt{\mathcal{N}^\sigma(J)}} \sum_{n\in\mathbb{Z}} \sqrt{\varpi^\sigma_n(J)}\, e^{-in\varphi} |e_n\rangle, \quad \mathcal{N}^\sigma(J) = \sum_{n\in\mathbb{Z}} \varpi^\sigma_n(J).$$

By virtue of the CSs quantization described in the general scheme, the quantum operator (acting on $\mathcal{H}$) associated with the classical observable $f(x)$ is obtained through:

$$A_f := \int_X f(x) |x\rangle\langle x| \mathcal{N}(x) \mu(dx) = \sum_{n,n'} (A_f)_{nn'}\, |e_n\rangle\langle e_{n'}|\,. \tag{3.2}$$

We focus in this paragraph on the building method to obtain the operators A_J and A_φ by following the general scheme of CS quantization. Then, we have the proposition:

Proposition 3.1. *Taking $f(J,\varphi) = J$ and $f(\varphi) = \varphi,\ 0 \leq \varphi \leq 2\pi$, the operators A_J and A_φ, satisfying $[A_J, A_\varphi] = i\sum_{n\neq n'} \varpi^\sigma_{n,n'}|e_n\rangle\langle e_{n'}|$, are obtained as:*

$$\begin{aligned} A_J &= \int_X \mathcal{N}^\sigma(J)\, J\, |J,\varphi\rangle\langle J,\varphi|\, \mu(dx) = \sum_{n\in\mathbb{Z}} n\, |e_n\rangle\langle e_n|, \\ A_\varphi &= \pi \mathbb{I}_{\mathcal{H}} + i \sum_{n\neq n'} \frac{\varpi^\sigma_{n,n'}}{n-n'}|e_n\rangle\langle e_{n'}|. \end{aligned} \tag{3.3}$$

Proof. Using (3.2), the matrix element $\langle x|A_f|x\rangle$ is:

$$\langle x|A_f|x\rangle = \int_X \mathcal{N}(x')f(x')|\langle x|x'\rangle|^2\mu(dx')$$

allowing, with $X = S^1\times\mathbb{R}$, $x = (J,\varphi), x' = (J',\phi)$ where $\mu(dx') = \frac{1}{2\pi}d\varphi dJ$ and $\phi_n(x) = \sqrt{\varpi^\sigma_n(J)}\, e^{in\varphi}\,, n\in\mathbb{Z}$ that

$$\left(A_{f(J)}\right)_{nn'} = \delta_{nn'}\int_{-\infty}^{+\infty} \varpi^\sigma e_n(J)\, f(J)dJ = \delta_{nn'}\langle f\rangle_{\varpi^\sigma_n}$$

where $\langle\cdot\rangle_{\varpi^\sigma_n}$ designates the mean value with respect to the distribution $J\mapsto \varpi^\sigma_n(J)$. Since

$$\begin{aligned} \left(A_{f(J)}\right)_{nn'} &= \int_{-\infty}^{+\infty}\sqrt{\varpi^\sigma_n(J)\,\varpi^\sigma_{n'}(J)}dJ\int_0^{2\pi} e^{-i(n-n')\varphi}f(J,\varphi)\frac{d\varphi}{2\pi}, \\ f(J,\varphi) &= J \Longrightarrow A_J = \sum_{n,n'\in\mathbb{Z}}\int_{-\infty}^{\infty} J\varpi^\sigma_n(J)\delta_{nn'}|e_n\rangle\langle e_{n'}|dJ \end{aligned}$$

such that by a change of variable $J' = J - n$, we have:

$$A_J = \sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty}(J'+n)\varpi^\sigma_0(J')|e_n\rangle\langle e_n|dJ' = \sum_{n\in\mathbb{Z}} n|e_n\rangle\langle e_n|.$$

From

$$(A_f)_{nn'} = \int_{-\infty}^{\infty}\sqrt{\varpi^\sigma_n(J)\,\varpi^\sigma_{n'}(J)}dJ\int_0^{2\pi}e^{-i(n-n')\varphi}f(J,\varphi)\frac{d\varphi}{2\pi}$$

taking $f(J,\varphi)\equiv f(\varphi)$, we get

$$(A_{f(\varphi)})_{nn'} = \varpi^\sigma_{nn'}\int_0^{2\pi} f(\varphi)e^{-i(n-n')\varphi}\frac{d\varphi}{2\pi} = \varpi^\sigma_{nn'}c_{n-n'}(f).$$

Then,

$$A_\varphi = \sum_{n\in\mathbb{Z}}\pi\left[\int_{-\infty}^{\infty}\varpi^\sigma_n(J)dJ\right]|e_n\rangle\langle e_n| + \sum_{n\neq n'}\frac{i}{n-n'}\left[\int_{-\infty}^{\infty}\sqrt{\varpi^\sigma_n(J)\,\varpi^\sigma_{n'}(J)}dJ\right]|e_n\rangle\langle e_{n'}|$$

$$= \pi \mathbb{I}_{\mathcal{H}} + i \sum_{n \neq n'} \frac{\varpi^{\sigma}_{n,n'}}{n-n'} |e_n\rangle\langle e_{n'}|.$$

which completes the proof of (3.3). □

4. GENERALIZED PHOTON-ADDED CSS RELATED TO ASSOCIATED HYPERGEOMETRIC FUNCTIONS

The photon-added coherent states (PACSs) represent interesting states generalizing both the Fock states and CSs. Indeed, they are obtained by repeatedly operating the photon creation operator on an ordinary CS, see (1.1). In some previous works, the PACSs were assimilated to nonlinear CSs. Their various generalizations were also performed [15]. They evidence some nonclassical effects, for e.g, amplitude squeezing, sub-Poissonian behaviour, nonclassical quasi-probability distribution [16].

4.1. **Coherent states construction.** In this paragraph, we apply the quantization procedure to the generalized photon-added associated hypergeometric CSs (GPAH-CSs) developed in [9] for Jacobi polynomials and hypergeometric functions. We introduce as illustration the analytical function (see [9] for the notations):

$$f(r_{m,n}(k)) = \sqrt{(\xi(r_{m,n}(k);1,-1))^2}. \tag{4.1}$$

This function (4.1) has not been treated in [9]. We obtain the expansion coefficient as:

$$|K^p_n(m)| = \sqrt{\frac{\Gamma(n+1)^2\Gamma(n+m+2\nu)^2\Gamma(m+\nu)^2}{\Gamma(2m+2\nu)\Gamma(n+p+1)\Gamma(n+p+2m+2\nu)\Gamma(n+m+\nu)^2}}.$$

The normalization factor gives in terms of Meijer's G-function:

$$\mathcal{N}_p(|z|^2;m) = \left[\frac{\Gamma(2m+2\nu)}{\Gamma(m+\nu)^2}\, G^{1,4}_{4,4}\left(-|z|^2 \,\middle|\, \begin{matrix} -p, 1-p-2m-2\nu, -m-\nu-1, -m-\nu-1; \\ 0\ ;\ 0, 1-2m-2\nu, 1-2m-2\nu \end{matrix}\right)\right]^{-1/2}.$$

The explicit form of the GPAH-CSs defined for $|z|<1$ follows as:

$$|z;m\rangle_p = \mathcal{N}_p(|z|^2;m)\sqrt{\frac{\Gamma(2m+2\nu)}{\Gamma(m+\nu)^2}}$$

$$\times \sum_{n=0}^{\infty} \left\{ \left[\frac{\Gamma(n+p+1)\Gamma(n+m+\nu)^2\Gamma(n+p+2m+2\nu)}{\Gamma(n+m+2\nu)^2} \right]^{1/2} \frac{z^n}{n!} |n+p\rangle \right\}. \tag{4.2}$$

The GPAH-CSs (4.2) fullfill the overcompleteness relation:

$$\int_{\mathbb{C}} d^2z \; |z;m\rangle_p \, \omega_p(|z|^2;m) \; {}_p\langle z;m| = \mathbb{I}_{\mathfrak{H}_{m,p}} \equiv \sum_{n=0}^{\infty} |n+p\rangle \, |n+p| \,, \tag{4.3}$$

with the weight function:

$$\begin{aligned} \omega_p(|z|^2;m) &= \frac{1}{\pi} G^{1,4}_{4,4}\left(-|z|^2 \,\middle|\, \begin{matrix} -p, 1-p-2m-2\nu, -m-\nu-1, -m-\nu-1; \\ 0\,;\; 0, 1-2m-2\nu, 1-2m-2\nu \end{matrix} \right) \\ &\quad \times G^{4,0}_{4,4}\left(|z|^2 \,\middle|\, \begin{matrix} ;\; p, -1+m+\nu, -1+m+\nu, -1+p+2m+2\nu \\ 0, 0, -1+m+2\nu, -1+m+2\nu\,; \end{matrix} \right). \end{aligned} \tag{4.4}$$

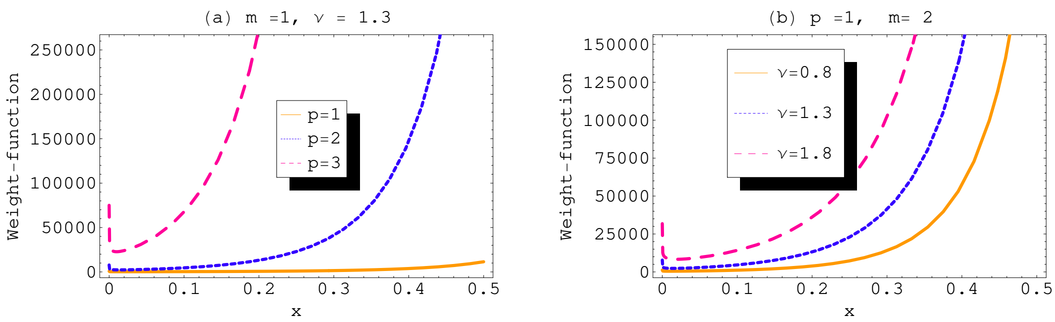

FIGURE 1. *Plots of the weight function (4.4) of the GPAH-CSs (4.2) versus $x = |z|^2$: (a) with the parameters $p = 1$, $\nu = 1.3$ and for different values of p; (b) with the parameters $p = 1$, $m = 2$ and for different values of ν.*

In Fig.1-a, we plot the weight function (4.4) versus $x = |z|^2$ for $m = 1$, $\nu = 1.3$ and for different values of the photon-added number $p = 1, 2, 3$. All the curves are positive, this confirms the positivity of the weight function for the parameter $\nu > 0$. Fig.1-b shows that the polynome parameter ν does not affect the general behaviour of the curves but increases their amplitude.

4.2. **Quantization in the complex plane.** The resolution of the identity (4.3) allows us to implement CS quantization also named Berezin-Klauder-Toeplitz or anti-Wick quantization of the complex plane by associating a function $\mathbb{C} \ni z \mapsto f(z)$. Define the operator on the Hilbert space $\mathfrak{H}_{m,p}$:

$$f(z) \mapsto A_f = \int_{\mathbb{C}} f(z) \, |z;m\rangle_p \;{}_p\langle z;m| \, \mathcal{N}_p(|z|^2;m)\omega_p(|z|^2;m),$$

such that:

$$\begin{aligned} A_f &= \sum_{k,n=0}^{\infty} \frac{|n+p\rangle\langle k+p|}{k!n!} \left[\frac{\Gamma(n+p+1)\Gamma(n+m+\nu)^2\Gamma(n+p+2m+2\nu)}{\Gamma(n+m+2\nu)^2}\right]^{1/2} \\ &\times \left[\frac{\Gamma(k+p+1)\Gamma(k+m+\nu)^2\Gamma(k+p+2m+2\nu)}{\Gamma(k+m+2\nu)^2}\right]^{1/2} \frac{\Gamma(2m+2\nu)}{\Gamma(m+\nu)^2} \\ &\times \int_{\mathbb{C}} \mathcal{N}_p(|z|^2;m)\omega_p(|z|^2;m)f(z)z^n\bar{z}^k d^2z. \end{aligned}$$

Then, we get the following matrix elements, via the maps $z \mapsto A_z$ and $\bar{z} \mapsto A_{\bar{z}}$ defined on the Hilbert space $\mathfrak{H}_{m,p}$:

$$\begin{aligned} (A_z)_{k,n} &= \sum_{k,n=0}^{\infty} \frac{|n+p\rangle\langle k+p|}{k!n!} \left[\frac{\Gamma(n+p+1)\Gamma(n+m+\nu)^2\Gamma(n+p+2m+2\nu)}{\Gamma(n+m+2\nu)^2}\right]^{1/2} \\ &\times \left[\frac{\Gamma(k+p+1)\Gamma(k+m+\nu)^2\Gamma(k+p+2m+2\nu)}{\Gamma(k+m+2\nu)^2}\right]^{1/2} \frac{\Gamma(2m+2\nu)}{\Gamma(m+\nu)^2} \\ &\times \int_0^{2\pi}\int_0^{\infty} rdrd\theta e^{i(n+1-k)\theta} r^{n+1+k}\mathcal{N}_p(|z|^2;m)\omega_p(|z|^2;m), \end{aligned}$$

$$\begin{aligned} (A_{\bar{z}})_{k,n} &= \sum_{k,n=0}^{\infty} \frac{|n+p\rangle\langle k+p|}{k!n!} \left[\frac{\Gamma(n+p+1)\Gamma(n+m+\nu)^2\Gamma(n+p+2m+2\nu)}{\Gamma(n+m+2\nu)^2}\right]^{1/2} \\ &\times \left[\frac{\Gamma(k+p+1)\Gamma(k+m+\nu)^2\Gamma(k+p+2m+2\nu)}{\Gamma(k+m+2\nu)^2}\right]^{1/2} \frac{\Gamma(2m+2\nu)}{\Gamma(m+\nu)^2} \\ &\times \int_0^{2\pi}\int_0^{\infty} rdrd\theta e^{i(n-k-1)\theta} r^{n+1+k}\mathcal{N}_p(|z|^2;m)\omega_p(|z|^2;m). \end{aligned}$$

For $|z|^2$, we have:

$$\begin{aligned} (A_{|z|^2})_{k,n} &= \sum_{k,n=0}^{\infty} \frac{|n+p\rangle\langle k+p|}{k!n!} \left[\frac{\Gamma(n+p+1)\Gamma(n+m+\nu)^2\Gamma(n+p+2m+2\nu)}{\Gamma(n+m+2\nu)^2}\right]^{1/2} \\ &\times \left[\frac{\Gamma(k+p+1)\Gamma(k+m+\nu)^2\Gamma(k+p+2m+2\nu)}{\Gamma(k+m+2\nu)^2}\right]^{1/2} \frac{\Gamma(2m+2\nu)}{\Gamma(m+\nu)^2} \\ &\times \int_0^{2\pi}\int_0^{\infty} rdrd\theta e^{i(n-k)\theta} r^{n+2+k}\mathcal{N}_p(|z|^2;m)\omega_p(|z|^2;m), \end{aligned}$$

where the following relations

$$z = re^{i\theta}, \qquad \bar{z} = re^{-i\theta} \qquad \text{and} \qquad d^2 z = r dr d\theta$$

$$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 0 & \text{if} \quad m \neq n, \\ \\ 2\pi & \text{if} \quad m = n \end{cases}$$

are used. Then, using (4.5)-(4.2), we get:

$$A_z = \sum_{n=0}^{\infty} \frac{(n+m+2\nu)(n+1)}{(n+m+\nu)\sqrt{(n+p+2)(n+p+2m+2\nu)}} |n+p\rangle\langle n+1+p|,$$

$$A_{\bar{z}} = \sum_{n=0}^{\infty} \frac{(n+m+2\nu)(n+1)}{(n+m+\nu)\sqrt{(n+p+2)(n+p+2m+2\nu)}} |n+1+p\rangle\langle n+p|,$$

$$A_{|z|^2} = \sum_{n=0}^{\infty} \frac{(n+m+2\nu)^2(n+1)^2}{(n+m+\nu)^2(n+p+1)(n+p+2m+2\nu)} |n+p\rangle\langle n+p|.$$

Thereby, the commutator $[A_z, A_{\bar{z}}]$ is delivered as:

$$\begin{aligned}
&[A_z, A_{\bar{z}}] \\
=& \frac{(m+2\nu)^2}{(m+\nu)(p+2)(p+2m+2\nu)} |0+p\rangle\langle 0+p| \\
&+ \sum_{n=1}^{\infty} \left\{ \frac{[(n+m+2\nu)(n+1)]^2}{(n+m+\nu)(n+p+2)(n+p+2m+2\nu)} \right. \\
&- \left. \frac{[(n-1+m+2\nu)n]^2}{(n-1+m+\nu)(n-1+p+2)(n-1+p+2m+2\nu)} \right\} |n+p\rangle\langle n+p|.
\end{aligned}$$

4.3. **Mandel parameter.** Computing the relations which determine this quantity (see [9]) in the case of the GPAH-CSs (4.2), we obtain:

$$Q = (m+p) \left[\frac{{}_6\mathcal{F}_5(|z|^2; m, p)}{{}_5\mathcal{F}_4(|z|^2; m, p} - \frac{{}_5\mathcal{F}_4(|z|^2; m, p)}{{}_4\mathcal{F}_3(|z|^2; m, p)} \right] - 1, \tag{4.5}$$

where ${}_4\mathcal{F}_3$, ${}_5\mathcal{F}_4$ and ${}_6\mathcal{F}_5$ are the generalized hypergeometric functions:

$$ {}_4\mathcal{F}_3(|z|^2;m,p) = {}_4F_3\left(\begin{array}{l} 1+p, 2m+p+2\nu, m+\nu, m+\nu; \\ 1, 2m+2\nu, 2m+2\nu; |z|^2 \end{array}\right), $$

$$ {}_5\mathcal{F}_4(|z|^2;m,p) = {}_5F_4\left(\begin{array}{l} 1+p, m+p+1, p+2m+2\nu, m+\nu, m+\nu; \\ 1, m+p, 2m+2\nu, 2m+2\nu; |z|^2 \end{array}\right), $$

$$ {}_6\mathcal{F}_5(|z|^2;m,p) = {}_6F_5\left(\begin{array}{l} 1+p, m+p+1, m+p+1, p+2m+2\nu, m+\nu, m+\nu; \\ 1, m+p, m+p, 2m+2\nu, 2m+2\nu; |z|^2 \end{array}\right). $$

Fig.2 shows that the Mandel Q-parameter (4.5) increases with the amplitude $|z|$. Fig.2-b shows that the parameter ν does not influence the behaviour of the curves. For small values of $|z|$, the Mandel Q-parameter is negative and become positive for high values of $|z|$. Then the GPAH-CSs (4.2) exhibit sub-poissonian distribution for small values of the amplitude $|z|$ and for sufficiently high values of $|z|$ present super-Poissonian distribution.

Concluding remarks

In this work, we have reviewed the general procedure of CS quantization also known as the Berezin-Klauder-Toeplitz quantization for a given set X of parameters equipped with a measure μ. As illustrations, the case of the motion of a particle on a circle has first studied. Indeed, we have provided details of calculations of the operators associated to the action and angle variables. Next, the CS quantization procedure has been applied to a set of generalized photon-added

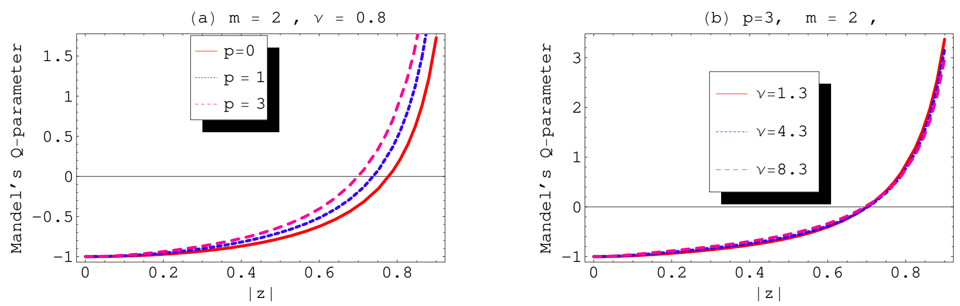

FIGURE 2. *Plots of the Mandel Q-parameter (4.5) of the GPAH-CSs (4.2) versus $|z|$ with the parameters : (a) $m = 2$ and $\nu = 0.8$ and for various values of the photon-added number p. (b) $p = 3$ and $m = 2$ for various values of ν.*

CSs related to associated hypergeometric functions, denoted GPAH-CSs. From the resolution of the identity satisfied by the constructed CSs obtained by a positive weight function, a CS quantization has been performed in the complex plane. In addition, the nonclassical behaviour of the GPAH-CSs has been discussed by investigating the Mandel Q-parameter.

References

[1] E. Schrödinger: *Der stetige Übergang von der Mikro-zur Makromechanik*, Naturwissenschaften **14** (1926), 664.

[2] J. R. Klauder and B. S. Skagerstam: *Coherent states. Applications in physics and mathematical physics*, World Scientific Publishing Co., Singapore (eds), 1985.

[3] A. M. Perelomov: *Generalized Coherent States and Their Applications*, Springer, Berlin, 1996.

[4] J. P. Gazeau: *Coherent States in Quantum Physics*, Wiley-VCH, Berlin, 2009.

[5] S. T. Ali, JP. Antoine, JP. Gazeau: *Coherent States, Wavelets and their Generalizations, 2nd ed., Theoretical and Mathematical Physics*, Springer, New York, 2014.

[6] F.A. Berezin : *General concept of quantization*, Commun. Math. Phys. **40** (1975), 153–74; J.R. Klauder: *Quantization without quantization*, Ann. Phys. **237** (1995), 147–60; S. T. Ali, M. Engliš: *Quantization methods: a guide for physicists and analysis*, arXiv: math-ph/0405065v1 (2004).

[7] I. Aremua, JP. Gazeau, M. N. Hounkonnou: *Action-angle coherent states for quantum systems with cylindrical phase space*, J. Phys. A: Math. Theor. **45** (2012), 335302.

[8] I. Aremua, M. N. Hounkonnou, E. Baloïtcha: *Coherent states for Landau levels: algebraic and thermodynamical properties*, Rep. Math. Phys. **76** (2) (2015), 247–269.

[9] K. Sodoga, I. Aremua, M. N. Hounkonnou: *Generalized photon-added associated hypergeometric coherent states: characterization and relevant properties*, Eur. Phys. J. D **72** (2018), 172.

[10] L. Mandel: *Sub-Poissonian photon statistics in resonance fluorescence*, Opt. Lett. **4** (1979), 205; L. Mandel, E. Wolf: *Optical coherence and quantum optics*, Cambridge University Press, Cambridge, 1995.

[11] H. J. Kimble, M. Dagenais, L. Mandel: *Photon Antibunching in Resonance Fluorescence*, Phys. Rev. Lett. **39** (1977), 691.

[12] J. Bajer, A. Miranowicz: *Sub-Poissonian photon statistics of higher harmonics: quantum predictions via classical trajectories*, J. Opt. B: Quantum Semiclass. Opt. **2** (2000), L10.

[13] G. Li, T. C. Zhang, Y. Li, J. M. Wang: *Photon statistics of light fields based on single-photon-counting modules*, Phys. Rev. A **71** (2005), 023807.

[14] X.-Z. ZHANG, Z.-H. WANG, , H. LI, Q. WU, B.-Q. TANG, F. GAO, J.-J. XU: *Characterization of Photon Statistical Properties with Normalized Mandel Parameter*, Chin. Phys. Lett. **25** (2008), 3976.

[15] H.M. LI, H.C. YUAN, H.Y. FAN: Int. J. Theor. Phys. **48** (2009), 2849; J.S. ZHANG, J. B. XU: Phys. Scr. **79** (2009), 025008

[16] V.V. DODONOV, YA A. KORENNOY, V.I. MAN'KO, Y.A. MOUKHIN: Quantum Semiclass. Opt. **8** (1996), 413; V.V. DODONOV: J. Opt. B: Quantum Semiclass. **4** (2002), R1; V.V. DODONOV, V.I. MAN'KO: *Theory of nonclassical states of light*, Taylor and Francis, London, 2003.

[1,2,3]DÉPARTEMENT DE PHYSIQUE, UNIVERSITÉ DE LOMÉ (UL)- LABORATOIRE DE PHYSIQUE DES MATÉRIAUX ET DES COMPOSANTS À SEMI-CONDUCTEURS-FACULTÉ DES SCIENCES (FDS) - B.P. 1515 LOMÉ, LOMÉ, TOGO.

INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS, (ICMPA–UNESCO CHAIR), 072 B.P. 50, COTONOU, REPUBLIC OF BENIN

Email address: claudisak@gmail.com

Email address: antoinekomisodoga@gmail.com

Email address: komtchak@yahoo.fr