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ON GRADED S-PRIME SUBMODULES OF GRADED MODULES OVER GRADED COMMUTATIVE RINGS

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ABSTRACT. Let G be an abelian group with identity e. Let R be a G-graded commutative ring with identity, M a graded R-module and $S \subseteq h(R)$ a multiplicatively closed subset of R. In this paper, we introduce the concept of graded S-prime submodules of graded modules over graded commutative rings. We investigate some properties of this class of graded submodules and their homogeneous components. Let N be a graded submodule of M such that $(N :_R M) \cap S = \emptyset$. We say that N is *a graded S-prime submodule of M* if there exists $s_g \in S$ and whenever $a_h m_i \in N$, then either $s_g a_h \in (N :_R M)$ or $s_g m_i \in N$ for each $a_h \in h(R)$ and $m_i \in h(M)$.

1. INTRODUCTION AND PRELIMINARIES

The concept of graded prime submodule was introduced by Atani in [13] and studied by many authors, see for example [11–14, 23]. In the literature, there are several different generalization of the notion of graded prime submodule in graded module.

The concept of graded classical prime submodule was introduced by Darani and Motmaen in [15] and studied in [7–10]. The concept of graded 2-absorbing submodule was introduced by Al-Zoubi and Abu-Dawwas in [2] and studied in

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[4–6]. Also, the concept of graded semiprime submodule was introduced and studied by many authors, see for example [3, 17, 19, 25].

Recently, The authors, in [24] studied S-prime submodules as a new generalization of prime submodules.

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, we introduce the concept of graded S-prime submodule as a new generalization of graded prime submodule and investigate several properties of graded S-prime submodule.

Throughout this paper all rings are commutative with identity and all modules are unitary.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [18] and [20–22]for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group with identity e. By a G-graded ring we mean a ring R together with direct sum decomposition (as abelian group) $R = \bigoplus_{\alpha\in G} R_\alpha$ with the property that $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all α, $\beta \in G$. The elements of R_α are called homogeneous of degree α and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{\alpha\in G} R_\alpha$. If $a \in R$, then a can be written uniquely as $\sum_{\alpha\in G} a_\alpha$, where a_α is called a homogeneous component of a in R_α. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. An ideal A of R is said to be a graded ideal if $A = \bigoplus_{\alpha\in G}(A\cap R_\alpha) := \bigoplus_{\alpha\in G} A_\alpha$ (see [22]).

Let R be a G-graded ring and M be an R-module. Then M is called *a G-graded R-module* if there exists a family of additive subgroups $\{M_\alpha\}_{\alpha\in G}$ of M such that $M = \bigoplus_{\alpha\in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha\beta}$ for all $\alpha, \beta \in G$. Also if an element of M belongs to $\cup_{\alpha\in G} M_\alpha = h(M)$, then it is called a homogeneous. Let R be a G-graded ring and M be a graded R-module. A submodule N of M is said to be *a graded submodule of M* if $N = \bigoplus_{\alpha\in G}(N \cap M_\alpha) := \bigoplus_{\alpha\in G} N_\alpha$. In this case, N_α is called the α-component of N (see [22]).

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus\limits_{\alpha\in G} (S^{-1}R)_\alpha$ where $(S^{-1}R)_\alpha = \{r/s : r \in R, s \in S$ and $\alpha = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R

and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fractions $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{\alpha \in G} (S^{-1}M)_\alpha$ where $(S^{-1}M)_\alpha = \{m/s : m \in M, s \in S \textit{ and } \alpha = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \bigcup_{\alpha \in G} (S^{-1}R)_\alpha$ and $h(S^{-1}M) = \bigcup_{\alpha \in G} (S^{-1}M)_\alpha$ (see [22]).

Let R be a G-graded ring, M a graded R-module, K be a graded submodule of M and I a graded ideal of R. Then $(K :_R M)$ is defined as $(K :_R M) = \{a \in R : aM \subseteq K\}$. It is shown in [13] that if K is a graded submodule of M, then $(K :_R M)$ is a graded ideal of R. The graded submodule $\{m \in M : mI \subseteq K\}$ will be denoted by $(K :_M I)$. A proper graded submodule P of M is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $r \in (P :_R M)$ or $m \in P$ (see [13]).

2. Results

Definition 2.1. *Let R be a G-graded ring and M a graded R-module, N a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R with $(N :_R M) \bigcap S = \emptyset$. Then N is said to be a graded S-prime submodule of M if there exists $s_g \in S$ and whenever $a_h m_i \in N$, then either $s_g a_h \in (N :_R M)$ or $s_g m_i \in N$ for each $a_h \in h(R)$ and $m_i \in h(M)$. In particular, a graded ideal I of R is said to be* a graded S-prime ideal of R *if I is a graded S-prime submodule of R-module R.*

Theorem 2.1. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded prime submodule of M and $(N :_R M) \bigcap S = \emptyset$, then N is a graded S-prime submodule of M.*

Proof. It is clear. □

The converse of Theorem 2.1 is not true in general, see the following example.

Example 1. *Let $G = (\mathbb{Z}, +)$ and $R = (\mathbb{Z}, +, .)$. Define*

$$R_g = \left\{ \begin{array}{ll} \mathbb{Z}, & \textit{if} g = 0 \\ 0, & \textit{otherwise} \end{array} \right\}.$$

Then R is a G-graded ring. Let $M = \mathbb{Z} \times \mathbb{Z}_2$. Then M is a G-graded R-module with

$$M_g = \left\{ \begin{array}{cc} \mathbb{Z} \times \{0\} & \text{if } g = 0 \\ \{0\} \times \mathbb{Z}_2 & \text{if } g = 1 \\ \{0\} \times \{0\} & \text{otherwise} \end{array} \right\}.$$

Now, consider a graded submodule $N = \{0\} \times \{0\}$ of M. Then N is not a graded prime submodule of M since $2(0,1) = (0,0) \in N$, $2 \notin (N :_{\mathbb{Z}} M) = \{0\}$ and $(0,1) \notin N$. Let $S = \mathbb{Z} - \{0\} \subseteq \mathbb{Z}_0 \subseteq h(R)$ be a multiplicatively closed subset of R and put $s_g = 2$. Then $S \cap (N :_R M) = \emptyset$. Now, let $r_h \in h(R)$ and $(a_i, b_i) \in h(M)$ with $r_h(a_i, b_i) = (r_h a_i, r_h b_i) \in N = \{0\} \times \{0\}$. Then $r_h a_i = 0$ and $r_h b_i = 0$. If $r_h = 0$, there is nothing to show, so assume that $a_i = 0$. Hence $s_g(a_i, b_i) = (s_g a_i, s_g b_i) = (0,0) \in N$. Thus N is a graded S-prime submodule of M.

Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then, $S^* = \{a_g \in h(R) : \frac{a_g}{1} \text{ is a unit of } S^{-1}R\}$ is a multiplicatively closed subset of R containing S.

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then:*

(i) *If $S_1 \subseteq S_2$ are multiplicatively closed subset of R such that $(N :_R M) \bigcap S_2 = \emptyset$ and N is a graded S_1-prime submodule of M, then N is a graded S_2-prime submodule of M.*

(ii) *N is a graded S-prime submodule of M if and only if N is a graded S^*-prime submodule of M.*

(iii) *If N is a graded S-prime submodule of M, then $S^{-1}N$ is a graded prime submodule of $S^{-1}M$.*

Proof.

(i) It is clear.

(ii) Assume that N is a graded S-prime submodule of M. First, we want to show that $(N :_R M) \bigcap S^* = \emptyset$. Suppose there exists $a_g \in (N :_R M) \bigcap S^*$. Then $\frac{a_g}{1}$ is a unit of $S^{-1}R$, it follows that there exist $b_h \in h(R)$ and $s_i \in S$ such that $\frac{a_g}{1} \frac{b_h}{s_i} = 1$. Hence $t_j s_i = t_j a_g b_h$ for some $t_j \in S$. So $t_j s_i = t_j a_g b_h \in (N :_R M) \bigcap S$, a contradiction. Therefore $(N :_R M) \bigcap S^* = \emptyset$. Now, by (i) we get N is a graded S^*-prime submodule of M. Conversely, assume that N is a graded S^*-prime submodule of M. Let $a_g m_h \in N$ for some $a_g \in h(R)$ and $m_h \in h(M)$. Then there exists $s_i^* \in S^*$ so that either, $s_i^* a_g \in (N :_R M)$ or $s_i^* m_h \in N$. Since $\frac{s_i^*}{1}$ is a unit of $S^{-1}R$, there exist $b_j \in h(R)$ and $s_k, t_l \in S$ such that $t_l s_k = t_l s_i^* b_j$. It follows that either $(t_l s_k)a_g = t_l s_i^* b_j a_g \in (N :_R M)$ or $(t_l s_k)m_h = t_l s_i^* b_j m_h \in N$. Therefore, N is a graded S-prime submodule of M.

(iii) Assume that N is a graded S-prime submodule of M. Let $\frac{a_{g_1}}{s_{h_1}} \in h(S^{-1}R)$ and $\frac{m_{g_2}}{s_{h_2}} \in h(S^{-1}M)$ such that $\frac{a_{g_1}}{s_{h_1}} \frac{m_{g_2}}{s_{h_2}} \in S^{-1}N$. Then, there exists $s_{h_3} \in S$ such that $s_{h_3} a_{g_1} m_{g_2} \in N$. Since N is a graded S-prime submodule of M, then there exists $s_{h_4} \in S$ such that $s_{h_4} s_{h_3} a_{g_1} \in (N :_R M)$ or $s_{h_4} m_{g_2} \in N$. Therefore, $\frac{a_{g_1}}{s_{h_1}} = \frac{s_{h_4} s_{h_3}}{s_{h_4} s_{h_3}} \frac{a_{g_1}}{s_{h_1}} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $\frac{m_{g_2}}{s_{h_2}} = \frac{s_{h_4} m_{g_2}}{s_{h_4} s_{h_2}} \in S^{-1}N$. Therefore, $S^{-1}N$ is a graded prime submodule of $S^{-1}M$. □

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R with $(N :_R M) \bigcap S = \emptyset$. Then the following statements are equivalent:*

(i) *N is a graded S-prime submodule of M.*
(ii) *For every graded ideal I of R and graded submodule K of M with $IK \subseteq N$, then there exists $s_g \in S$ such that either $s_g I \subseteq (N :_R M)$ or $s_g K \subseteq N$.*

Proof.

(i)⇒ (ii) Assume that N is a graded S-prime submodule of M. Let $IK \subseteq N$ for some graded ideal I of R and graded submodule K of M. Since N is a graded S-prime submodule of M, there exists $s_g \in S$ so that $a_h m_i \in N$ implies $s_g a_h \in (N :_R M)$ or $s_g m_i \in N$ for each $a_h \in h(R)$ and $m_i \in h(M)$. Suppose that $s_g K \nsubseteq N$. Then, there exists $b_j \in h(K)$ such that $s_g b_j \notin N$. Since $IK \subseteq N$, for each $i_k \in h(I)$, we have $i_k b_j \in N$. Since N is a graded S-prime submodule of M and $s_g b_j \notin N$, we get $s_g i_k \in (N :_R M)$ for each $i_k \in h(I)$. It follows that $s_g I \subseteq (N :_R M)$.

(ii) ⇒ (i) Let $a_h \in h(R)$ and $m_i \in h(M)$ such that $a_h m_i \in N$. Put $I = Ra_h$ and $K = Rm_i$, then I is a graded ideal of R and K is a graded submodule of M. Therefore, $IK = Ra_h m_i \subseteq N$. By our assumption, there exists $s_g \in S$ such that either $s_g I = Rs_g a_h \subseteq (N :_R M)$ or $s_g K = Rs_g m_i \subseteq N$. Thus, either $s_g a_h \in (N :_R M)$ or $s_g m_i \in N$. Therefore, N is a graded S-prime submodule of M. □

As an immediate consequence of Theorem 2.3 we have the following corollary.

Corollary 2.1. *Let R be a G-graded ring, I a graded ideal of R and $S \subseteq h(R)$ be a multiplicatively closed subset of R with $I \bigcap S = \emptyset$. Then the following statements are equivalent:*

(i) *I is a graded S-prime ideal of R.*

(ii) *For every graded ideals J and L of R with $JL \subseteq I$, then there exists $s_g \in S$ such that either $s_g J \subseteq I$ or $s_g L \subseteq I$.*

Let R be a G-graded ring and M, M' graded R-modules. Let $f : M \longrightarrow M'$ be an R-module homomorphism. Then f is said to be a graded homomorphism if $f(M_g) \subseteq M'_g$ for all $g \in G$ (see [22].)

Theorem 2.4. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be a graded epimorphism. Let $S \subseteq h(R)$ be a multiplicatively closed subset of R.*

(i) *If N' is a graded S-prime submodule of M', then $\varphi^{-1}(N')$ is a graded S-prime submodule of M.*

(ii) *If N is a graded S-prime submodule of M with $Ker\varphi \subseteq N$, then $\varphi(N)$ is a graded S-prime submodule of M'.*

Proof.

(i) Assume that N' is a graded S-prime submodule of M'. First, we want to show that $(\varphi^{-1}(N') :_R M) \bigcap S = \emptyset$. Suppose on the contrary that there exists $s_g \in (\varphi^{-1}(N') :_R M) \bigcap S$. Hence $s_g M \subseteq \varphi^{-1}(N')$, it follows that $s_g \varphi(M) = s_g M' \subseteq N'$, which is a contradiction since $(N' :_R M') \bigcap S = \emptyset$. Now, let $a_h m_i \in \varphi^{-1}(N')$ for some $a_h \in h(R)$ and $m_i \in h(M)$. Hence $\varphi(a_h m_i) = a_h \varphi(m_i) \in N'$. Then there exists $s_g \in S$ such that $s_g a_h M' = \varphi(s_g a_h M) \subseteq N'$ or $s_g \varphi(m_i) = \varphi(s_g m_i) \in N'$ as N' is a graded S-prime submodule of M'. Hence either $s_g a_h M \subseteq \varphi^{-1}(N')$ or $s_g m_i \in \varphi^{-1}(N')$. Therefore $\varphi^{-1}(N')$ is a graded S-prime submodule of M.

(ii) First, we want to show that $(\varphi(N) :_R M') \bigcap S = \emptyset$. Suppose on the contrary that there exists $s_g \in (\varphi(N) :_R M') \bigcap S$. Hence $s_g M' \subseteq \varphi(N)$, it follows that $\varphi(s_g M) = s_g \varphi(M) \subseteq s_g M' \subseteq \varphi(N)$. Which implise, $s_g M \subseteq s_g M + Ker\varphi \subseteq N + Ker\varphi = N$. Thus, $s_g M \subseteq N$ and so, $s_g \in (N :_R M)$, which is a contradiction since $(N :_R M) \bigcap S = \emptyset$. Now, let $a_h m'_i \in \varphi(N)$ for some $a_h \in h(R)$ and $m'_i \in h(M')$. Then, there exists $n_j \in N \bigcap h(M)$ such that $a_h m'_i = \varphi(n_j)$. Since φ is a graded epimorphism and $m'_i \in h(M')$, there exists $m_k \in h(M)$ such that $m'_i = \varphi(m_k)$.

Hence $\varphi(n_j) = a_h m'_i = a_h \varphi(m_k) = \varphi(a_h m_k)$, it follows that $n_j - a_h m_k \in Ker(\varphi) \subseteq N$ and so $a_h m_k \in N$. Then there exists $s_g \in S$ such that $s_g a_h \in (N :_R M)$ or $s_g m_k \in N$ as N is a graded S-prime submodule of M. Since $(N :_R M) \subseteq (\varphi(N) :_R M')$, we get either $s_g a_h \in (\varphi(N) :_R M')$ or $s_g m'_i = s\varphi(m_k) = \varphi(s m_k) \in \varphi(N)$. Therefore, $\varphi(N)$ is a graded S-prime submodule of M'. □

Recall that a graded R-module M is called *a graded multiplication* if for each graded submodule N of M, $N = IM$ for some graded ideal I of R. If N is graded submodule of a graded multiplication module M, then $N = (N :_R M)M$, (see [16]).

Theorem 2.5. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R.*

(i) *If N is a graded S-prime submodule of M, then $(N :_R M)$ is a graded S-prime ideal of R.*
(ii) *If M is a graded multiplication R-module and $(N :_R M)$ is a graded S-prime ideal of R, then $N$is a graded S-prime submodule of M.*

Proof.
(i) Assume that N is a graded S-prime submodule of M. Let $a_g b_h \in (N :_R M)$ for some a_g, $b_h \in h(R)$. Then, $a_g b_h M \subseteq N$ and so $a_g b_h m_i \in N$ for all $m_i \in h(M)$. Then then there exists $s_j \in S$ such that either $s_j a_g \in (N :_R M)$ or $s_j b_h m_i \in N$ as N is a graded S-prime submodule of M. If $s_j a_g \in (N :_R M)$, then we are done. Assume that $s_j a_g \notin (N :_R M)$. Then $s_j b_h m_i \in N$ for all $m_i \in h(M)$. This yields that $s_j b_h M \subseteq N$ and so $s_j b_h \in (N :_R M)$. Therefore, $(N :_R M)$ is a graded S-prime ideal of R.

(ii) Assume that M is a graded multiplication R-module and $(N :_R M)$ is a graded S-prime ideal of R. Let $a_h m_i \in N$ for some $a_h \in h(R)$ and $m_i \in h(M)$. Then, $I = Ra_h$ is a graded ideal of R and $K = Rm_i$ is a graded submodule of M and $IK \subseteq N$. Hence $I(K :_R M)M \subseteq N$ as M is a graded multiplication R-module, so $I(K :_R M) \subseteq (N :_R M)$. Since $(N :_R M)$ is a graded S-prime ideal of R, by Corollary 2.1, there exists $s_g \in S$ such that $s_g I \subseteq (N :_R M)$ or $s_g(K :_R M) \subseteq (N :_R M)$. Thus, either $s_g I \subseteq (N :_R M)$ or $s_g K = s_g(K :_R M)M \subseteq N$. It follows that either $s_g a_h \in (N :_R M)$ or $s_g m_i \in N$. Thus N is a graded S-prime submodule of M. □

Theorem 2.6. *Let R be a G-graded ring, M a graded multiplication R-module, $S \subseteq h(R)$ be a multiplicatively closed subset of R and N a graded S-prime submodule of M. Let L and K be a graded submodules of M with $K \bigcap L \subseteq N$, then either $s_g K \subseteq N$ or $s_g L \subseteq N$ for some $s_g \in S$.*

Proof. Assume that N a graded S-prime submodule of M and $K \bigcap L \subseteq N$. Then, there exists $s_g \in S$ so that $a_h m_i \in N$, implies $s_g a_h \in (N :_R M)$ or $s_g m_i \in N$ for each $a_h \in h(R)$ and $m_i \in h(M)$. Suppose that $s_g L \nsubseteq N$, then there exists $b_j \in h(L)$ such that $s_g b_j \notin N$. Let $c_k \in (K :_R M) \bigcap h(R)$. So, $c_k b_j \in (K :_R M)L \subseteq K \bigcap L \subseteq N$. Since N is a graded S-prime submodule of M and $s_g b_j \notin N$, we get $s_g c_k \in (N :_R M)$. This yield that $s_g(K :_R M) \subseteq (N :_R M)$. Since M is a graded multiplication R-modules, then $s_g K = s_g(K :_R M)M \subseteq (N :_R M)M = N$. □

Lemma 2.1. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded S-prime submodule of M. Then the following statements hold for some $s_g \in S$.*

(i) $(N :_M t_h) \subseteq (N :_M s_g)$ *for all* $t_h \in S$.

(ii) $((N :_R M) :_R t_h) \subseteq ((N :_R M) :_R s_g)$ *for all* $t_h \in S$.

Proof. Assume that N is a graded S-prime submodule of M. Then, there exists $s_g \in S$ so that $a_h m_i \in N$ implies $s_g a_h \in (N :_R M)$ or $s_g m_h \in N$ for each $a_h \in h(R)$ and $m_i \in h(M)$.

(i) Let $m_i \in (N :_M t_h) \bigcap h(M)$ where $t_h \in S$, it follows that $t_h m_i \in N$. Then either $s_g t_h \in (N :_R M)$ or $s_g m_i \in N$ as N is a graded S-prime submodule of M. Since $(N :_R M) \bigcap S = \Phi$, we get $s_g m_i \in N$ and so, $m_i \in (N :_M s_g)$. Therefore, $(N :_M t_h) \subseteq (N :_M s_g)$.

(ii) Let $a_i \in ((N :_R M) :_R t_h) \bigcap h(M)$ where $t_h \in S$, it follows that $a_i t_h \in (N :_R M)$. So $a_i t_h M \subseteq N$. By (i), we get $a_i s_g M \subseteq N$. Hence $a_i s_g \in (N :_R M)$ and so $a_i \in ((N :_R M) :_R s_g)$. Thus $((N :_R M) :_R t_h) \subseteq ((N :_R M) :_R s_g)$. □

Let R be a G-graded ring. A graded R-module M is called a graded finitely generated if $M = \sum_{i=1}^{n} R x_{g_i}$, where $x_{g_i} \in h(M)$ $(1 \leq i \leq n)$ (see [22]).

Theorem 2.7. *Let R be a G-graded ring, M a graded finitely generated R-module, $S \subseteq h(R)$ be a multiplicatively closed subset of R and N a proper graded submodule of M with $(N :_R M) \bigcap S = \emptyset$. Then the following statements are equivalent:*

(i) *N is a graded S-prime submodule of M.*

(ii) *$S^{-1}N$ is a graded prime submodule of $S^{-1}M$ and there exists $s_g \in S$ such that $(N :_M t_h) \subseteq (N :_M s_g)$ for all $t_h \in S$.*

Proof.

(i) $\Rightarrow$ (ii) By Theorem 2.2 (iii) and Lemma 2.1 (i).

(ii) $\Rightarrow$ (i) Assume that $S^{-1}N$ is a graded prime submodule of $S^{-1}M$ and there exists $s_g \in S$ such that $(N :_M t_h) \subseteq (N :_M s_g)$ for all $t_h \in S$. Let $a_i m_j \in N$ for some $a_i \in h(R)$ and $m_j \in h(M)$. Hence $\frac{a_i}{1}\frac{m_j}{1} \in S^{-1}N$. Since $S^{-1}N$ is a graded prime submodule of $S^{-1}M$ and M is a graded finitely generated, we get either $\frac{a_i}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}((N :_R M))$ or $\frac{m_j}{1} \in S^{-1}N$. Thus either $s_{1_k} a_i \in (N :_R M)$ or $s_{2_l} m_j \in N$ for some $s_{1_k}, s_{2_l} \in S$. If $s_{1_k} a_i \in (N :_R M)$, then we get $a_i M \subseteq (N :_R s_{1_k}) \subseteq (N :_M s_g)$ and so, $s_g a_i \in (N :_R M)$. If $s_{2_l} m_j \in N$, then $m_j \in (N :_M s_{2_l}) \subseteq (N :_M s_g)$ and so $s_g m_j \in N$. Therefore, N is a graded S-prime submodule of M. □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module, $S \subseteq h(R)$ be a multiplicatively closed subset of R and N a proper graded submodule of M with $(N :_R M) \bigcap S = \emptyset$. Then N is a graded S-prime submodule of M if and only if $(N :_M s_g)$ is a graded prime submodule of M for some $s_g \in S$.*

Proof. Assume that N is a graded S-prime submodule of M. Then, there exists $s_g \in S$ so that $a_h m_i \in N$ implies that either $a_h s_g \in (N :_R M)$ or $s_g m_i \in N$ for each $a_h \in h(R)$ and $m_i \in h(M)$. Now let $b_h m_i \in (N :_M s_g)$ for some $b_h \in h(R)$ and $m_i \in h(M)$. Hence $(s_g b_h) m_i \in N$. Then either $s_g^2 b_h \in (N :_R M)$ or $s_g m_i \in N$ as N is a graded S-prime submodule of M. If $s_g m_i \in N$, then $m_i \in (N :_M s_g)$. We are done. If $s_g^2 b_h \in (N :_R M)$, then $b_h \in ((N :_R M) :_R s_g^2) \subseteq ((N :_R M) :_R s_g)$ by Lemma 2.1. It follows that $b_h \in ((N :_M s_g) :_R M)$. Hence, $(N :_M s_g)$ is a graded prime submodule of M. Conversely, assume that $(N :_M s_g)$ is a graded prime submodule of M for some $s_g \in S$. Let $a'_h m'_i \in N$ for some $a'_h \in h(R)$ and $m'_i \in h(M)$. It follows that $a'_h m'_i \in (N :_M s_g)$. Then either $a'_h \in ((N :_M s_g) :_R M)$ or $m'_i \in (N :_M s_g)$ as $(N :_M s_g)$ is a graded prime submodule of M. Which implise that, either $a'_h s_g \in (N :_R M)$ or $s_g m'_i \in N$. Therefore, N is a graded S-prime submodule of M. □

REFERENCES

[1] K. AL-ZOUBI, R. ABU-DAWWAS: *On graded quasi-prime submodules*, Kyungpook Math. J. **55**(2) (2015), 259–26.

[2] K. AL-ZOUBI, R. ABU-DAWWAS:*On graded 2-absorbing and weakly graded 2-absorbing submodules*, J. Math. Sci. Adv. Appl., **28** (2014), 45–60.

[3] K. AL-ZOUBI, R. ABU-DAWWAS AND I. AL-AYYOUB: *Graded semiprime submodules and graded semi-radical of graded submodules in graded modules*, Ricerche mat., **66**(2) (2017), 449–455.

[4] K. AL-ZOUBI, I. AL-AYYOUB, M. AL-DOLAT: *On graded 2-absorbing compactly packed modules*, Adv. Stud. Contemp. Math. (Kyungshang), **28**(3) (2018), 479–486.

[5] K. AL-ZOUBI, M. AL-AZAIZEH: *On graded classical 2-absorbing submodules of graded modules over graded commutative rings*, Rend. Istit. Mat. Univ. Trieste, **50** (2018), 37–46.

[6] K. AL-ZOUBI, M. AL-AZAIZEH: *Some properties of graded 2-absorbing and graded weakly 2-absorbing submodules*, J. Nonlinear Sci. Appl., **12**(8) (2019), 503–508.

[7] K. AL-ZOUBI, F. AL-TURMAN: *On graded 2-classical prime submodules of graded modules over graded commutative rings,* Proyecciones Journal of Mathematics, **38**(4) (2019), 781–790.

[8] K. AL-ZOUBI, M. JARADAT, R. ABU-DAWWAS: *On graded classical prime and graded prime submodules*, Bull. Iranian Math. Soc., **41**(1) (2015), 217–225.

[9] K. AL-ZOUBI, M. JARADAT: *The Zariski Topology on the graded classical prime spectrum of a graded module over a graded commutative ring*, Matematički Vesnik, **70**(4) (2018), 303–313.

[10] K. AL-ZOUBI, M. JARADAT: *On the graded classical prime spectrum of a graded module*, Proyecciones Journal of Mathematics, **37**(3) (2018), 519–533.

[11] K. AL-ZOUBI, J. PASEKA: *On Graded Coprimely Packed Modules,* Adv. Stud. Contemp. Math. (Kyungshang), **29**(2) (2019), 271–279.

[12] K. AL-ZOUBI, F. QARQAZ: *An Intersection condition for graded prime submodules in Gr-multiplication modules*, Math. Reports, **20**(3) (2018), 329–336.

[13] S.E. ATANI: *On graded prime submodules*, Chiang Mai J. Sci., **33**(1) (2006), 3–7.

[14] S.E. ATANI, F. FARZALIPOUR: *On graded secondary modules*, Turk. J. Math., **31** (2007), 371–378.

[15] A. Y. DARANI, S. MOTMAEN: *Zariski topology on the spectrum of graded classical prime submodules*, Appl. Gen. Topol., **14**(2) (2013), 159–169.

[16] J. ESCORIZA, B. TORRECILLAS: *Multiplication Objects in Commutative Grothendieck Categories*, Comm. in Algebra, **26**(6) (1998), 1867–1883.

[17] F. FARZALIPOUR, P. GHIASVAND: *On Graded Semiprime Submodules*, Word Acad. Sci. Eng. Technol. **68** (2012), 694–697.

[18] R. HAZRAT: *Graded Rings and Graded Grothendieck Groups*, Cambridge University Press, Cambridge, 2016.

[19] S.C LEE, R. VARMAZYAR: *Semiprime submodules of Graded multiplication modules*, J. Korean Math. Soc. **49**(2) (2012), 435–447.

[20] C. NASTASESCU, F. VAN OYSTAEYEN:*Graded and filtered rings and modules*, Lecture notes in mathematics 758, Berlin-New York: Springer-Verlag, 1982.

[21] C. NASTASESCU, F. VAN OYSTAEYEN: *Graded Ring Theory*, Mathematical Library 28, North Holand, Amsterdam, 1982.

[22] C. NASTASESCU, F. VAN OYSTAEYEN: *Methods of Graded Rings*, LNM 1836. Berlin-Heidelberg: Springer-Verlag, 2004.

[23] K.H. ORAL, U. TEKIR, A.G. AGARGUN: *On Graded prime and primary submodules*, Turk. J. Math., **35** (2011), 159–167.

[24] E. Ş. SEVIM, T. ARABACI, Ü. TEKIR, S. KOC: *On S-prime submodules*, Turk. J. Math. **43**(2) (2019), 1036–1046.

[25] H. A. TAVALLAEE, M. ZOLFAGHARI: *Graded weakly semiprime submodules of graded multiplication modules*, Lobachevskii J. Math. **34**(1) (2013), 61–67.

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