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# CONFIDENCE INTERVALS OF THE ADJUSTED TAIL CONDITIONAL EXPECTATION RISK MEASURE FOR A STATIONARY SERIE

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ABSTRACT. In this paper we present a semi-parametric estimator of the adjusted tail conditional expectation risk measure based on the theory of extreme values for a stationary serie. We prove its asymptotic normality and we construct the confidence intervals. The accuracy of these intervals is evaluated through a simulation study.

## 1. INTRODUCTION

The principle of insurance is based on the concept of risk transfer: for a premium, the insured protects himself from a random financial risk. In order to be solvent, an insurer must have a certain level of equity, and even add a security charge to the net premium. To do this, it needs to measure the insured risk.

Some of the early risk measures in actuarial science were based on the so called premium principles. The purpose is to develop an appropriate premium to charge for a given risk.

Let $X > 0$ be loss random variable. The measure of risk is a function

$$\mathcal{R}(X) : X \to [0, \infty).$$

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The use of risk measures in actuarial science was in the development of the principle of calculating premiums. According to Artzner *et al* [1] a risk measure $\mathcal{R}(.)$ is said to be coherent when it satisfies the following four coherence properties:

- *(P1)-Positive homogeneity:*

$$\mathcal{R}(\zeta X) = \zeta \mathcal{R}(X),\ \zeta > 0.$$

- *(P2)-Translation invariance:*

$$\mathcal{R}(X + \delta) = \mathcal{R}(X) + \delta,\ \delta \in \mathbf{R}.$$

- *(P3)-Sub-additivity:* For any random loss variables $X$, $Y$:

$$\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y).$$

- *(P4)-Monotony:* For any random loss variables $X$, $Y$, with $X \leq Y$ in probability

$$\mathcal{R}(X) \leq \mathcal{R}(Y).$$

Let $X$ be an insurance risk, that is, a non-negative random variable representing the total claim amount of an insurance policy in a given period of time. When a risk measure $\mathcal{R}$ is used for premium calculation, it is often assumed that the premium coincides with the risk measure of $X$.

Another property that must be checked by risk measures is that they contain a safety load,

$$\mathcal{R}(X) \geq E(X).$$

There is growing interest among insurance and investment experts in the use of the tail conditional expectation (TCE) as a measure of risk because of its desirable properties and its flexibility. To define this premium principle, we suppose $X$ has distribution function $F(x)$ and survival function given by $S(x) = 1 - F(x)$. The tail conditional expectation premium calculation principle is defined as

$$TCE_{1-p}(X) = E[X | X > VaR_{1-p}],$$

where (VaR) is the Value-at-Risk defined by the following qauntile function

$$VaR_{1-p}(X) = Q(1-p) = F^{-1}(1-p).$$

The probability level $p$ is usually taken to be close to 0.

The $TCE$ is a coherent risk measure, it takes into account the whole information contained in the upper part of the tail distribution and, contrary to the $VaR$, on the heaviness of the tail distribution.

TCE is an important tail characteristic, which is most often studied in insurance and finance (see [9], [10], [19]).

Once the degree of riskiness is known, there still is the problem of incorporating a risk loading to be added to the net premium ($E(X)$). This led Denneberg [2] and Wang [21] to develop the following distorted premium calculation principle.

The idea is indeed to make the tail of the distribution of the variable of interest heavier in order to generate a load relative to the net premium. This transformation of the distribution function will be performed using a distortion function $g$, be an increasing concave function defined on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. Wang's premium is given by

$$\Pi_g = \int_0^{+\infty} g(S(x))dx. \tag{1.1}$$

The distortion risk measure in (1.1) has been studied by many authors such that [5], [16], [17] and [18].

The $TCE$ is a particular case of $\Pi_g$ using the following distortion function in formula (1.1),

$$g_{TCE}(x) = \begin{cases} \dfrac{x}{1-p} & \text{if} \quad x \leq 1-p, \\ 1 & \text{if} \quad x > 1-p. \end{cases}$$

Li Zhu and Haijun Li [24] propose the so-called (ATCE): risk-adjusted or distorted version of $TCE$. Their approach is inspired by Denneberg [2] and Wang [21] in order to obtain a risk-loaded premium.

This measure is defined of any non-negative random variable $X$ as follows:

$$ATCE_{1-p} = \int_0^{\infty} g(\overline{F}_{X|X>VaR_{1-p}}(x))dx, \tag{1.2}$$

with $\overline{F}_{X|X>s}(x) = 1 - F_{X|X>s}(x) = 1 - \mathbf{P}(X \leq x | X > s)$.

Premium calculation is one of the most essential and complex tasks of an insurer. Premium flow must guarantee payments of claims, but on the other hand, premiums must be competitive.

Very often loss distributions are skewed and have so-called fat tails. In this case, the use of methods based on a priori assumptions about the normal distributions is untenable, and it makes sense to use Extreme Value Theory (EVT).

The objective of this paper is to propose an semi asymptotically normal of $ATCE$ for mixing heavy-tailed process using the EVT approach, since in insurance and finance the real data sets are most often dependent.

## 2. Heavy tailed $\beta$-mixing sequences

Let $\{X_i\}$ a stationary sequence with common distribution function $F$ of an insured risk $X > 0$ satisfy the following condition of $\beta$-mixing dependence structure

$$\beta(l) := \sup_{m\in\mathbf{N}} \mathbf{E}\left(\sup_{A\in\mathcal{B}_{m+l+1}^{\infty}} |P(A|\mathcal{B}_1^m) - P(A)|\right) \to 0,$$

as $l \to \infty$, where $\mathcal{B}_1^m$ and $\mathcal{B}_{m+l+1}^{\infty}$ denote the $\sigma$-fields generated by $(X_i)_{1\leq i\leq m}$ and $(X_i)_{m+l+1\leq i}$, respectively.

We assume that the tail $S(x) = 1 - F(x)$ has regular variation function near infinity with index $-\alpha$, that is, for all $x > 0$,

$$\lim_{t\to\infty} \frac{S(tx)}{S(t)} = x^{-\alpha}, \tag{2.1}$$

where $\alpha > 0$ is the tail index. It follows that the survival function can be expressed as

$$S(x) = x^{-\alpha}\mathcal{L}(x),\ x > 0, \tag{2.2}$$

where $\mathcal{L}(x)$ is a slowly varying function at infinity:

$$\lim_{t\to\infty} \frac{\mathcal{L}(tx)}{\mathcal{L}(t)} = 1. \tag{2.3}$$

Such distribution function constitute a major subclass of the family of heavy-tailed distributions, reflecting the extremely high variability that they capture. These distributions have applications in finance, insurance, telecommunications, and many other fields (see Embrechts et al. [6]).

Several estimators of $\alpha$ have been proposed. One of the famous estimators was introduced by Hill [12] and defined by

$$\widehat{\alpha}^H = \left(\frac{1}{k}\sum_{i=1}^{k} \log X_{n-i,n} - \log X_{n-k+1,n}\right)^{-1}, \tag{2.4}$$

where $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ are the order statistics and $k = k_n$ is an intermediate sequence such that

$$k \to \infty\ ,\ k/n \to 0,\ n \to \infty. \tag{2.5}$$

Weissman [22] proposed the following semi-parametric estimator of a high quantile

$$\widehat{VaR}^H_{1-p} = \widehat{Q}^H(1-p) = X_{n-k,n}\left(\frac{np}{k}\right)^{-1/\widehat{\alpha}^H}. \tag{2.6}$$

We present now our main regularity conditions that will be maintained throughout.

(C1) Assumed that there exists a sequence $l_n$, $n \in \mathbf{N}$, such that

$$\lim_{n\to\infty} \frac{\beta(l_n)}{l_n} n + l_n k^{-1/2} \log^2 k = 0.$$

(C2) A regularity condition for the joint tail of $(X_1, X_{1+m})$:

$$c_m(x,y) = \lim_{n\to\infty} \frac{n}{k} P\left[X_1 > F^{-1}\left(1 - \frac{k}{n}x\right), X_{1+m} > F^{-1}\left(1 - \frac{k}{n}y\right)\right]$$

for all $m \in \mathbf{N},\ x > 0,\ y \le 1+\varepsilon,\ \varepsilon > 0$ and $F^{-1}$ denoting the inverse function of $F$.

(C3) A uniform bound on the probability that both $X_1$ and $X_{1+m}$ belong to an extreme interval:

$$\frac{n}{k} P\left(X_1 \in I_n(x,y), X_{1+m} \in I_n(x,y)\right) \le (y-x)\left(\widetilde{\rho}(m) + D_1 \frac{k}{n}\right),$$

for all $m \in \mathbf{N}$, $0 < x,\ y \le 1+\varepsilon$, where $D_1 \ge 0$ is a constant, $\widetilde{\rho}(m)$, is a sequence satisfying $\sum_{m=1}^{\infty} \widetilde{\rho}(m) < \infty$ and $I_n(x,y) = ]F^{-1}(1-yk/n), F^{-1}(1-xk/n)]$.

(C4) The quantile function admits the following representation:

$$F^{-1}(1-t) = dt^{-\alpha^{-1}}(1 - r(t)),$$

with $|r(t)| \le \Phi(t)$, for some constant $d > 0$ and a function $\Phi$ which is $\tau$-varying at 0 for $\tau > 0$, or $\tau = 0$ and $\Phi$ is non decreasing with $\lim_{t\downarrow 0} \Phi(t) = 0$

(C5) A limiting behavior for $k$

$$\lim_{n\to\infty} \sqrt{k}\Phi(k/n) \to 0.$$

## 3. ESTIMATION OF $ATCE$

Let $\{X_i\}$ be a positive stationary $\beta$-mixing process satisfying (2.3) and (C1)-(C5).

3.1. **Empirical estimator.** If $g(x) = x^\rho$, with $0 < \rho \le 1$ called the distortion parameter or the risk-aversion index, we find the Proportional Hazards (PH) transform which has been extensively studied in insurance applications (see Wang [20]), then from Li Zhu and Haijun Li [24], we can rewrite (1.2) as

$$ATCE_{1-p} = VaR_{1-p} + \frac{\int_{VaR_{1-p}}^{\infty} (S(x))^\rho dx}{p^\rho}. \tag{3.1}$$

By a change of variable, we have

$$ATCE_{1-p} = VaR_{1-p} - \frac{\int_0^p t^\rho dQ(1-t)}{p^\rho}$$

Then

$$\widehat{ATCE}_{1-p}^{emp} = X_{[n(1-p)],n} - \frac{\int_0^p t^\rho dQ_n(1-t)}{p^\rho},$$

where $Q_n(t)$ is the empirical quantile function defined as

$$Q_n(t) = X_{i,n},\ \frac{i-1}{n} < t < \frac{i}{n},\ i = 1, \dots, n,$$

with $Q_n(0) = X_{1,n}$. By integration by parts we get

$$\begin{aligned}
-\int_0^p t^\rho dQ_n(1-t) &= \rho \int_0^p t^{\rho-1} Q_n(1-t)dt - p^\rho X_{[n(1-p)],n} \\
&= \rho \int_{1-p}^1 (1-t)^{\rho-1} Q_n(t)dt - p^\rho X_{[n(1-p)],n} \\
&= \sum_{i=[n(1-p)]}^{n} \int_{\frac{i}{n}}^{\frac{i-1}{n}} (1-t)^\rho X_{i,n} - p^\rho X_{[n(1-p)],n} \\
&= \sum_{i=[n(1-p)]}^{n} \left[ \left(\frac{n-i+1}{n}\right)^\rho - \left(\frac{n-i}{n}\right)^\rho \right] X_{i,n} - p^\rho X_{[n(1-p)],n}.
\end{aligned}$$

Then

$$\widehat{ATCE}^{emp}_{1-p} = \frac{1}{p^\rho} \sum_{i=[n(1-p)]}^{n} \left[ \left( \frac{n-i+1}{n} \right)^\rho - \left( \frac{n-i}{n} \right)^\rho \right] X_{i,n}. \tag{3.2}$$

3.2. **Semi parametric estimator using Hill estimator.** Recall, from Karamata's Theorem (see de Haan and Ferreira [8]), that for $1/\alpha < \rho \leq 1$ we have

$$\frac{\int_{VaR_{1-p}}^{\infty} (S(x))^\rho dx}{VaR_{1-p} p^\rho} \to \frac{1}{\alpha\rho - 1},\ as\ p \to 0,$$

since $(S(x))^\rho$ is regular varying with index $-\alpha\rho < -1$ and $VaR_{1-p} \to \infty$ as $p \to 0$. Hence we have

$$ATCE_{1-p} = \frac{\alpha\rho}{\alpha\rho - 1} VaR_{1-p}\,, \tag{3.3}$$

with $1/\alpha < \rho \leq 1$.

From (3.3) we have the following estimator

$$\widehat{ATCE}^{H}_{1-p} = \frac{\widehat{\alpha}^H \rho}{\widehat{\alpha}^H \rho - 1} \widehat{VaR}^{H}_{1-p} = \frac{\widehat{\alpha}^H \rho}{\widehat{\alpha}^H \rho - 1} X_{n-k,n} \left( \frac{np}{k} \right)^{-1/\widehat{\alpha}^H}. \tag{3.4}$$

**Theorem 3.1.** *Under the conditions (C1)-(C5) with $l_n k/n \to 0$ as $n \to \infty$ and $k/np \to \infty$. For $p \to 0$, then*

$$\frac{\sqrt{k}}{\log(k/np)} \left( \log \frac{\widehat{ATCE}^{H}_{1-p}}{ATCE_{1-p}} \right) \xrightarrow{D} \mathcal{N}\left(0, \sigma_H^2\right),$$

*where*

$$\sigma_H^2 = \alpha^2 c(1,1). \tag{3.5}$$

*and*

$$c(x,y) = \min(x,y) + \sum_{m=1}^{\infty} [c_m(x,y) + c_m(y,x)]. \tag{3.6}$$

*Proof.* We start with

$$\widehat{ATCE}^{H}_{1-p} - ATCE_{1-p} = \frac{\widehat{\alpha}^H \rho}{\widehat{\alpha}^H \rho - 1} (\widehat{VaR}^{H}_{1-p} - VaR_{1-p}) + VaR_{1-p} \left( \frac{\widehat{\alpha}^H \rho}{\widehat{\alpha}^H \rho - 1} - \frac{\alpha\rho}{\alpha\rho - 1} \right),$$

then we have

$$\frac{\sqrt{k}}{VaR_{1-p}\log(k/np)}(\widehat{ATCE}_{1-p}^{H}-ATCE_{1-p})$$
$$=\left(\frac{\widehat{\alpha}^{H}\rho}{\widehat{\alpha}^{H}\rho-1}\right)\frac{\sqrt{k}}{\log(k/np)}\left(\frac{\widehat{VaR}_{1-p}^{H}}{VaR_{1-p}}-1\right)$$
$$+\frac{\sqrt{k}}{\log(k/np)}\left(\frac{\widehat{\alpha}^{H}\rho}{\widehat{\alpha}^{H}\rho-1}-\frac{\alpha\rho}{\alpha\rho-1}\right).$$

Hence we have

$$\frac{\sqrt{k}}{\log(k/np)}\left(\frac{\widehat{ATCE}_{1-p}^{H}}{ATCE_{1-p}}-1\right)=H_1+H_2,$$

where

$$H_1=\left(\frac{\alpha\rho-1}{\alpha\rho}\right)\left(\frac{\widehat{\alpha}^{H}\rho}{\widehat{\alpha}^{H}\rho-1}\right)\frac{\sqrt{k}}{\log(k/np)}\left(\frac{\widehat{VaR}_{1-p}^{H}}{VaR_{1-p}}-1\right)$$

and

$$H_2=\left(\frac{\alpha\rho-1}{\alpha\rho}\right)\frac{\sqrt{k}}{\log(k/np)}\left(\frac{\widehat{\alpha}^{H}\rho}{\widehat{\alpha}^{H}\rho-1}-\frac{\alpha\rho}{\alpha\rho-1}\right).$$

From Theorem 2.2 of Drees [4] and Drees [3] we have

$$\frac{\sqrt{k}}{\log(k/np)}\left(\frac{\widehat{VaR}_{1-p}^{H}}{VaR_{1-p}}-1\right)\sim\sqrt{k}(\widehat{\alpha}^{H}-\alpha)\xrightarrow{D}\mathcal{N}\left(0,\sigma_H^2\right), \tag{3.7}$$

where

$$\sigma_H^2=\alpha^2c(1,1). \tag{3.8}$$

Since $\widehat{\alpha}$ is a consistent estimator for $\alpha$ (see Hsing [13]), then for all large $n$ and $k/np\to\infty,\ p\to 0,$

$$H_1\xrightarrow{D}\mathcal{N}\left(0,\sigma_H^2\right). \tag{3.9}$$

By the application of the delta method, we find

$$\sqrt{k}\left(\frac{\widehat{\alpha}^{H}\rho}{\widehat{\alpha}^{H}\rho-1}-\frac{\alpha\rho}{\alpha\rho-1}\right)\xrightarrow{D}\mathcal{N}\left(0,\frac{\rho^2}{(\alpha\rho-1)^2}\sigma_H^2\right).$$

Hence

$$H_2\to 0. \tag{3.10}$$

By combining (3.9) and (3.10) and using that $\log(1+x) \sim x$ as $x \to 0$, we find the result of the Theorem 3.1. $\square$

## 4. SIMULATION

In this section, we carry out a simulation study by means of the statistical software R (see Ihaka and Gentleman [14]).

Consider now the stationary solution of the ARMAX(1) equation

$$X_t = \max(\lambda X_{t-1}, Z_t),\ 1 \leq t \leq n, \tag{4.1}$$

where $0 < \lambda < 1$ and $\{Z_t\}$ are independent and identically distributed, with tail distribution $1 - F_Z(x) = 1 - \exp(-(1-\lambda^\alpha)x^{-\alpha})$.

Ferreira and Canto e Castro [7] showed that

$$c(x,y) = \min(x,y) + \sum_{m=1}^{w-1}[c_m(x,y) + c_m(y,x)] + (x+y)\frac{\lambda^{w\alpha}}{1-\lambda^\alpha},$$

for

$$w \equiv w_{x,y} = [\max\{\alpha^{-1}\log(x/y)/\log\lambda,\ \alpha^{-1}\log(y/x)/\log\lambda\}] + 1$$

Then the variance in (3.8) becomes

$$\sigma_H^2 = \alpha^2\left(\frac{1+\lambda^\alpha}{1-\lambda^\alpha}\right). \tag{4.2}$$

To calculate the confidence intervals for the $ATCE$, we fix the risk aversion $\rho = 0.8$ and $\rho = 0.9$, then we follow the following steps:

(1) To select the optimal sample fraction $k$ in Hill estimator, we remark that

$$P(X > x) = 1 - \exp(-x^{-\alpha}),$$

so from Hall [11], Meerschaert and Scheffler [15] we can choose the optimal value as $k_{opt} = [n^{2/3}]$.

(2) Compute the corresponding values of $\widehat{\alpha}^H, \widehat{ATCE}_{1-p}^H$ denoted by $\widehat{\alpha}^{H,k_{opt}}$, $\widehat{ATCE}_{1-p}^{H,k_{opt}}$.

(3) Compute the estimator of $\lambda$ by $\widehat{\lambda} = \bigwedge_{i=2}^{n} \frac{X_i}{X_{i-1}}$, (see Zarepour and Banjevic [23]) and the corresponding value of $\widehat{\sigma}_H$ by

$$\widehat{\sigma}_H^{k_{opt}} = \widehat{\alpha}^{H,k_{opt}} \sqrt{\frac{1 + \widehat{\lambda}^{\widehat{\alpha}^{H,k_{opt}}}}{1 - \widehat{\lambda}^{\widehat{\alpha}^{H,k_{opt}}}}}$$

Then for $\theta \in (0,1)$ the $(1-\theta)$ confidence intervals of $ATCE$, will be $[lb, ub]$, where $lb$: "lower bound" and $up$: "upper bound" with

$$lb = \widehat{ATCE}_{1-p}^{H,k_{opt}} \times \exp\left(-z_{\theta/2} \times \widehat{\sigma}_H^{k_{opt}} \times k_{opt}^{-0.5} \times \log\left(k_{opt}/np\right)\right)$$

and

$$up = \widehat{ATCE}_{1-p}^{H,k_{opt}} \times \exp\left(z_{\theta/2} \times \widehat{\sigma}_H^{k_{opt}} \times k_{opt}^{-0.5} \times \log\left(k_{opt}/np\right)\right).$$

We generate 100 replications of the time series $(X_1, \ldots, X_n)$ for different sample sizes $(800, 900)$, where $X_t$ is an maximum autoregressive process of order one satisfying (4.1), with $c = 0.3$, using two tail indices $\alpha = 3$ and $\alpha = 4$. The simulation results are presented in the Table 1-2, we compare also in terms of absolute bias (abias) and root mean squared error (RMSE), the performances of the semi-parametric estimator $\widehat{ATCE}_{1-p}^{H}$ and the empirical estimator $\widehat{ATCE}_{1-p}^{emp}$. We conclude that the semi-parametric estimator has smaller bias and RMSE, it performs better than the empirical one.

Table 1: Point estimates and 95%-confidence intervals for $ATCE_{0.90}$.

| $\alpha$ | 3 | | 4 | |
|---|---|---|---|---|
| $\rho$ | 0.8 | 0.9 | 0.8 | 0.9 |
| $n = 800$ | | | | |
| $ATCE_{0.90}$ | 3.673326 | 3.400684 | 2.574858 | 2.449829 |
| $\widehat{ATCE}_{0.90}^{emp}$ | 3.515685 | 3.373349 | 2.555977 | 2.425367 |
| abias | 0.1576415 | 0.02733578 | 0.01888046 | 0.02446198 |
| RMSE | 0.3842232 | 0.30548 | 0.1697781 | 0.1800666 |
| $\widehat{ATCE}_{0.90}^{H}$ | 3.634467 | 3.414872 | 2.592479 | 2.42709 |
| abias | 0.03885992 | 0.01418773 | 0.01762137 | 0.02273939 |
| RMSE | 0.3486875 | 0.2960125 | 0.1559095 | 0.1579191 |

| | | | | |
|---|---|---|---|---|
| $lb$ | 3.441217 | 3.237414 | 2.417836 | 2.258819 |
| $ub$ | 3.838686 | 3.602166 | 2.779876 | 2.6081 |
| length | 0.3974697 | 0.364751 | 0.36204 | 0.3492816 |
| $n = 900$ | | | | |
| $\widehat{ATCE}_{0.90}^{emp}$ | 3.56382 | 3.380674 | 2.558335 | 2.463877 |
| abias | 0.1095064 | 0.02001071 | 0.01652297 | 0.01404715 |
| RMSE | 0.3253109 | 0.2747316 | 0.1357082 | 0.1266738 |
| $\widehat{ATCE}_{0.90}^{H}$ | 3.71199 | 3.387614 | 2.586034 | 2.462764 |
| abias | 0.03866322 | 0.01307011 | 0.01117669 | 0.01293508 |
| RMSE | 0.320591 | 0.2465466 | 0.1287492 | 0.1211489 |
| $lb$ | 3.614472 | 3.298168 | 2.498082 | 2.378489 |
| $ub$ | 3.812163 | 3.479503 | 2.67711 | 2.550056 |
| length | 0.1976909 | 0.1813347 | 0.1790284 | 0.1715677 |

Table 2: Point estimates and 95%-confidence intervals for $ATCE_{0.95}$.

| $\alpha$ | 3 | | 4 | |
|---|---|---|---|---|
| $\rho$ | 0.8 | 0.9 | 0.8 | 0.9 |
| $n = 800$ | | | | |
| $ATCE_{0.95}$ | 4.640824 | 4.298005 | 3.069092 | 2.920814 |
| $\widehat{ATCE}_{0.95}^{emp}$ | 4.505302 | 4.391646 | 3.038673 | 2.969765 |
| abias | 0.1355219 | 0.09364121 | 0.03041934 | 0.04895023 |
| RMSE | 0.6257293 | 0.8293191 | 0.2588149 | 0.2843255 |
| $\widehat{ATCE}_{0.95}^{H}$ | 4.750579 | 4.377251 | 3.082221 | 2.952297 |
| abias | 0.1097549 | 0.07924582 | 0.01312849 | 0.03148261 |
| RMSE | 0.6185121 | 0.4694091 | 0.2487155 | 0.2460353 |
| lb | 2.917828 | 2.688598 | 1.614364 | 1.557195 |
| ub | 7.751327 | 7.141143 | 5.911403 | 5.629174 |
| length | 4.8335 | 4.452546 | 4.297039 | 4.071979 |
| $n = 900$ | | | | |
| $\widehat{ATCE}_{0.95}^{emp}$ | 4.521222 | 4.263283 | 3.041758 | 2.932621 |

| | | | | |
|---|---|---|---|---|
| abias | 0.1196024 | 0.03472163 | 0.02733434 | 0.01180642 |
| RMSE | 0.5134408 | 0.43833 | 0.2461871 | 0.2338621 |
| $\widehat{ATCE}_{0.95}^{H}$ | 4.735155 | 4.329422 | 3.082136 | 2.931111 |
| abias | 0.09433061 | 0.03141699 | 0.01304396 | 0.01029647 |
| RMSE | 0.4840349 | 0.3629902 | 0.223165 | 0.2083294 |
| $lb$ | 3.02836 | 2.76905 | 1.704176 | 1.630524 |
| $ub$ | 7.414861 | 6.777707 | 5.593159 | 5.287705 |
| length | 4.386502 | 4.008657 | 3.888983 | 3.657181 |

## 5. Conclusion

The question of assessing insurance risk is a crucial one. Different risk measures have been proposed in the literature.

Once these risk measures have been defined, the question arises of knowing how to give an analytical expression or to propose estimators of these quantities.

The actuarial context requires the study of the tail of the distributions. Indeed, by nature, a risk which has a significant probability of causing large losses is dangerous and insurance companies need to assess this level of dangerousness.

In this work we have proposed an estimator of the adjusted tail conditional expectation measure for extreme risks under dependence. The behavior of the resulting estimator has been analyzed using numerical simulations and real data.

Many distortion functions depend on a risk-aversion parameter $\rho$, which in turn determines the risk-adjusted premium. Theory is often unable to identify this parameter. It is important to consider the estimation of the risk-aversion parameter $\rho$, and test the equality of the risk measures.

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