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EXISTENCE OF SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR THE FOUR VELOCITY BROADWELL MODEL

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ABSTRACT. Existence and boundedness is proved for the solutions of a boundary value problem for the two-dimensional Broadwell model in a rectangle. The influence of the orientation of the model in relation to the sides of the rectangle on the form of the boundary value problem is analysed. Exact solutions are found and use to determine accommodation coefficients at the boundaries of a fluid flow in a rectangular box.

1. INTRODUCTION

The rapid development of industrial applications of microfluidics in recent years leads to the introduction of flows in micro-channels and micro-tubes in various fields of technology. The flows in these micro devices are in slip or transitional regimes and rarefied gas flow phenomena such as velocity slip and temperature jump are observed. Theoretical studies of such kind of flow are thus done in the scope of the kinetic theory of gases. Due to the complexity of the Boltzmann equation, simplified approximating models such as the discrete velocity models [3] have been proposed. The plane four velocity discrete model of Broadwell seems

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to be the simplest of them and has been used to study initial and boundary value problems [7] and flow problems in one dimension [11]. However the modelling of flows in micro-channels or micro-tubes deserves, in order to take into account notably longitudinal and transversal directions, the use of at least two-dimensional models. The first papers on the boundary value problem for the two-dimensional Broadwell model [7], [8] consider the following boundary value problem:

$$\begin{cases} c\dfrac{\partial N_1'}{\partial x'} = -c\dfrac{\partial N_4'}{\partial x'} = Q_0' \\ c\dfrac{\partial N_2'}{\partial y'} = -c\dfrac{\partial N_3'}{\partial y'} = -Q_0' \\ N_1'(0,y') = \phi_1'(y') \\ N_2'(a,y') = \phi_2'(y') \\ N_3'(x',0) = \phi_3'(x') \\ N_4'(x',b) = \phi_4'(x') \end{cases} \tag{1.1}$$

$$Q_0' = 2cs(N_3'N_4' - N_1'N_2')$$

which models in an orthonormal reference $(O, \vec{e}_1, \vec{e}_2)$ of the plane $\mathbb{R}^2$, the flow of a gas in a rectangular box, when the velocities of the discrete velocity model are $\vec{u}_1 = c\vec{e}_1$, $\vec{u}_2 = c\vec{e}_2$, $\vec{u}_3 = -\vec{u}_2$, $\vec{u}_4 = -\vec{u}_1$ and the origin O is chosen so that the edges of the box are located on the lines $x' = 0$, $x' = a$, $y' = 0$ and $y' = b$, $0 < b \leq a$. We denote as usual by $N_i(t', x', y')$ the number density of particles of velocity $\vec{u}_i$ in point $M(t', x', y')$ at time t'. The velocities of the general four velocity discrete model of Broadwell in the basis $(\vec{e}_1, \vec{e}_2)$ of reference are in fact $\vec{u}_1 = c(\cos\theta, \sin\theta)$, $\vec{u}_2 = c(-\sin\theta, \cos\theta)$, $\vec{u}_3 = -\vec{u}_2$, $\vec{u}_4 = -\vec{u}_1$, where the angle θ between the velocity $\vec{u}_1$ and the unit vector $\vec{e}_1$ accounts of the orientation of the discrete velocity model with respect to the reference. When $\theta \in \left\{0, \dfrac{\pi}{2}, \dfrac{\pi}{4}\right\}$ the model is isotropic with respect to the reference in contrast with the cases where $\theta \notin \left\{0, \dfrac{\pi}{2}, \dfrac{\pi}{4}\right\}$. The boundary value problem has the form (1.1) if and only if $\theta \in \left\{0, \dfrac{\pi}{2}\right\}$. For $\theta \in \left]0, \dfrac{\pi}{2}\right[$, $\cos\theta$ and $\sin\theta$ are non zero and the form of the kinetic equations changes.

The aim of this paper is to investigate the boundary value problem for the two-dimensional Broadwell model for $\theta = \dfrac{\pi}{4}$ and compare the results with those found in [7] and [8].

The paper is organized as follows. In section 2 we set the boundary value problem of the Broadwell model for $\theta = \frac{\pi}{4}$ and present the main result of the paper which is proved in section 3. We seek in section 4 exact solutions and compare them to those of [8]. In section 5, an application to the determination of accommodation coefficients is performed for a gas flow in a box.

2. STATEMENT OF THE PROBLEM

The components of the velocities of the model for $\theta = \frac{\pi}{4}$ are: $\vec{u}_1 = \frac{c}{\sqrt{2}}(1;1)$, $\vec{u}_2 = \frac{c}{\sqrt{2}}(-1;1)$, $\vec{u}_3 = \frac{c}{\sqrt{2}}(1;-1)$, $\vec{u}_4 = \frac{c}{\sqrt{2}}(-1;-1)$. The kinetic equations of the Broadwell model in consideration are given by:

$$
\text{(2.1)} \qquad \left\{ \begin{array}{rcl}
\dfrac{\partial N_1'}{\partial t'} + \dfrac{c}{\sqrt{2}}\dfrac{\partial N_1'}{\partial x'} + \dfrac{c}{\sqrt{2}}\dfrac{\partial N_1'}{\partial y'} & = & Q' \\
\dfrac{\partial N_2'}{\partial t'} - \dfrac{c}{\sqrt{2}}\dfrac{\partial N_2'}{\partial x'} + \dfrac{c}{\sqrt{2}}\dfrac{\partial N_2'}{\partial y'} & = & -Q' \\
\dfrac{\partial N_3'}{\partial t'} + \dfrac{c}{\sqrt{2}}\dfrac{\partial N_3'}{\partial x'} - \dfrac{c}{\sqrt{2}}\dfrac{\partial N_3'}{\partial y'} & = & -Q' \\
\dfrac{\partial N_4'}{\partial t'} - \dfrac{c}{\sqrt{2}}\dfrac{\partial N_4'}{\partial x'} - \dfrac{c}{\sqrt{2}}\dfrac{\partial N_4'}{\partial y'} & = & Q'
\end{array} \right.
$$

with

$$
Q' = 2cs\left(N_2'N_3' - N_1'N_4'\right).
$$

We consider gas flow described by this model in a rectangular box of lenght a and width b $(0 < b \leq a)$. Arranging as usual the velocities of the model into three groups corresponding to emerging, grazing and impinging particles in relation with each edge of box [11], we derive the following boundary conditions:

$$
\text{(2.2)} \qquad \left\{ \begin{array}{rcl}
N_1'(0,y') & = & \phi_1'(y') \\
N_2'(a,y') & = & \phi_2'(y') \\
N_3'(0,y') & = & \phi_3'(y') \\
N_4'(a,y') & = & \phi_4'(y') \\
N_1'(x',0) & = & \phi_5'(x') \\
N_2'(x',0) & = & \phi_6'(x') \\
N_3'(x',b) & = & \phi_7'(x') \\
N_4'(x',b) & = & \phi_8'(x')
\end{array} \right.
$$

The boundary value problem is the system (2.1) with the boundary conditions (2.2). The boundary conditions (2.2) are twice as numerous as the unknowns. The problem (2.1)-(2.2), is an overdetermined two point boundary value problem. It is generally necessary to have compatibility conditions between the boundary data for a solution to exist.

Obviously, we can see that the change in the orientation of the model in relation with the reference results in a change in the form of the kinetic equations and the boundary conditions. The boundary value problems for $\theta = 0$ and $\theta = \frac{\pi}{2}$ are totally different from the one for $\theta = \frac{\pi}{4}$.

2.1. **The non dimensional problem.** The problem is put in dimensionless form. The chosen reference values are: c for the velocity, n_0 for the densities, a and b for the lenght. We thus introduce the following non dimensional quantities:

$$N_i = N_i'/n_0 \quad i = 1,2,3,4 \quad , \quad x = x'/a \quad , \quad y = y'/b,$$
$$\varepsilon = b/a \quad , \quad Kn = (sn_0a)^{-1} \quad , \quad \phi_j = \phi_j'/n_0 \; j = 1,\ldots,8.$$

the Knudsen number Kn provides information on the degree of rarefaction of the flow while ε which is the channel aspect ratio provides information on the relative length. The boundary value problem takes the form:

$$\left\{\begin{array}{lcl}
\dfrac{\partial N_1}{\partial x} + \dfrac{1}{\varepsilon}\dfrac{\partial N_1}{\partial y} = -\dfrac{\partial N_4}{\partial x} - \dfrac{1}{\varepsilon}\dfrac{\partial N_4}{\partial y} & = & Q \\
-\dfrac{\partial N_2}{\partial x} + \dfrac{1}{\varepsilon}\dfrac{\partial N_2}{\partial y} = \dfrac{\partial N_3}{\partial x} - \dfrac{1}{\varepsilon}\dfrac{\partial N_3}{\partial y} & = & -Q \\
N_1(0,y) & = & \phi_1(y) \\
N_2(1,y) & = & \phi_2(y) \\
N_3(0,y) & = & \phi_3(y) \\
N_4(1,y) & = & \phi_4(y) \\
N_1(x,0) & = & \phi_5(x) \\
N_2(x,0) & = & \phi_6(x) \\
N_3(x,1) & = & \phi_7(x) \\
N_4(x,1) & = & \phi_8(x)
\end{array}\right. \tag{2.3}$$

with

$$Q = 2\sqrt{2}\left(N_2N_3 - N_1N_4\right)/Kn.$$

We prove in the sequel the following result:

Theorem 2.1. *The problem* (2.3) *has continuous, derivable and bounded solution if the boundary data* ϕ_i , $i = 1, \ldots, 8$ *and their first derivatives are continuous and bounded.*

In order to simplify the form of (2.3), we perform the following change of variables.

$$\mathscr{L} : (x, y) \mapsto (\alpha_1, \alpha_2) \text{ such that } \alpha_1 = \frac{x + \varepsilon y}{\sqrt{2}} \ , \ \alpha_2 = \frac{-x + \varepsilon y}{\sqrt{2}}.$$

$\mathscr{L}$ is an isomorphism of $[0, 1] \times [0, 1]$ on $\left[0, \frac{1+\varepsilon}{\sqrt{2}}\right] \times \left[\frac{-1}{\sqrt{2}}, \frac{\varepsilon}{\sqrt{2}}\right]$. The α_j, $j = 1, 2$ are the new variables and x, y are the old ones. The vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, $D = (0, 1)$ of the rectangle in which the flow takes place are transformed into $A' = (0, 0)$, $B' = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $C' = \left(\frac{1+\varepsilon}{\sqrt{2}}, \frac{-1+\varepsilon}{\sqrt{2}}\right)$, $D' = \left(\frac{\varepsilon}{\sqrt{2}}, \frac{\varepsilon}{\sqrt{2}}\right)$ and the rectangle $ABCD$ is transformed into the losenge $A'B'C'D'$ by the transformation $\mathscr{L}$. The vectors $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$ associated with the initial coordinate system are transformed into $\vec{u}_1 = (1, 1)$ and $\vec{u}_2 = (-1, 1)$. The lines $x = 0$, $x = 1$, $y = 0$ and $y = 1$ are respectively transformed into $\alpha_1 - \alpha_2 = 0$, $\alpha_1 - \alpha_2 = \sqrt{2}$, $\alpha_1 + \alpha_2 = 0$, $\alpha_1 + \alpha_2 = \varepsilon\sqrt{2}$. In the new coordinate system the velocities of the model are normal to the vertices of the losenge and the boundary value problem (2.3) takes the following form:

$$\text{(2.4)} \qquad \begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = -\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} & = \widetilde{Q} \\ -\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} = \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} & = \widetilde{Q} \\ \widetilde{N}_1 \, (0, \alpha_2) & = \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2 \, (\alpha_1, 0) & = \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3 \, (\alpha_1, 1) & = \widetilde{\phi}_3(\alpha_1) \\ \widetilde{N}_4 \, (1, \alpha_2) & = \widetilde{\phi}_4(\alpha_2) \end{cases}$$

with

$$\widetilde{Q} = 2\left(\widetilde{N}_2 \widetilde{N}_3 - \widetilde{N}_1 \widetilde{N}_4\right)/Kn.$$

Although $\mathscr{L}$ is an isomorphism, the change of variable completely modifies the structure of the boundary conditions of the problem (2.3). Hence to ensure that a solution of the problem (2.4) is a solution of (2.3) we must choose the data $\widetilde{\phi}_i$, $i = 1, \ldots, 4$ conveniently so that it satisfies in the old variables the boundary conditions of the problem (2.3).

3. Resolution of the problem (2.4)

We put $J = \left[0, \frac{1+\varepsilon}{\sqrt{2}}\right] \times \left[\frac{-1}{\sqrt{2}}, \frac{\varepsilon}{\sqrt{2}}\right]$ and denote respectively by $\mathscr{C}(J)$ and $\mathscr{C}_+(J)$ the set of continuous functions defined on J and its subset of positive functions. We defined the following norms.

If $\alpha = (\alpha_1, \alpha_2) \in J$ and $M = (M_1, \ldots, M_4) \in \mathscr{C}(J)^4$ then

$$\|\alpha\| = |\alpha_1| + |\alpha_2| \ , \ \|M_i\|_0 = \sup_{\alpha \in J} |M_i(\alpha)| \ , \ \|M\|_1 = \sup_{i \in \Lambda} |M_i|_0 \text{ with } \Lambda = \{1, 2, 3, 4\}.$$

We denote $|M| = (|M_1|, \ldots, |M_4|)$.

3.1. **Positivity of the solutions of problem** (2.4).

Proposition 3.1. *The solution* $(\widetilde{N}_1, \ldots, \widetilde{N}_4)$ *of the problem* (2.4) *when it exists, belongs to* $\mathscr{C}(J)^4_+$.

Proof. Let

$$\begin{aligned}
\overline{N}_1(\alpha_1, \alpha_2) &= \exp\left[\int_0^{\alpha_1} \rho(\widetilde{N})(\alpha_1, s)ds\right] \widetilde{N}_1(\alpha_1, \alpha_2) \\
\overline{N}_2(\alpha_1, \alpha_2) &= exp\left[\int_0^{\alpha_2} \rho(\widetilde{N})(s, \alpha_2)ds\right] \widetilde{N}_2(\alpha_1, \alpha_2) \\
\overline{N}_3(\alpha_1, \alpha_2) &= exp\left[\int_1^{\alpha_2} \rho(\widetilde{N})(s, \alpha_2)ds\right] \widetilde{N}_3(\alpha_1, \alpha_2) \\
\overline{N}_4(\alpha_1, \alpha_2) &= exp\left[\int_1^{\alpha_1} \rho(\widetilde{N})(\alpha_1, s)ds\right] \widetilde{N}_4(\alpha_1, \alpha_2) \\
\sigma_0 &= 2/Kn.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial \overline{N}_1}{\partial \alpha_1} &= exp\left[\int_0^{\alpha_1} \rho(\widetilde{N})(s, \alpha_2)ds\right] \sigma_0 \widetilde{N}_2 \widetilde{N}_3 + \overline{N}_1 \left[\rho(\widetilde{N}) - \sigma_0 \widetilde{N}_4\right] \\
\frac{\partial \overline{N}_4}{\partial \alpha_1} &= -\exp\left[\int_1^{\alpha_1} \rho(\widetilde{N})(s, \alpha_2)ds\right] \sigma_0 \widetilde{N}_2 \widetilde{N}_3 + \overline{N}_4 \left[\rho(\widetilde{N}) + \sigma_0 \widetilde{N}_1\right] \\
\frac{\partial \overline{N}_2}{\partial \alpha_2} &= exp\left[\int_0^{\alpha_2} \rho(\widetilde{N})(\alpha_1, s)ds\right] \sigma_0 \widetilde{N}_1 \widetilde{N}_4 + \overline{N}_2 \left[\rho(\widetilde{N}) - \sigma_0 \widetilde{N}_3\right] \\
\frac{\partial \overline{N}_3}{\partial \alpha_2} &= -exp\left[\int_1^{\alpha_2} \rho(\widetilde{N})(\alpha_1, s)ds\right] \sigma_0 \widetilde{N}_1 \widetilde{N}_4 + \overline{N}_3 \left[\rho(\widetilde{N}) + \sigma_0 \widetilde{N}_2\right].
\end{aligned}$$

Putting $F(\widetilde{N}) = \widetilde{N}_1\widetilde{N}_4\sigma_0$ and $G(\widetilde{N}) = \widetilde{N}_2\widetilde{N}_3\sigma_0$ we get

$$\begin{aligned}
\overline{N}_1(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_1(\alpha_2) + \int_0^{\alpha_1} \exp\left[\int_0^s \sigma_0\widetilde{N}_4(a,\alpha_2)da\right] G(\widetilde{N})(s,\alpha_2)ds\right) \times \\
&\quad \exp\left[\int_0^{\alpha_1}\left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_4\right](s,\alpha_2)ds\right] \\
\overline{N}_2(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_2(\alpha_1) + \int_0^{\alpha_2} \exp\left[\int_0^s \sigma_0\widetilde{N}_3(\alpha_1,a)da\right] F(\widetilde{N})(\alpha_1,s)ds\right) \times \\
&\quad \exp\left[\int_0^{\alpha_2}\left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_3\right](\alpha_1,s)ds\right] \\
\overline{N}_3(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_3(\alpha_1) + \int_1^{\alpha_2} \exp\left[\int_1^{\alpha_2} \sigma_0\widetilde{N}_2(\alpha_1,a)da\right] F(\widetilde{N})(\alpha_1,s)ds\right) \times \\
&\quad \exp\left[\int_1^{\alpha_2}\left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_2\right](\alpha_1,s)ds\right] \\
\overline{N}_4(\alpha_1,\alpha_2) &= \left(\widetilde{\phi}_4(\alpha_2) + \int_1^{\alpha_1} \exp\left[\int_1^s \sigma_0\widetilde{N}_1(a,\alpha_2)da\right] G(\widetilde{N})(s,\alpha_2)ds\right) \times \\
&\quad \exp\left[\int_1^{\alpha_1}\left[\rho(\widetilde{N}) - \sigma_0\widetilde{N}_1\right](s,\alpha_2)ds\right].
\end{aligned}$$

As $\widetilde{\phi}_i$, $i \in \Lambda$ are positive then $\overline{N}_k$, $k \in \Lambda$ are positive and so are $\widetilde{N}_k$, $k \in \Lambda$. Hence if a solution of (2.4) exists then it is positive. □

3.2. **Definition of an auxiliary problem.** We put $\rho^+(\widetilde{N}) = \widetilde{N}_1 + \widetilde{N}_4$ and $\rho^-(\widetilde{N}) = \widetilde{N}_2 + \widetilde{N}_3$ and consider for $\sigma > 0$ the following problem:

$$\left\{\begin{aligned}
&\frac{\partial\widetilde{N}_1}{\partial\alpha_1} + \sigma\widetilde{N}_1\rho^+(\widetilde{N}) = \widetilde{Q} + \sigma\widetilde{N}_1\rho^+(\widetilde{N}) = Q_1^\sigma(\widetilde{N}) \\
&\frac{\partial\widetilde{N}_2}{\partial\alpha_2} + \sigma\widetilde{N}_2\rho^-(\widetilde{N}) = -\widetilde{Q} + \sigma\widetilde{N}_2\rho^-(\widetilde{N}) = Q_2^\sigma(\widetilde{N}) \\
&\frac{\partial\widetilde{N}_3}{\partial\alpha_2} + \sigma\widetilde{N}_3\rho^-(\widetilde{N}) = \widetilde{Q} + \sigma\widetilde{N}_3\rho^-(\widetilde{N}) = Q_3^\sigma(\widetilde{N}) \\
&\frac{\partial\widetilde{N}_4}{\partial\alpha_1} + \sigma\widetilde{N}_4\rho^+(\widetilde{N}) = -\widetilde{Q} + \sigma\widetilde{N}_4\rho^+(\widetilde{N}) = Q_4^\sigma(\widetilde{N}) \\
&\widetilde{N}_1(0,\alpha_2) = \widetilde{\phi}_1(\alpha_2) \\
&\widetilde{N}_2(\alpha_1,0) = \widetilde{\phi}_2(\alpha_1) \\
&\widetilde{N}_3(\alpha_1,1) = \widetilde{\phi}_3(\alpha_1) \\
&\widetilde{N}_4(1,\alpha_2) = \widetilde{\phi}_4(\alpha_2)
\end{aligned}\right. \tag{3.1}$$

The problem (3.1) is obtained from problem (2.4) by adding $\sigma\widetilde{N}_i\rho^{\pm}(\widetilde{N})$ to the two members of the kinetic equation for $\widetilde{N}_i$, $i \in \Lambda$ so the two systems of equations are equivalent.

3.2.1. *Existence of solutions of* (3.1).

Proposition 3.2. *The problem* (3.1) *has a solution which belongs to* $\mathscr{C}(J)^4_+$ *for sufficiently large* σ.

Proof. Consider for $M \in \mathscr{C}(J)^4$ the following boundary value problem:

$$(3.2)\qquad \begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} + \sigma\widetilde{N}_1\rho^{+}(|M|) & = & Q_1^{\sigma}(|M|) \\ \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \sigma\widetilde{N}_2\rho^{-}(|M|) & = & Q_2^{\sigma}(|M|) \\ \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \sigma\widetilde{N}_3\rho^{-}(|M|) & = & Q_3^{\sigma}(|M|) \\ \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} + \sigma\widetilde{N}_4\rho^{+}(|M|) & = & Q_4^{\sigma}(|M|) \\ \widetilde{N}_1\,(0,\alpha_2) & = & \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2\,(\alpha_1,0) & = & \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3\,(\alpha_1,1) & = & \widetilde{\phi}_3(\alpha_1) \\ \widetilde{N}_4\,(1,\alpha_2) & = & \widetilde{\phi}_4(\alpha_2)) \end{cases}$$

Lemma 3.1. *The problem* (3.2) *has for given* $M \in \mathscr{C}(J)^4_+$ *an unique solution which belongs to* $\mathscr{C}(J)^4_+$ *for sufficiently large* σ.

Proof. The problem (3.2) is a linear problem associated with the problem (3.1) and it is solved by splitting it into the two following boundary value problems:

$$(3.3)\qquad \begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} + \sigma\widetilde{N}_1\rho^{+}(|M|) & = & Q_1^{\sigma}(|M|) \\ \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} + \sigma\widetilde{N}_4\rho^{+}(|M|) & = & Q_4^{\sigma}(|M|) \\ \widetilde{N}_1\,(0,\alpha_2) & = & \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_4\,(1,\alpha_2) & = & \widetilde{\phi}_4(\alpha_2)) \end{cases}$$

$$
\left\{
\begin{array}{lll}
\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \sigma \widetilde{N}_2 \rho^-(|M|) & = & Q_2^\sigma(|M|) \\
\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \sigma \widetilde{N}_3 \rho^-(|M|) & = & Q_3^\sigma(|M|) \\
\widetilde{N}_2\,(\alpha_1, 0) & = & \widetilde{\phi}_2(\alpha_1) \\
\widetilde{N}_3\,(\alpha_1, 1) & = & \widetilde{\phi}_3(\alpha_1)
\end{array}
\right.
\tag{3.4}
$$

The unique solution of (3.2) is:

$$
\begin{array}{lll}
\widetilde{N}_1(\alpha_1, \alpha_2) & = & \widetilde{\phi}_1(\alpha_2) g^+(\alpha_1, \alpha_2) \\
& & + \displaystyle\int_0^{\alpha_1} Q_1^\sigma(|M|)(s, \alpha_2) f^+(\alpha_1 - s, \alpha_2) ds \\
\widetilde{N}_2(\alpha_1, \alpha_2) & = & \widetilde{\phi}_2(\alpha_1) g^-(\alpha_1, \alpha_2) \\
& & + \displaystyle\int_0^{\alpha_2} Q_2^\sigma(|M|)(\alpha_1, s) f^-(\alpha_1, \alpha_2 - s) ds \\
\widetilde{N}_3(\alpha_1, \alpha_2) & = & \widetilde{\phi}_3(\alpha_1) f^-(\alpha_1, \alpha_2 - 1) \\
& & + \displaystyle\int_1^{\alpha_2} Q_3^\sigma(|M|)(\alpha_1, s) f^-(\alpha_1, \alpha_2 - s) ds \\
\widetilde{N}_4(\alpha_1, \alpha_2) & = & \widetilde{\phi}_4(\alpha_2) f^+(\alpha_1 - 1, \alpha_2) \\
& & + \displaystyle\int_1^{\alpha_1} Q_4^\sigma(|M|)(s, \alpha_1) f^+(\alpha_1 - s, \alpha_2) ds
\end{array}
\tag{3.5}
$$

with

$$
\begin{aligned}
g^+(\alpha_1, \alpha_2) &= exp\left[-\sigma \int_{\alpha_2}^{\alpha_1} \rho^+(|M|)(s, \alpha_2) ds\right] \\
g^-(\alpha_1, \alpha_2) &= exp\left[-\sigma \int_{\alpha_1}^{\alpha_2} \rho^-(|M|)(\alpha_1, s) ds\right] \\
f^+(\alpha_1 - s, \alpha_2) &= \frac{g^+(\alpha_1, \alpha_2)}{g^+(s, \alpha_2)} \\
f^-(\alpha_1, \alpha_2 - s) &= \frac{g^-(\alpha_1, \alpha_2)}{g^-(\alpha_1, s)}.
\end{aligned}
$$

For sufficiently large σ, Q^σ is positive $\forall i \in \Lambda$. Hence as $\widetilde{\phi}_i$ is positive $\forall i \in \Lambda$, $\widetilde{N}_i(\alpha_1, \alpha_2) > 0$, $i = 1, 2$, $\forall (\alpha_1, \alpha_2) \in J$ and $\widetilde{N}_i(\alpha_1, \alpha_2) > 0$, $i = 3, 4$, $\forall (\alpha_1, \alpha_2) \in J$ if and only if:

$$
\begin{aligned}
&\int_{\alpha_2}^{1} Q_3^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds \leqslant sup_{(\alpha_1,\alpha_2)\in J}Q_3^\sigma(|M|)\\
&\int_{\alpha_1}^{1} Q_4^\sigma(|M|)(s,\alpha_1)f^+(\alpha_1-s,\alpha_2)ds \leqslant sup_{(\alpha_1,\alpha_2)\in J}Q_4^\sigma(|M|)
\end{aligned}
\tag{3.6}
$$

and it is sufficient that $\widetilde{\phi}_3 > sup_{(\alpha_1,\alpha_2)\in J}Q_3^\sigma(|M|)$ and $\widetilde{\phi}_4 > sup_{(\alpha_1,\alpha_2)\in J}Q_4^\sigma(|M|)$ to have $\widetilde{N} \in \mathscr{C}(J)_+^4$. □

3.2.2. *Compatibility relations.* We derive in this paragraph the conditions that must fulfil the solution (3.5) to satisfy in the old variables the boundary conditions of problem (2.3). We have:

$$
\widetilde{N}_i(\alpha_1,\alpha_2) = \widetilde{N}_i \circ \mathscr{L}(x,y) = N_i(x,y),\ i = 1,\dots,4. \tag{3.7}
$$

and

$$
\begin{aligned}
\widetilde{\phi}_1(\alpha_2) &= \frac{\widetilde{N}_1(\alpha_1,\alpha_2)}{g^+(\alpha_1,\alpha_2)} - \int_0^{\alpha_1} \frac{Q_1^\sigma(|M|)(a,\alpha_2)}{g^+(a,\alpha_2)}da\\
\widetilde{\phi}_2(\alpha_1) &= \frac{\widetilde{N}_2(\alpha_1,\alpha_2)}{g^-(\alpha_1,\alpha_2)} - \int_0^{\alpha_2} \frac{Q_2^\sigma(|M|)(\alpha_1,a)}{g^-(\alpha_1,a)}da\\
\widetilde{\phi}_3(\alpha_1) &= \frac{\widetilde{N}_3(\alpha_1,\alpha_2)}{g^-(\alpha_1,\alpha_2)} - \int_1^{\alpha_2} \frac{Q_3^\sigma(|M|)(\alpha_1,a)}{g^-(\alpha_1,a)}da\\
\widetilde{\phi}_4(\alpha_2) &= \frac{\widetilde{N}_4(\alpha_1,\alpha_2)}{g^+(\alpha_1,\alpha_2)} - \int_1^{\alpha_1} \frac{Q_4^\sigma(|M|)(a,\alpha_2)}{g^+(a,\alpha_2)}da.
\end{aligned}
\tag{3.8}
$$

For two sets A and B we denote by $\mathscr{P}_i$, $i = 1,2$, the first and the second projections; that is the mappings of $A\times B$ onto A and B respectively defined by: $\mathscr{P}_1 : (x,y) \mapsto x$ and $\mathscr{P}_2 : (x,y) \mapsto y$. Then in the old variables the system (3.8) reads as $\mathscr{L}$ and $\mathscr{P}_i$, $i = 1,2$ are linear:

$$
\begin{aligned}
\widetilde{\phi}_1 \circ \mathscr{P}_2 \circ \mathscr{L}(x,y) &= \frac{\widetilde{N}_1 \circ \mathscr{L}(x,y)}{g^+ \circ \mathscr{L}(x,y)} - \int_0^{x} \frac{Q_1^\sigma(|M|)(\mathscr{L}(a,y))}{g^+(\mathscr{L}(a,y))}\mathscr{P}_1 \circ \mathscr{L}(a,y)da\\
\widetilde{\phi}_2 \circ \mathscr{P}_1 \circ \mathscr{L}(x,y) &= \frac{\widetilde{N}_2 \circ \mathscr{L}(x,y)}{g^- \circ \mathscr{L}(x,y)} - \int_0^{y} \frac{Q_2^\sigma(|M|)(\mathscr{L}(x,a))}{g^-(\mathscr{L}(x,a))}\mathscr{P}_2 \circ \mathscr{L}(x,a)da\\
\widetilde{\phi}_3 \circ \mathscr{P}_1 \circ \mathscr{L}(x,y) &= \frac{\widetilde{N}_3 \circ \mathscr{L}(x,y)}{g^- \circ \mathscr{L}(x,y)} - \int_1^{y} \frac{Q_3^\sigma(|M|)(\mathscr{L}(x,a))}{g^-(\mathscr{L}(x,a))}\mathscr{P}_2 \circ \mathscr{L}(x,a)da\\
\widetilde{\phi}_4 \circ \mathscr{P}_2 \circ \mathscr{L}(x,y) &= \frac{\widetilde{N}_4 \circ \mathscr{L}(x,y)}{g^+ \circ \mathscr{L}(x,y)} - \int_1^{x} \frac{Q_4^\sigma(|M|)(\mathscr{L}(a,y))}{g^+(\mathscr{L}(a,y))}\mathscr{P}_1 \circ \mathscr{L}(a,y)da.
\end{aligned}
\tag{3.9}
$$

where:

$$g^{+}(\mathscr{L}(x,y)) = exp\left[-\sigma \int_{y}^{x} \rho^{+}(|M|)(\mathscr{L}(a,y))\mathscr{P}_1 \circ \mathscr{L}(a,y)da\right]$$
$$g^{-}(\mathscr{L}(x,y)) = exp\left[-\sigma \int_{x}^{y} \rho^{-}(|M|)(\mathscr{L}(x,a))\mathscr{P}_2 \circ \mathscr{L}(x,a)da\right].$$

We thus have:

$$\begin{aligned}
\widetilde{\phi}_1 \circ \mathscr{P}_2 \circ \mathscr{L}(0,y) &= \frac{\widetilde{N}_1 \circ \mathscr{L}(0,y)}{g^{+} \circ \mathscr{L}(0,y)} = \frac{N_1(0,y)}{g^{+} \circ \mathscr{L}(0,y)} \\
\widetilde{\phi}_1 \circ \mathscr{P}_2 \circ \mathscr{L}(x,0) &= \frac{\widetilde{N}_1 \circ \mathscr{L}(x,0)}{g^{+} \circ \mathscr{L}(x,0)} - \int_{1}^{x} \frac{Q_1^{\sigma}(|M|)(\mathscr{L}(a,0))}{g^{+}(\mathscr{L}(a,0))}\mathscr{P}_1 \circ \mathscr{L}(a,0)da \\
&= \frac{N_1(x,0)}{g^{+} \circ \mathscr{L}(x,0)} - \int_{1}^{x} \frac{Q_1^{\sigma}(|M|)(\mathscr{L}(a,0))}{g^{+}(\mathscr{L}(a,0))}\mathscr{P}_1 \circ \mathscr{L}(a,0)da \\
\widetilde{\phi}_2 \circ \mathscr{P}_1 \circ \mathscr{L}(1,y) &= \frac{\widetilde{N}_2 \circ \mathscr{L}(1,y)}{g^{-} \circ \mathscr{L}(1,y)} - \int_{0}^{y} \frac{Q_2^{\sigma}(|M|)(\mathscr{L}(1,a))}{g^{-}(\mathscr{L}(1,a))}\mathscr{P}_2 \circ \mathscr{L}(1,a)da \\
&= \frac{N_2(1,y)}{g^{-} \circ \mathscr{L}(1,y)} - \int_{0}^{y} \frac{Q_2^{\sigma}(|M|)(\mathscr{L}(1,a))}{g^{-}(\mathscr{L}(1,a))}\mathscr{P}_2 \circ \mathscr{L}(1,a)da \\
\widetilde{\phi}_2 \circ \mathscr{P}_1 \circ \mathscr{L}(x,0) &= \frac{\widetilde{N}_2 \circ \mathscr{L}(x,0)}{g^{-} \circ \mathscr{L}(x,0)} = \frac{N_2(x,0)}{g^{-} \circ \mathscr{L}(x,0)} \\
\widetilde{\phi}_3 \circ \mathscr{P}_1 \circ \mathscr{L}(0,y) &= \frac{\widetilde{N}_3 \circ \mathscr{L}(0,y)}{g^{-} \circ \mathscr{L}(0,y)} - \int_{1}^{y} \frac{Q_3^{\sigma}(|M|)(\mathscr{L}(0,a))}{g^{-}(\mathscr{L}(0,a))}\mathscr{P}_2 \circ \mathscr{L}(0,a)da \\
&= \frac{N_3(0,y)}{g^{-} \circ \mathscr{L}(0,y)} - \int_{1}^{y} \frac{Q_3^{\sigma}(|M|)(\mathscr{L}(0,a))}{g^{-}(\mathscr{L}(0,a))}\mathscr{P}_2 \circ \mathscr{L}(0,a)da \\
\widetilde{\phi}_4 \circ \mathscr{P}_2 \circ \mathscr{L}(1,y) &= \frac{\widetilde{N}_4 \circ \mathscr{L}(1,y)}{g^{+} \circ \mathscr{L}(1,y)} = \frac{N_4(1,y)}{g^{+} \circ \mathscr{L}(1,y)} \\
\widetilde{\phi}_4 \circ \mathscr{P}_2 \circ \mathscr{L}(x,1) &= \frac{\widetilde{N}_4 \circ \mathscr{L}(x,1)}{g^{+} \circ \mathscr{L}(x,1)} - \int_{1}^{x} \frac{Q_4^{\sigma}(|M|)(\mathscr{L}(a,1))}{g^{+}(\mathscr{L}(a,1))}\mathscr{P}_1 \circ \mathscr{L}(a,1)da \\
&= \frac{N_4(x,1)}{g^{+} \circ \mathscr{L}(x,1)} - \int_{1}^{x} \frac{Q_4^{\sigma}(|M|)(\mathscr{L}(a,1))}{g^{+}(\mathscr{L}(a,1))}\mathscr{P}_1 \circ \mathscr{L}(a,1)da.
\end{aligned}$$

The solution (3.5) of the problem (3.2) satisfies the boundary conditions of (2.3) in the old variables only if:

$$
(3.10)\qquad
\begin{aligned}
\widetilde{\phi}_1(\frac{\varepsilon y}{\sqrt{2}}) &= \frac{\phi_1(y)}{g^+\circ\mathscr{L}(0,y)}\\
\widetilde{\phi}_1(\frac{-x}{\sqrt{2}}) &= \frac{\phi_5(x)}{g^+\circ\mathscr{L}(x,0)} - \int_1^x \frac{Q_1^\sigma(|M|)(\mathscr{L}(a,0))}{g^+(\mathscr{L}(a,0))}\mathscr{P}_1\circ\mathscr{L}(a,0)da\\
\widetilde{\phi}_2(\frac{1+\varepsilon y}{\sqrt{2}}) &= \frac{\phi_2(y)}{g^-\circ\mathscr{L}(1,y)} - \int_0^y \frac{Q_2^\sigma(|M|)(\mathscr{L}(1,a))}{g^-(\mathscr{L}(1,a))}\mathscr{P}_2\circ\mathscr{L}(1,a)da\\
\widetilde{\phi}_2(\frac{x}{\sqrt{2}}) &= \frac{\phi_6(x)}{g^-\circ\mathscr{L}(x,0)}\\
\widetilde{\phi}_3(\frac{\varepsilon y}{\sqrt{2}}) &= \frac{\phi_3(y)}{g^-\circ\mathscr{L}(0,y)} - \int_1^y \frac{Q_3^\sigma(|M|)(\mathscr{L}(0,a))}{g^-(\mathscr{L}(0,a))}\mathscr{P}_2\circ\mathscr{L}(0,a)da\\
\widetilde{\phi}_3(\frac{x+\varepsilon}{\sqrt{2}}) &= \frac{\phi_7(x)}{g^-\circ\mathscr{L}(x,1)}\\
\widetilde{\phi}_4(\frac{-1+\varepsilon y}{\sqrt{2}}) &= \frac{\phi_4(y)}{g^+\circ\mathscr{L}(1,y)}\\
\widetilde{\phi}_4(\frac{-x+\varepsilon}{\sqrt{2}}) &= \frac{\phi_8(x)}{g^+\circ\mathscr{L}(x,1)} - \int_1^x \frac{Q_4^\sigma(|M|)(\mathscr{L}(a,1))}{g^+(\mathscr{L}(a,1))}\mathscr{P}_1\circ\mathscr{L}(a,1)da.
\end{aligned}
$$

The compatibility conditions amount to prescribe the values of the functions $\widetilde{\phi}_i$, $i\in\Lambda$ in terms of the values of the functions ϕ_i, $i\in\Lambda$ in some points of the flow domain.

Let $\mathscr{T}$ be the mapping defined by $\mathscr{T}(M)=\widetilde{N}$ where $\widetilde{N}$ is the unique solution of problem (3.2) satisfying the compatibility conditions (3.10).

Lemma 3.2. *$\mathscr{T}$ is continuous and compact on $\mathscr{C}(J)^4$.*

Proof. We have $\mathscr{T}(M)=\widetilde{N}$ if and only if $\widetilde{N}$ is given by the relations (3.5) from which we deduce:

$$
\begin{aligned}
\left|\widetilde{N}_1(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_1(\alpha_2)\right|\left|g^+(\alpha_1,\alpha_2)\right| + \left|\int_0^{\alpha_1} Q_1^\sigma(|M|)(s,\alpha_2)f^+(\alpha_1-s,\alpha_2)ds\right|\\
\left|\widetilde{N}_2(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_2(\alpha_1)\right|\left|g^-(\alpha_1,\alpha_2)\right| + \left|\int_0^{\alpha_2} Q_2^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds\right|\\
\left|\widetilde{N}_3(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_3(\alpha_1)\right|\left|f^-(\alpha_1,\alpha_2-1)\right| + \left|\int_1^{\alpha_2} Q_3^\sigma(|M|)(\alpha_1,s)f^-(\alpha_1,\alpha_2-s)ds\right|\\
\left|\widetilde{N}_4(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_4(\alpha_2)\right|\left|f^+(\alpha_1-1,\alpha_2)\right| + \left|\int_1^{\alpha_1} Q_4^\sigma(|M|)(s,\alpha_1)f^+(\alpha_1-s,\alpha_2)ds\right|.
\end{aligned}
$$

In one hand using the Generalized Mean Value Theorem, as f^+ and f^- are strictly positive functions, we can find

$$c_1 \in]0, \alpha_1[\ ; \ c_2 \in]0, \alpha_2[\ ; \ c_3 \in]\alpha_2, 1[\ ; \ c_4 \in]\alpha_1, 1[$$

such that

$$\begin{aligned}
\int_0^{\alpha_1} Q_1^\sigma(|M|)(s, \alpha_2) f^+(\alpha_1 - s, \alpha_2) ds &= Q_1^\sigma(|M|)(c_1, \alpha_2) \int_0^{\alpha_1} f^+(\alpha_1 - s, \alpha_2) ds \\
\int_0^{\alpha_2} Q_2^\sigma(|M|)(\alpha_1, s) f^-(\alpha_1, \alpha_2 - s) ds &= Q_2^\sigma(|M|)(\alpha_1, c_2) \int_0^{\alpha_2} f^-(\alpha_1, \alpha_2 - s) ds \\
\int_{\alpha_2}^{1} Q_3^\sigma(|M|)(\alpha_1, s) f^-(\alpha_1, \alpha_2 - s) ds &= Q_3^\sigma(|M|)(\alpha_1, c_3) \int_{\alpha_2}^{1} f^-(\alpha_1, \alpha_2 - s) ds \\
\int_{\alpha_1}^{1} Q_4^\sigma(|M|)(s, \alpha_2) f^+(\alpha_1 - s, \alpha_2) ds &= Q_4^\sigma(|M|)(c_4, \alpha_2) \int_{\alpha_1}^{1} f^+(\alpha_1 - s, \alpha_2) ds.
\end{aligned}$$

In the other hand in accordance with the Mean Value Theorem we can find $d_1 \in]0, \alpha_1[$, $d_2 \in]0, \alpha_2[$, $d_3 \in]\alpha_2, 1[$ and $d_4 \in]\alpha_1, 1[$ such that

$$\begin{aligned}
\int_0^{\alpha_1} f^+(\alpha_1 - s, \alpha_2) ds &= \alpha_1 f^+(\alpha_1 - d_1, \alpha_2) \\
\int_0^{\alpha_2} f^-(\alpha_1, \alpha_2 - s) ds &= \alpha_2 f^-(\alpha_1, \alpha_2 - d_2) \\
\int_{\alpha_2}^{1} f^-(\alpha_1, \alpha_{-} s) ds &= (1 - \alpha_2)\, f^-(\alpha_1, \alpha_2 - d_3) \\
\int_{\alpha_1}^{1} f^+(\alpha_1 - s, \alpha_2) ds &= (1 - \alpha_1)\, f^+(\alpha_1 - d_4, \alpha_2).
\end{aligned}$$

Hence letting:

$$A^+(\alpha_2) = \exp\left(\sigma \int_0^1 \rho^+(|M|)(s - 1, \alpha_2) ds\right)$$

,

$$A^-(\alpha_1) = \exp\left(\sigma \int_0^1 \rho^+(|M|)(\alpha_1, s - 1) ds\right)$$

we get:

$$\begin{aligned}\left|\widetilde{N}_1(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_1(\alpha_2)\right| + |Q_1^\sigma(|M|)(,c_1,\alpha_2)|\\ \left|\widetilde{N}_2(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_2(\alpha_1)\right| + |Q_2^\sigma(|M|)(\alpha_1,c_2)|\\ \left|\widetilde{N}_3(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_3(\alpha_1)\right| \left|A^-(\alpha_1)\right| + |Q_3^\sigma(|M|)(\alpha_1,c_3)|\\ \left|\widetilde{N}_4(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_4(\alpha_2)\right| \left|A^+(\alpha_2)\right| + |Q_4^\sigma(|M|)(c_4,\alpha_2)|\,.\end{aligned}$$

since $|g^\pm(\alpha_1,\alpha_2)| > 1$. From which we infer:

$$\begin{aligned}\|\mathscr{T}(M)\|_1 \leqslant{}& \max\left(\left\|\widetilde{\phi}_1\right\|_0, \left\|\widetilde{\phi}_2\right\|_0, \left\|\widetilde{\phi}_3\right\|_0 \|A^-\|_0, \left\|\widetilde{\phi}_4\right\|_0 \|A^+\|_0\right)\\ &+ \|Q^\sigma(|M|)\|_1\end{aligned} \tag{3.11}$$

Thus $\mathscr{T}$ is continuous and bounded since $\widetilde{\phi}_i$, $i \in \Lambda$ $A^\pm$ and Q^σ are continuous and bounded. Hence if $M \in \mathscr{C}(J)^4$ is bounded then $\widetilde{N} = \mathscr{T}(M)$ is bounded.

Otherwise if $\widetilde{N}$ is the solution of the problem (3.2) then $\forall M \in \mathscr{C}(J)^4$

$$\begin{cases}\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = Q_1^\sigma(|M|) - \sigma \widetilde{N}_1 \rho^+(|M|)\\ \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} = Q_2^\sigma(|M|) - \sigma \widetilde{N}_2 \rho^-(|M|)\\ \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} = Q_3^\sigma(|M|) - \sigma \widetilde{N}_3 \rho^-(|M|)\\ \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} = Q_4^\sigma(|M|) - \sigma \widetilde{N}_4 \rho^+(|M|)\end{cases}.$$

Thus

$$\begin{cases}\left|\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1}\right| \leqslant |Q_1^\sigma(|M|)| + \sigma \left|\widetilde{N}_1 \rho^+(|M|)\right|\\ \left|\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2}\right| \leqslant |Q_2^\sigma(|M|)| + \sigma \left|\widetilde{N}_2 \rho^-(|M|)\right|\\ \left|\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2}\right| \leqslant |Q_3^\sigma(|M|)| + \sigma \left|\widetilde{N}_3 \rho^-(|M|)\right|\\ \left|\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1}\right| \leqslant |Q_4^\sigma(|M|)| + \sigma \left|\widetilde{N}_4 \rho^+(|M|)\right|\end{cases}. \tag{3.12}$$

Thus for bounded $M \in \mathscr{C}(J)^4$ $\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1}$, $\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2}$, $\dfrac{\partial \widetilde{N}_3}{\partial \alpha_2}$ and $\dfrac{\partial \widetilde{N}_4}{\partial \alpha_1}$ are uniformly bounded in J. From the kinetic equations of (3.2), we derive the conservation equations

$$\left\{\begin{array}{lcl} \dfrac{\partial}{\partial \alpha_2}(\widetilde{N}_2 - \widetilde{N}_3) + \dfrac{\partial}{\partial \alpha_1}(\widetilde{N}_1 - \widetilde{N}_4) & = & 0 \\ \dfrac{\partial}{\partial \alpha_2}(\widetilde{N}_2 + \widetilde{N}_3) & = & 0 \\ \dfrac{\partial}{\partial \alpha_1}(\widetilde{N}_1 + \widetilde{N}_4) & = & 0 \end{array}\right. . \tag{3.13}$$

From (3.13) we deduce the system

$$\left\{\begin{array}{l} \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = 0 \\ \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} = 0 \end{array}\right. . \tag{3.14}$$

We differentiate the equations (3.14) with respect to α_2 and get as the $\widetilde{N}_i$, $i \in \Lambda$ are differentiable functions of α_1 and α_2 the system:

$$\left\{\begin{array}{l} \dfrac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2} + \dfrac{\partial^2 \widetilde{N}_1}{\partial \alpha_2 \partial \alpha_1} = 0 \\ \dfrac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2} + \dfrac{\partial^2 \widetilde{N}_4}{\partial \alpha_2 \partial \alpha_1} = 0 \end{array}\right. . \tag{3.15}$$

We integrate the Eqs. (3.15) with respect to α_1 and get:

$$\begin{array}{lcl} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_2} & = & -\displaystyle\int_0^{\alpha_1} \dfrac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(s, \alpha_2) ds + \Theta_1(\alpha_2) \\ \dfrac{\partial \widetilde{N}_4}{\partial \alpha_2} & = & -\displaystyle\int_1^{\alpha_1} \dfrac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(s, \alpha_2) ds + \Theta_4(\alpha_2) . \end{array}$$

Then we integrate with respect to α_2 and we have:

$$\begin{array}{lcl} \widetilde{N}_1(\alpha_1, \alpha_2) & = & -\displaystyle\int_0^{\alpha_2} \int_0^{\alpha_1} \dfrac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(s, t) ds dt + \int_0^{\alpha_2} \Theta_1(t) dt + \chi_1(\alpha_1) \\ \widetilde{N}_4(\alpha_1, \alpha_2) & = & -\displaystyle\int_0^{\alpha_2} \int_1^{\alpha_1} \dfrac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(s, t) ds dt + \int_0^{\alpha_2} \Theta_4(t) dt + \chi_4(\alpha_1) . \end{array}$$

Using the boundary conditions we have the system:

$$\begin{aligned}\widetilde{N}_1(0,\alpha_2) &= \int_0^{\alpha_2} \Theta_1(t)\,dt + \chi_1(0) = \widetilde{\phi}_1(\alpha_2)\\ \widetilde{N}_4(1,\alpha_2) &= \int_0^{\alpha_2} \Theta_4(t)dt + \chi_4(1) = \widetilde{\phi}_4(\alpha_2)\,.\end{aligned}$$

From which we get by differentiation $\Theta_j = \dfrac{d\widetilde{\phi}_j}{d\alpha_2}$, $j = 1, 4$. We thus have:

$$\begin{aligned}\frac{\partial \widetilde{N}_1}{\partial \alpha_2}(\alpha_1,\alpha_2) &= -\int_0^{\alpha_1} \frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(s,\alpha_2)ds + \frac{d\widetilde{\phi}_1}{d\alpha_2}(\alpha_2)\\ \frac{\partial \widetilde{N}_4}{\partial \alpha_2}(\alpha_1,\alpha_2) &= -\int_1^{\alpha_1} \frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(s,\alpha_2)ds + \frac{d\widetilde{\phi}_4}{d\alpha_2}(\alpha_2)\,.\end{aligned}$$

Similarly we obtain:

$$\begin{aligned}\frac{\partial \widetilde{N}_2}{\partial \alpha_1}(\alpha_1,\alpha_2) &= -\int_0^{\alpha_2} \frac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}(\alpha_1,s)ds + \frac{d\widetilde{\phi}_2}{d\alpha_1}(\alpha_1)\\ \frac{\partial \widetilde{N}_3}{\partial \alpha_1}(\alpha_1,\alpha_2) &= -\int_1^{\alpha_2} \frac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,s)ds + \frac{d\widetilde{\phi}_3}{d\alpha_1}(\alpha_1)\,.\end{aligned}$$

Using the expressions (3.5) of $\widetilde{N}_i$, $i \in \Lambda$ we have

$$\begin{aligned}\frac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_1(\alpha_2)\frac{\partial^2 g^+}{\partial \alpha_1^2}(\alpha_1,\alpha_2) + \frac{\partial Q_1^\sigma(|M|)}{\partial \alpha_1}(\alpha_1,\alpha_2)\\ \frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_2(\alpha_1)\frac{\partial^2 g^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2) + \frac{\partial Q_2^\sigma(|M|)}{\partial \alpha_2}(\alpha_1,\alpha_2)\\ \frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_3(\alpha_1)\frac{\partial^2 f^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2-1) + \frac{\partial Q_3^\sigma(|M|)}{\partial \alpha_2}(\alpha_1,\alpha_2)\\ \frac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,\alpha_2) &= \widetilde{\phi}_4(\alpha_1)\frac{\partial^2 f^+}{\partial \alpha_1^2}(\alpha_1-1,\alpha_2) + \frac{\partial Q_4^\sigma(|M|)}{\partial \alpha_1}(\alpha_1,\alpha_2)\,.\end{aligned}$$

As

$$\frac{\partial \rho^+}{\partial \alpha_1} = \frac{\partial \rho^-}{\partial \alpha_2} = 0$$

we have

$$\begin{aligned} \frac{\partial^2 g^+}{\partial \alpha_1^2}(\alpha_1,\alpha_2) &= \sigma^2 \rho^{+2}(|M|) g^+(\alpha_1,\alpha_2) \\ \frac{\partial^2 g^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2) &= \sigma^2 \rho^{-2}(|M|) g^-(\alpha_1,\alpha_2) \\ \frac{\partial^2 f^+}{\partial \alpha_1^2}(\alpha_1-1,\alpha_2) &= \sigma^2 \rho^{+2}(|M|) f^+(\alpha_1-1,\alpha_2) \\ \frac{\partial^2 f^-}{\partial \alpha_2^2}(\alpha_1,\alpha_2-1) &= \sigma^2 \rho^{-2}(|M|) f^-(\alpha_1,\alpha_2-1)\,. \end{aligned}$$

Hence

$$\begin{aligned} \left|\frac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_1(\alpha_2)\right| \sigma^2 \rho^{+2}(|M|)(\alpha_1,\alpha_2) g^+(\alpha_1,\alpha_2) \\ &\quad + \left|\frac{\partial Q_1^\sigma}{\partial \alpha_1}(\alpha_1,\alpha_2)\right| \\ \left|\frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_2(\alpha_1)\right| \sigma^2 \rho^{-2}(|M|)(\alpha_1,\alpha_2) g^-(\alpha_1,\alpha_2) \\ &\quad + \left|\frac{\partial Q_2^\sigma}{\partial \alpha_2}(\alpha_1,\alpha_2)\right| \\ \left|\frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_3(\alpha_1)\right| \sigma^2 \rho^{-2}(|M|)(\alpha_1,\alpha_2) g^-(\alpha_1,\alpha_2) \\ &\quad + \left|\frac{\partial Q_3^\sigma(|M|)}{\partial \alpha_2}(\alpha_1,\alpha_2)\right| \\ \left|\frac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\widetilde{\phi}_4(\alpha_2)\right| \sigma^2 \rho^{+2}(|M|)(\alpha_1,\alpha_2) g^+(\alpha_1,\alpha_2) \\ &\quad + \left|\frac{\partial Q_4^\sigma(|M|)}{\partial \alpha_1}(\alpha_1,\alpha_2)\right| \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \left|\frac{\partial \widetilde{N}_1}{\partial \alpha_2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\frac{\partial^2 \widetilde{N}_2}{\partial \alpha_2^2}\,(\alpha_1,\alpha_2)\right| + \left|\frac{d\widetilde{\phi}_1}{d\alpha_2}(\alpha_2)\right| \\ \left|\frac{\partial \widetilde{N}_2}{\partial \alpha_1}(\alpha_1,\alpha_2)\right| &\leqslant \left|\frac{\partial^2 \widetilde{N}_1}{\partial \alpha_1^2}\,(\alpha_1,\alpha_2)\right| + \left|\frac{d\widetilde{\phi}_2}{d\alpha_1}(\alpha_1)\right| \end{aligned} \tag{3.17}$$

$$\begin{aligned}\left|\frac{\partial \widetilde{N}_3}{\partial \alpha_1}(\alpha_1,\alpha_2)\right| &\leqslant \left|\frac{\partial^2 \widetilde{N}_4}{\partial \alpha_1^2}(\alpha_1,\alpha_2)\right| + \left|\frac{d\widetilde{\phi}_3}{d\alpha_1}(\alpha_1)\right| \\ \left|\frac{\partial \widetilde{N}_4}{\partial \alpha_2}(\alpha_1,\alpha_2)\right| &\leqslant \left|\frac{\partial^2 \widetilde{N}_3}{\partial \alpha_2^2}(\alpha_1,\alpha_2)\right| + \left|\frac{d\widetilde{\phi}_4}{d\alpha_2}(\alpha_2)\right| .\end{aligned}$$

The inequalities (3.16) and (3.17) show that $\left|\frac{\partial \widetilde{N}_1}{\partial \alpha_1}\right|$, $\left|\frac{\partial \widetilde{N}_2}{\partial \alpha_2}\right|$, $\left|\frac{\partial \widetilde{N}_3}{\partial \alpha_2}\right|$ and $\left|\frac{\partial \widetilde{N}_4}{\partial \alpha_1}\right|$ are bounded for bounded $M \in \mathscr{C}(J)^4$ provided that $\left|\frac{d\widetilde{\phi}_1}{d\alpha_2}\right|$, $\left|\frac{d\widetilde{\phi}_2}{d\alpha_1}\right|$, $\left|\frac{d\widetilde{\phi}_3}{d\alpha_1}\right|$ and $\left|\frac{d\widetilde{\phi}_4}{d\alpha_2}\right|$ are bounded.

We thus prove that if $M \in \mathscr{C}(J)^4$ is bounded then $\frac{\partial \widetilde{N}_i}{\partial \alpha_1}$ and $\frac{\partial \widetilde{N}_i}{\partial \alpha_2}$, $i \in \Lambda$ are uniformly bounded if $\frac{d\widetilde{\phi}_j}{d\alpha_2}$ $j = 1, 4$ and $\frac{d\widetilde{\phi}_k}{d\alpha_1}$ $k = 2, 3$ are bounded. Then it exists $\beta > 0$ and $\gamma > 0$ such that $\forall i \in \Lambda$,

$$\left|\frac{\partial \widetilde{N}_i}{\partial \alpha_2}\right| < \beta \quad \text{in } \left[-\frac{1}{\sqrt{2}}, \frac{\varepsilon}{\sqrt{2}}\right]$$

and

$$\left|\frac{\partial \widetilde{N}_i}{\partial \alpha_1}\right| < \gamma \quad \text{in } \left[0, \frac{1+\varepsilon}{\sqrt{2}}\right].$$

Given $\alpha^1 = (\alpha_1^1, \alpha_2^1) \in J$ and $\alpha^2 = (\alpha_1^2, \alpha_2^2) \in J$. We deduce from the Mean Value Theorem, that its exists $\alpha^0 = (\alpha_1^0, \alpha_2^0) \in [\alpha^1, \alpha^2] \subset J$ such that

$$\widetilde{N}_i\left(\alpha^1\right) - \widetilde{N}_i\left(\alpha^2\right) = d\widetilde{N}_i\left(\alpha^0\right)\left(\alpha^1 - \alpha^2\right), \quad i \in \Lambda$$

with

$$\left[\alpha^1, \alpha^2\right] = \left\{\alpha \in \mathbb{R}^2 / \alpha = t\left(\alpha^1 - \alpha^2\right) + \alpha^2,\ t \in [0,1]\right\}$$

and

$$d\widetilde{N}_i\left(\alpha^0\right)(h) = \frac{\partial \widetilde{N}_i}{\partial \alpha_1}\left(\alpha^0\right) h_1 + \frac{\partial \widetilde{N}_i}{\partial \alpha_2}\left(\alpha^0\right) h_2 \quad \forall h = (h_1, h_2) \in \mathbb{R}^2.$$

Hence

$$\begin{aligned}\left|\widetilde{N}_i\left(\alpha^1\right) - \widetilde{N}_i\left(\alpha^2\right)\right| &= \left|d\widetilde{N}_i\left(\alpha^0\right)\left(\alpha^1 - \alpha^2\right)\right| \\ &\leqslant \left\|d\widetilde{N}_i\left(\alpha^0\right)\right\|_0 \left\|\alpha^1 - \alpha^2\right\|\end{aligned}$$

with

$$\left\|d\widetilde{N}_i\left(\alpha^0\right)\right\|_0 = \sup_{\|h\|\leqslant 1} \frac{\left|d\widetilde{N}_i\left(\alpha^0\right)h\right|}{\|h\|} = \sup_{\|h\|\leqslant 1} \frac{\left|\frac{\partial \widetilde{N}_i}{\partial \alpha_1}\left(\alpha^0\right)h_1 + \frac{\partial \widetilde{N}_i}{\partial \alpha_2}\left(\alpha^0\right)h_2\right|}{|h_1|+|h_2|}.$$

But

$$\begin{aligned}\left|\frac{\partial \widetilde{N}_i}{\partial \alpha_1}\left(\alpha^0\right)h_1 + \frac{\partial \widetilde{N}_i}{\partial \alpha_2}\left(\alpha^0\right)h_2\right| &\leqslant \left\|\frac{\partial \widetilde{N}_i}{\partial \alpha_1}\left(\alpha^0\right)\right\|_0 |h_1| + \left\|\frac{\partial \widetilde{N}_i}{\partial \alpha_2}\left(\alpha^0\right)\right\|_0 |h_2| \\ &\leqslant \max\left(\left\|\frac{\partial \widetilde{N}_i}{\partial \alpha_1}\left(\alpha^0\right)\right\|_0, \left\|\frac{\partial \widetilde{N}_i}{\partial \alpha_2}\left(\alpha^0\right)\right\|_0\right)\|h\| \\ &\leqslant \max(\beta,\gamma)\|h\|.\end{aligned}$$

That is $\left\|d\widetilde{N}_i\left(\alpha^0\right)\right\|_0 \leqslant \max(\beta,\gamma)$. Then $\left|\widetilde{N}_i\left(\alpha^1\right) - \widetilde{N}_i\left(\alpha^2\right)\right| \leqslant \max(\beta,\gamma)\,\|\alpha^1-\alpha^2\|$. Let $\varepsilon > 0$, it is sufficient that $\|\alpha^1-\alpha^2\| < \dfrac{\varepsilon}{\max(\beta,\alpha)}$ to have $\left|\widetilde{N}_i\left(\alpha^1\right) - \widetilde{N}_i\left(\alpha^2\right)\right| < \varepsilon$, $\forall\, i \in \wedge$.

We prove that for all solution $\widetilde{N}$ of (3.2),

$$\forall\,\varepsilon > 0\,,\ \exists \eta > 0\,,\ \left\|\alpha^1 - \alpha^2\right\| < \eta \Longrightarrow \left|\widetilde{N}_i\left(\alpha^1\right) - \widetilde{N}_i\left(\alpha^2\right)\right| < \varepsilon \quad \forall\,\alpha^1\,,\ \alpha^2 \in J.$$

The set of the solutions of (3.2) is thus equicontinuous so $\mathscr{T}$ is compact on every bounded subset of $\mathscr{C}_+(J)^4$. □

Lemma 3.3. *Every solution of the equation* $\widetilde{N} = \lambda\mathscr{T}(\widetilde{N})$, $0<\lambda<1$, *is bounded.*

Proof. $\widetilde{N}$ is a solutions of $\widetilde{N} = \lambda\mathscr{T}(\widetilde{N})$ if, and only if,

$$(3.18)\qquad \begin{cases} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} + \sigma \widetilde{N}_1 \rho^+(\widetilde{N}) = \lambda Q_1^\sigma(\widetilde{N}) & (3.18)-(1) \\ \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} + \sigma \widetilde{N}_2 \rho^-(\widetilde{N}) = \lambda Q_2^\sigma(\widetilde{N}) & (3.18)-(2) \\ \dfrac{\partial \widetilde{N}_3}{\partial \alpha_2} + \sigma \widetilde{N}_3 \rho^-(\widetilde{N}) = \lambda Q_3^\sigma(\widetilde{N}) & (3.18)-(3) \\ \dfrac{\partial \widetilde{N}_4}{\partial \alpha_1} + \sigma \widetilde{N}_4 \rho^+(\widetilde{N}) = \lambda Q_4^\sigma(\widetilde{N}) & (3.18)-(4) \\ \widetilde{N}_1\,(0,\alpha_2) = \lambda\widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2\,(\alpha_1,0) = \lambda\widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3\,(\alpha_1,1) = \lambda\widetilde{\phi}_3(\alpha_1) \\ \widetilde{N}_4\,(1,\alpha_2) = \lambda\widetilde{\phi}_4(\alpha_2). \end{cases}.$$

Making the sums (3.18)-(1)+(3.18)-(4) and (3.18)-(2)+(3.18)-(3), we obtain for the determination of the partial macroscopic densities ρ^+ and ρ^- the following system of partial differential equations:

$$(3.19)\qquad \begin{cases} \dfrac{\partial\left[\rho^+(\widetilde{N})\right]}{\partial \alpha_1} + (1-\lambda)\sigma\left[\rho^+(\widetilde{N})\right]^2 = 0 \\ \dfrac{\partial\left[\rho^-(\widetilde{N})\right]}{\partial \alpha_2} + (1-\lambda)\sigma\left[\rho^-(\widetilde{N})\right]^2 = 0 \end{cases}.$$

The unique solution of system (3.19) is obviously

$$(3.20)\qquad \begin{cases} \rho^+(\widetilde{N})(\alpha_1,\alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_1 + h^+(\alpha_2)} \\ \\ \rho^-(\widetilde{N})(\alpha_1,\alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_2 + h^-(\alpha_1)} \end{cases}.$$

The problem (3.18) is a two point boundary value problem and only a part of the data are given at each boundary. Hence we have $\widetilde{N}_1\,(0,\alpha_2)$ on the line $\alpha_1 = 0$, $\widetilde{N}_4\,(1,\alpha_2)$ on the line $\alpha_1 = 1$, $\widetilde{N}_2\,(\alpha_1,0)$ on the line $\alpha_2 = 0$ and $\widetilde{N}_3\,(\alpha_1,1)$ on the line $\alpha_2 = 1$.

We thus introduce the positive functions of α_2, λ_k^+ and the positive functions of de α_1, λ_k^- , $k = 0, 1$ such that

$$(3.21)\qquad \begin{array}{rcl} \widetilde{N}_4\,(0,\alpha_2) &=& \lambda_0^+(\alpha_2)\widetilde{N}_1\,(0,\alpha_2) \\ \widetilde{N}_1\,(1,\alpha_2) &=& \lambda_1^+(\alpha_2)\widetilde{N}_4\,(1,\alpha_2) \\ \widetilde{N}_3\,(\alpha_1,0) &=& \lambda_0^-(\alpha_1)\widetilde{N}_2\,(\alpha_1,0) \\ \widetilde{N}_2\,(\alpha_1,1) &=& \lambda_1^-(\alpha_1)\widetilde{N}_3\,(\alpha_1,1)\,. \end{array}$$

The relations (3.21) which are by no means reflection laws and are obtained just by comparing functions of the same variables at the boundaries of the domain J allow to compute the values of ρ^+ and ρ^- at the boundaries:

$$\begin{array}{rcl} \rho^+(\widetilde{N})\,(0,\alpha_2) &=& \left[1+\lambda_0^+(\alpha_2)\right]\lambda\widetilde{\phi}_1(\alpha_2) \\ \rho^+(\widetilde{N})\,(1,\alpha_2) &=& \left[1+\lambda_1^+(\alpha_2)\right]\lambda\widetilde{\phi}_4(\alpha_2) \\ \rho^-(\widetilde{N})\,(\alpha_1,0) &=& \left[1+\lambda_0^-(\alpha_1)\right]\lambda\widetilde{\phi}_2(\alpha_1) \\ \rho^-(\widetilde{N})\,(\alpha_1,1) &=& \left[1+\lambda_1^-(\alpha_1)\right]\lambda\widetilde{\phi}_3(\alpha_1). \end{array}$$

From which we infer:

$$\begin{array}{l} h^+(\alpha_2) = \dfrac{1}{\left[1+\lambda_0^+(\alpha_2)\right]\lambda\widetilde{\phi}_1(\alpha_2)} = \dfrac{1}{\left[1+\lambda_1^+(\alpha_2)\right]\lambda\widetilde{\phi}_4(\alpha_2)} + \sigma(\lambda-1) \\ h^-(\alpha_1) = \dfrac{1}{\left[1+\lambda_0^-(\alpha_1)\right]\lambda\widetilde{\phi}_2(\alpha_1)} = \dfrac{1}{\left[1+\lambda_1^-(\alpha_1)\right]\lambda\widetilde{\phi}_3(\alpha_1)} + \sigma(\lambda-1). \end{array}$$

Hence

$$(3.22)\qquad \begin{cases} \rho^+(\widetilde{N})(\alpha_1,\alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_1 + \dfrac{1}{\left[1+\lambda_0^+(\alpha_2)\right]\lambda\widetilde{\phi}_1(\alpha_2)}} \\ \rho^-(\widetilde{N})(\alpha_1,\alpha_2) = \dfrac{1}{(1-\lambda)\sigma\alpha_2 + \dfrac{1}{\left[1+\lambda_0^-(\alpha_1)\right]\lambda\widetilde{\phi}_2(\alpha_1)}} \end{cases}.$$

Thus for $0 < \lambda < 1$, ρ^+ and ρ^- are continuous and bounded as $\widetilde{\phi}_i$, $i = 1, 2$ and $\lambda_0^{\pm}$ and so are the number densities $\widetilde{N}_i$, $i \in \Lambda$. □

Remark 3.1. *For* $\lambda = 1$ *the solutions* (3.22) *are not singular. Moreover they satisfy patial conservation equations and depend upon one variable.*

Finally we conclude to the existence of the solutions of problem (3.1) by using the fixed point theorem of Schaefer [12].

Theorem 3.1. *Let T be a continuous and compact mapping of a Banach X into itself, such that the set $\{x \in X\ ,\ x = \lambda T(x)\}$ is bounded $\forall\, \lambda,\ 0 < \lambda < 1$. Then T has a fixed point.*

The $\widetilde{N}_i\,, i \in \Lambda$ which are positive functions of α_1 and α_2 exist, are continuous, bounded and satisfy the compatibility conditions (3.10). Thus the problem (2.3) possesses a solution N continuous and bounded. □

4. Exact solutions

For $\lambda = 1$, ρ^+ and ρ^- are known and we have:

$$\rho^+(\widetilde{N})(\alpha_2) = \left(\widetilde{N}_1 + \widetilde{N}_4\right)(\alpha_2) \quad \text{and} \quad \rho^-(\widetilde{N})(\alpha_1) = \left(\widetilde{N}_2 + \widetilde{N}_3\right)(\alpha_1).$$

Then

$$\left\{\begin{array}{lcl} \widetilde{N}_4(\alpha_1,\alpha_2) & = & \rho^+(\widetilde{N})(\alpha_2) - \widetilde{N}_1(\alpha_1,\alpha_2) \\ \widetilde{N}_3(\alpha_1,\alpha_2) & = & \rho^-(\widetilde{N})(\alpha_1) - \widetilde{N}_2(\alpha_1,\alpha_2) \end{array}\right. , \tag{4.1}$$

and the system (2.4) becomes:

$$\left\{\begin{array}{lcl} \dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} & = & \sigma_0\left[\left(\widetilde{N}_1 - \dfrac{\rho^+}{2}\right)^2 - \left(\widetilde{N}_2 - \dfrac{\rho^-}{2}\right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4}\right] \\ \dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} & = & -\sigma_0\left[\left(\widetilde{N}_1 - \dfrac{\rho^+}{2}\right)^2 - \left(\widetilde{N}_2 - \dfrac{\rho^-}{2}\right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4}\right] \\ \widetilde{N}_4(\alpha_1,\alpha_2) & = & \rho^+(\widetilde{N})(\alpha_2) - \widetilde{N}_1(\alpha_1,\alpha_2) \\ \widetilde{N}_3(\alpha_1,\alpha_2) & = & \rho^-(\widetilde{N})(\alpha_1) - \widetilde{N}_2(\alpha_1,\alpha_2) \\ \widetilde{N}_1\,(0,\alpha_2) & = & \widetilde{\phi}_1(\alpha_2) \\ \widetilde{N}_2\,(\alpha_1,0) & = & \widetilde{\phi}_2(\alpha_1) \\ \widetilde{N}_3\,(\alpha_1,1) & = & \widetilde{\phi}_3(\alpha_1) \\ \widetilde{N}_4\,(1,\alpha_2) & = & \widetilde{\phi}_4(\alpha_2) \end{array}\right. . \tag{4.2}$$

The boundary value problem for the numbers densities $\widetilde{N}_i\,,\ i = 1, 2$ is thus:

$$
(4.3)\quad \begin{cases}
\dfrac{\partial \widetilde{N}_1}{\partial \alpha_1} = -\dfrac{\partial \widetilde{N}_2}{\partial \alpha_2} & = \sigma_0 \left[\left(\widetilde{N}_1 - \dfrac{\rho^+}{2} \right)^2 - \left(\widetilde{N}_2 - \dfrac{\rho^-}{2} \right)^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4} \right] \\
& = Q_1(\widetilde{N}) \\
\widetilde{N}_1\,(0, \alpha_2) & = \widetilde{\phi}_1(\alpha_2) \\
\widetilde{N}_1\,(1, \alpha_2) & = \rho^+(\alpha_2) - \widetilde{\phi}_4(\alpha_2) \\
\widetilde{N}_2\,(\alpha_1, 0) & = \widetilde{\phi}_2(\alpha_1) \\
\widetilde{N}_2\,(\alpha_1, 1) & = \rho^-(\alpha_1) - \widetilde{\phi}_3(\alpha_1)
\end{cases} .
$$

Letting

$$
F_1(\alpha_1, \alpha_2) = \widetilde{N}_1(\alpha_1, \alpha_2) - \frac{\rho^+(\alpha_2)}{2} \quad \text{and} \quad F_2(\alpha_1, \alpha_2) = \widetilde{N}_2(\alpha_1, \alpha_2) - \frac{\rho^-(\alpha_1)}{2}
$$

the system (4.3) take the form:

$$
(4.4)\quad \begin{cases}
\dfrac{\partial F_1}{\partial \alpha_1} = -\dfrac{\partial F_2}{\partial \alpha_2} & = \sigma_0 \left[F_1^2 - F_2^2 + \dfrac{\rho^{-2} - \rho^{+2}}{4} \right] = Q_1(\widetilde{N}) \\
F_1\,(0, \alpha_2) & = \widetilde{\phi}_1(\alpha_2) - \dfrac{\rho^+(\alpha_2)}{2} \\
F_1\,(1, \alpha_2) & = -\widetilde{\phi}_4(\alpha_2) + \dfrac{\rho^+(\alpha_2)}{2} \\
F_2\,(\alpha_1, 0) & = \widetilde{\phi}_2(\alpha_1) - \dfrac{\rho^-(\alpha_1)}{2} \\
F_2\,(\alpha_1, 1) & = -\widetilde{\phi}_3(\alpha_1) + \dfrac{\rho^-(\alpha_1)}{2}
\end{cases} .
$$

The system (4.4) has a simpler form but its exact resolution is complicated. However it permits to find exact solutions of the problem (4.3) in particular cases.

4.1. **The Maxwellian solutions.** An obvious solution of system (4.4) is: $F_1(\alpha_1, \alpha_2) = \dfrac{\rho^+(\alpha_2)}{2}$ and $F_2(\alpha_1, \alpha_2) = \dfrac{\rho^-(\alpha_1)}{2}$ which leads to $\widetilde{\phi}_1(\alpha_2) = \rho^+(\alpha_2)$, $\widetilde{\phi}_2(\alpha_1) = \rho^-(\alpha_1)$, $Q_1(\widetilde{N}) = 0$ and $\widetilde{\phi}_3 = \widetilde{\phi}_4 = 0$. As the number densities are strictly positive this solution is not admissible.

The solution $F_1(\alpha_1, \alpha_2) = \frac{1}{2}\sqrt{\rho^{+2}(\alpha_2) - 4c_1}$, $F_2(\alpha_1, \alpha_2) = \frac{1}{2}\sqrt{\rho^{-2}(\alpha_1) - 4c_1}$ for $c_1 \geqslant 0$ is also a maxwellian solution. Hence:

(4.5)
$$\begin{aligned}
\widetilde{N}_1(\alpha_1, \alpha_2) &= \frac{\rho^+(\alpha_2)}{2} + \frac{1}{2}\sqrt{\rho^{+2}(\alpha_2) - 4c_1} \\
\widetilde{N}_2(\alpha_1, \alpha_2) &= \frac{\rho^-(\alpha_1)}{2} + \frac{1}{2}\sqrt{\rho^{-2}(\alpha_1) - 4c_1} \\
\widetilde{N}_3(\alpha_1, \alpha_2) &= \frac{\rho^-(\alpha_1)}{2} - \frac{1}{2}\sqrt{\rho^{-2}(\alpha_1) - 4c_1} \\
\widetilde{N}_4(\alpha_1, \alpha_2) &= \frac{\rho^+(\alpha_2)}{2} - \frac{1}{2}\sqrt{\rho^{+2}(\alpha_2) - 4c_1}.
\end{aligned}$$

Taking into account the boundary conditions, we get:

(4.6)
$$\begin{aligned}
&\rho^+(\alpha_2) = \widetilde{\phi}_1(\alpha_2) + \widetilde{\phi}_4(\alpha_2) \\
&\rho^-(\alpha_1) = \widetilde{\phi}_2(\alpha_1) + \widetilde{\phi}_3(\alpha_1) \\
&c_1 = \widetilde{\phi}_1(\alpha_2).\widetilde{\phi}_4(\alpha_2) = \widetilde{\phi}_2(\alpha_1).\widetilde{\phi}_3(\alpha_1).
\end{aligned}$$

The validity of the third relation (4.6) imposes the dependence of the boundary data in the form:

(4.7)
$$\left\{\begin{aligned}
\widetilde{\phi}_4(\alpha_2) &= \frac{c_1}{\widetilde{\phi}_1(\alpha_2)} \\
\widetilde{\phi}_3(\alpha_1) &= \frac{c_1}{\widetilde{\phi}_2(\alpha_1)}
\end{aligned}\right. .$$

The Maxwellian solutions are thus:

(4.8)
$$\begin{aligned}
\widetilde{N}_1(\alpha_1, \alpha_2) &= \widetilde{\phi}_1(\alpha_2) \\
\widetilde{N}_2(\alpha_1, \alpha_2) &= \widetilde{\phi}_2(\alpha_1) \\
\widetilde{N}_3(\alpha_1, \alpha_2) &= \frac{c_1}{\widetilde{\phi}_2(\alpha_1)} \\
\widetilde{N}_4(\alpha_1, \alpha_2) &= \frac{c_1}{\widetilde{\phi}_1(\alpha_2)} \\
\widetilde{\phi}_1^2 > c1 \quad &; \quad \widetilde{\phi}_2^2 > c1\,.
\end{aligned}$$

The solutions (4.8) are associated to the macroscopic variables:

$$\begin{aligned}\rho &= \widetilde{\phi}_1+\widetilde{\phi}_2+\frac{c_1}{\widetilde{\phi}_1}+\frac{c_1}{\widetilde{\phi}_2}\\ \rho U &= c\left[\widetilde{\phi}_1-\widetilde{\phi}_4+\frac{c_1}{\widetilde{\phi}_2}-\frac{c_1}{\widetilde{\phi}_1}\right]\\ \rho V &= c\left[\widetilde{\phi}_1+\widetilde{\phi}_2-\frac{c_1}{\widetilde{\phi}_2}-\frac{c_1}{\widetilde{\phi}_1}\right].\end{aligned} \tag{4.9}$$

So they are merely particular expressions of the unique maxwellian solutions of the model associated to the macroscopic variables ρ, U and V defined by:

$$\begin{aligned}\widetilde{N}_{1M} &= \frac{\rho}{4}(1+u)(1+v)\\ \widetilde{N}_{2M} &= \frac{\rho}{4}(1-u)(1+v)\\ \widetilde{N}_{3M} &= \frac{\rho}{4}(1+u)(1-v)\\ \widetilde{N}_{4M} &= \frac{\rho}{4}(1-u)(1-v)\\ u=\frac{U}{c},&\qquad v=\frac{V}{c}.\end{aligned} \tag{4.10}$$

4.2. **Non maxwellian solutions.**

4.2.1. *The mean densities are linear functions of* α_j, $j=1,2$. For $\rho^-(\alpha_2)=\rho_0\alpha_2+\rho_0^-$ and $\rho^+(\alpha_1)=\rho_0\alpha_1+\rho_0^+$ we seek the solutions of in the form:

$$\widetilde{N}_1 = X+\frac{\rho^+}{2},\qquad \widetilde{N}_2 = Y+\frac{\rho^-}{2} \tag{4.11}$$

with

$$X=\frac{\rho^+}{2}+\frac{k(\alpha_1,\alpha_2)}{m(\alpha_1,\alpha_2)},\qquad Y=\frac{\rho^-}{2}+\frac{k(\alpha_1,\alpha_2)}{m(\alpha_1,\alpha_2)}. \tag{4.12}$$

We find after computations the solutions:

$$X=\frac{\rho^+}{2}+\frac{1}{m(\alpha_1,\alpha_2)},\qquad Y=\frac{\rho^-}{2}+\frac{1}{m(\alpha_1,\alpha_2)} \tag{4.13}$$

with

$$m(\alpha_1,\alpha_2)=\lambda\text{exp}\left[\frac{\rho_0}{Kn}\left(\alpha_1-\alpha_2\right)\right)^2-\frac{2(\rho_0^+-\rho_0^-)}{Kn}(\alpha_1-\alpha_2)\right].$$

4.2.2. *The mean densities are constant.* For $\rho^+ = \rho_0^+$, $\rho^- = \rho_0^-$ with $\rho_0^\pm$ constant and $h = \rho_0^+/\rho_0^-$ we have the solution

$$X = \frac{\rho^+}{2} + \frac{k(\alpha_1, \alpha_2)}{m(\alpha_1, \alpha_2)}, \qquad Y = \frac{\rho^-}{2} + \frac{h(\alpha_1, \alpha_2)k(\alpha_1, \alpha_2)}{m(\alpha_1, \alpha_2)} \tag{4.14}$$

with

$$m(\alpha_1, \alpha_2) = \frac{2({\rho^-}^2 - {\rho^+}^2)}{\rho^- Kn} \left(\frac{\alpha_1}{\rho^+} - \frac{\alpha_2}{\rho^-} \right) k(\alpha_1, \alpha_2).$$

Hence

$$\left\{ \begin{array}{lcl} X & = & \dfrac{\rho^+}{2} + \dfrac{\rho^- Kn}{2({\rho^-}^2 - {\rho^+}^2)\left(\dfrac{\alpha_1}{\rho^+} - \dfrac{\alpha_2}{\rho^-} \right)} \\ Y & = & \dfrac{\rho^-}{2} + \dfrac{\rho^+ Kn}{2({\rho^-}^2 - {\rho^+}^2)\left(\dfrac{\alpha_1}{\rho^+} - \dfrac{\alpha_2}{\rho^-} \right)} \end{array} \right. . \tag{4.15}$$

When ρ_0 is zero in the solution of the first case, we have ρ^+ and ρ^- which are constant but the solution of the first case is different from the solution of the second case. Which proves the non-uniqueness of the solution of problem (2.3).

5. Steady flow in box

We investigate in this section the flow of a discrete gas in box in order to compute accommodation coefficients. In this statement of a flow problem, in contrast to the boundary value problem (2.4) in which they are assumed known, the boundary conditions $\widetilde{\phi}_i$ depend upon the accommodation coefficients which describe the interactions between the particles of the gas and those of the boundaries of the flow domain. The accommodation coefficients are unknowns of the problem and classically one has to prescribe reflection laws to get additional relations for their determination which is achieved only when the mathematical problem is solved [8] [9].

The macroscopic variables of the flow are the mean density ρ, the tangential velocity u and the transversal velocity v given by:

$$\begin{aligned} \rho &= \widetilde{N}_1 + \widetilde{N}_2 + \widetilde{N}_3 + \widetilde{N}_4 &&= \rho^+(\alpha_2) + \rho^-(\alpha_1) \\ \rho u &= \widetilde{N}_1 - \widetilde{N}_2 + \widetilde{N}_3 - \widetilde{N}_4 &&= 2\left(\widetilde{N}_1 - \widetilde{N}_2\right) - \rho^+(\alpha_2) + \rho^-(\alpha_1) \\ \rho v &= \widetilde{N}_1 + \widetilde{N}_2 - \widetilde{N}_3 - \widetilde{N}_4 &&= 2\left(\widetilde{N}_1 + \widetilde{N}_2\right) - \rho^+(\alpha_2) - \rho^-(\alpha_1). \end{aligned} \tag{5.1}$$

The maxwellian densities of the model associated with the macroscopic variables ρ, u and v are:

$$\begin{aligned} \widetilde{N}_{1M} &= \frac{\rho}{4}(1+u)(1+v) \\ \widetilde{N}_{2M} &= \frac{\rho}{4}(1-u)(1+v) \\ \widetilde{N}_{3M} &= \frac{\rho}{4}(1+u)(1-v) \\ \widetilde{N}_{4M} &= \frac{\rho}{4}(1-u)(1-v). \end{aligned} \tag{5.2}$$

The microscopic densities of the discrete gas in maxwellian equilibrium with a wall are the maxwellian densities associated with 1, the tangential and transversal velocities of the wall. Assume that the macroscopic velocity of the box is $\overrightarrow{U}_w = (u_w(\alpha_1,\alpha_2), v_w(\alpha_1,\alpha_2))$. The microscopic densities of the gas in Maxwellian equilibrium with the box are:

$$\begin{aligned} \widetilde{N}_{1M} &= \frac{1}{4}(1+u_w)(1+v_w) \\ \widetilde{N}_{2M} &= \frac{1}{4}(1-u_w)(1+v_w) \\ \widetilde{N}_{3M} &= \frac{1}{4}(1+u_w)(1-v_w) \\ \widetilde{N}_{4M} &= \frac{1}{4}(1-u_w)(1-v_w). \end{aligned} \tag{5.3}$$

It is usually assumed, when the exchanges of mass or energy of a gas and its surrounding only result from the collisions of its particles with its boundaries, that only the microscopic densities of the reflected particles are known near the walls [5]. We can compare these densities to those of the fictituous gas in equilibrium with each wall and introduce the functions $l_i(\alpha_2)$, $i = 1,4$ et $l_j(\alpha_1)$, $j = 2,3$ such that:

$$\begin{array}{lrcl} (5.4) & \widetilde{N}_1(0,\alpha_2) & = & \dfrac{l_1(\alpha_2)}{4}\,(1+u_w(0,\alpha_2))\,(1+v_w(0,\alpha_2)) \\ & \widetilde{N}_2(\alpha_1,0) & = & \dfrac{l_2(\alpha_1)}{4}\,(1-u_w(\alpha_1,0))\,(1+v_w(\alpha_1,0)) \\ & \widetilde{N}_3(\alpha_1,1) & = & \dfrac{l_3(\alpha_1)}{4}\,(1+u_w(\alpha_1,1))\,(1-v_w(\alpha_1,1)) \\ & \widetilde{N}_4(1,\alpha_2) & = & \dfrac{l_4(\alpha_2)}{4}\,(1-u_w(1,\alpha_2))\,(1-v_w(1,\alpha_2))\,. \end{array}$$

Using the form (4.8) of the maxwellian solutions we have, taking $c_1 = 1/4$,

$$\begin{array}{lrcl} (5.5) & \widetilde{N}_1(0,\alpha_2) & = & \widetilde{\phi}_1(\alpha_2) \\ & \widetilde{N}_2(\alpha_1,0) & = & \widetilde{\phi}_2(\alpha_1) \\ & \widetilde{N}_3(\alpha_1,1) & = & \dfrac{1}{4\widetilde{\phi}_1(\alpha_1)} \\ & \widetilde{N}_4(1,\alpha_2) & = & \dfrac{1}{4\widetilde{\phi}_1(\alpha_2)}. \end{array}$$

We can thus explicitly determine the functions l_i, $i = 1,4$ et l_j, $j = 2,3$ which are given by:

$$\begin{array}{lrcl} (5.6) & l_1(\alpha_2) & = & \dfrac{4\widetilde{\phi}_1(\alpha_2)}{[1+u_w(0,\alpha_2)]\,[1+v_w(0,\alpha_2)]} \\ & l_2(\alpha_1) & = & \dfrac{4\widetilde{\phi}_2(\alpha_1)}{[1-u_w(\alpha_1,0)]\,[1+v_w(\alpha_1,0)]} \\ & l_3(\alpha_1) & = & \dfrac{1}{\widetilde{\phi}_2(\alpha_1)\,[1+u_w(\alpha_1,1)]\,[1-v_w(\alpha_1,1)]} \\ & l_4(\alpha_2) & = & \dfrac{1}{\widetilde{\phi}_1(\alpha_2)\,[1-u_w(1,\alpha_2)]\,[1-v_w(1,\alpha_2)]}. \end{array}$$

We introduce now reflection laws. We prescribe that particles of opposite velocities are reflected with the same accommodation coefficients. That is:

$$(5.7) \qquad \left\{ \begin{array}{rcll} l_1(\alpha_2) & = & l_4(\alpha_2)\,, & \forall \alpha_2 \in \left[\dfrac{-1}{\sqrt{2}}, \dfrac{\varepsilon}{\sqrt{2}}\right] \\ l_2(\alpha_1) & = & l_3(\alpha_1)\,, & \forall \alpha_1 \in \left[0, \dfrac{1+\varepsilon}{\sqrt{2}}\right]. \end{array} \right.$$

We infer from these additional relations:

$$
\begin{aligned}
\widetilde{\phi}_1(\alpha_2) &= \frac{1}{2}\sqrt{\frac{[1+u_w(0,\alpha_2)]\,[1+v_w(0,\alpha_2)]}{[1-u_w(1,\alpha_2)]\,[1-v_w(1,\alpha_2)]}} \\
\widetilde{\phi}_2(\alpha_1) &= \frac{1}{2}\sqrt{\frac{[1-u_w(\alpha_1,0)]\,[1+v_w(\alpha_1,0)]}{[1+u_w(\alpha_1,1)]\,[1-v_w(\alpha_1,1)]}} \\
l_1(\alpha_2) &= \frac{2}{\sqrt{[1+u_w(0,\alpha_2)]\,[1+v_w(0,\alpha_2)]\,[1-u_w(1,\alpha_2)]\,[1-v_w(1,\alpha_2)]}} \\
l_2(\alpha_1) &= \frac{2}{\sqrt{[1+u_w(\alpha_1,1)]\,[1-v_w(\alpha_1,1)]\,[1-u_w(\alpha_1,0)]\,[1+v_w(\alpha_1,0)]}}.
\end{aligned} \tag{5.8}
$$

The relations (5.8) give the boundary data $\widetilde{\phi}_j$ in terms of the macroscopic variables of the box's walls. In fact the walls do not move freely as we assume in our computations. Thus when we take into account the fact that for a solid box all the walls have the same constant velocity we have:

$$
\begin{aligned}
\widetilde{\phi}_1 &= \frac{1}{2}\sqrt{\frac{[1+u_w]\,[1+v_w]}{[1-u_w]\,[1-v_w]}} \\
\widetilde{\phi}_2 &= \frac{1}{2}\sqrt{\frac{[1-u_w]\,[1+v_w]}{[1+u_w]\,[1-v_w]}} \\
l_1 &= \frac{2}{\sqrt{[1-u_w^2]\,[1-v_w^2]}} \\
l_2 &= \frac{2}{\sqrt{[1-u_w^2]\,[1-v_w^2]}}.
\end{aligned} \tag{5.9}
$$

The accommodation coefficients are equal although the boundary conditions are different in this more realistic case.

6. Conclusion

We show that the boundary value problem for the two dimensional Broadwell four velocity discrete model has bounded solution for $\theta = \frac{\pi}{4}$. Due to the geometry of the model the boundary value problem is overdetermined in the sense that there are more boundary conditions than unknowns. The result is that, unlike the cases $\theta \in \left\{0, \frac{\pi}{2}\right\}$, compatibility relations between the boundary data must be added to the positivity and boundedness assumptions. The solution is not unique in general. Some exact analytic maxwellian and non maxwellian solutions are built and compared to those of the cases $\theta \in \left\{0, \frac{\pi}{2}\right\}$. An application to the determination of the accommodation coefficients on the boundaries of a gas flow in a box is performed. The fact that we have two completely analytic expressions of the maxwellian densities permits to compute exactly the accommodation coefficients.

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