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SOLITONS AND OTHER SOLUTIONS FOR THE (2+1)-DIMENSIONAL HEISENBERG FERROMAGNETIC SPIN CHAIN EQUATION USING IMPROVED MODIFIED EXTENDED TANH-FUNCTION METHOD

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ABSTRACT. This paper studies $(2+1)$-dimensional Heisenberg ferromagnetic spin chain model by using improved modified extended tanh-function method. Various types of solutions are extracted such as bright solitons, singular solitons, dark solitons, singular periodic solutions, Weierstrass elliptic periodic type solutions and exponential function solutions. Moreover, some of the obtained solutions are represented graphically.

1. INTRODUCTION

Nonlinear evolution equations play a major role in a variety of scientific and engineering fields, such as ocean engineering, optical fiber communications, plasma physics and fluid dynamics. The studies of Soliton solutions for non-linear evolution equation attracted many researchers and one can review the articles (see [1-12]). The$(2+1)$-dimensional Heisenberg ferromagnetic spin chain equation has been studied by many authors (see[13-17]).

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The $(2+1)$-dimensional Heisenberg ferromagnetic spin chain (HFSC) model by using improved modified extended tanh-function method has not yet been considered in the literature, and this fact motivates this work. In this paper, we consider the $(2+1)$-dimensional Heisenberg ferromagnetic spin chain in the following form (see[17]):

(1.1)
$$iq_t(x,y,t) + aq_{xx}(x,y,t) + \varrho q_{yy}(x,y,t) + \vartheta q_{xy}(x,y,t) - hq(x,y,t)|q(x,y,t)|^2 = 0,$$

where $a = \rho^4(J + J_2), \varrho = \rho^4(J_1 + J_2), \vartheta = 2\rho^4 J_2, h = 2\rho^4 A.$

The parameter ρ describes the lattice parameter, the bilinear exchange interactions coefficients along X and Y directions are represented by J and J_1 respectively and the neighboring interaction on the diagonal is denoted by J_2 while the uniaxial crystal field anisotropy parameter is denoted by $A.$

In this work, the proposed method gives more and variety types of solutions than other methods. These solutions including bright solitons, singular solitons, dark solitons, singular periodic solutions, Weierstrass elliptic periodic type solutions and exponential function solutions. In the end of the paper, two-dimensional and three-dimensional graphs of some solutions are introduced for knowing the physical interpretation.

2. Improved modified extended tanh-function method

In this section, the improved modified extended tanh-function method is described as follows (see [18-19]).

We Consider the following nonlinear partial differential equation with two independent variables (x, t),

$$F(h, h_t, h_x, h_{xx}, \ldots) = 0, \tag{2.1}$$

where $h = h(x, t)$ is an unknown function, F is a polynomial in h and its various partial derivatives h_t, h_x with respect to t, x respectively, in which the highest order derivatives and nonlinear terms are involved.

Step 1: Using the traveling wave transformation:

$$h(x,t) = H(\xi), \qquad \xi = \kappa(x - vt), \tag{2.2}$$

where κ and v are constant to be determined later.

Then equation (2.1) can be transformed to the following nonlinear ordinary differential equation

$$P(H, \kappa v H', \kappa^2 H'', \ldots) = 0. \tag{2.3}$$

Step 2: We assume that the solution of equation (2.3) can be expressed in the form

$$H(\xi) = \alpha_0 + \sum_{\ell=1}^{N} (\alpha_\ell \psi^\ell + \beta_\ell \psi^{-\ell}), \tag{2.4}$$

where ω satisfies

$$\psi' = \varepsilon \sqrt{g_0 + g_1\psi + g_2\psi^2 + g_3\psi^3 + g_4\psi^4}, \tag{2.5}$$

Where $\varepsilon = \pm 1$. This equation give various kinds of fundamental solutions. From these solutions, more new exact solutions for (2.1) can be obtained.

Step 3: Determine the positive integer number N in (2.4) from balancing the nonlinear term and the highest order linear term in equation (2.3).

Step 4: Substitute the solution (2.4) which satisfies the condition (2.5) into equation (2.3). As a result of this substitution, we get a polynomial of ψ. In this polynomial we gather all terms of same powers and equating them to be zero, we get a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown parameters $\kappa,\ v,\ \alpha_i$ and β_i, (i=1,2,...). Consequently, we obtain the exact solutions of (2.1).

3. Solitons and other solutions to the proposed model

In order to solve the (2 + 1)- dimensional HFSC equation. we consider the traveling wave transformation:

$$q(x, y, t) = H(\xi) e^{i\Re}, \tag{3.1}$$

where $\xi = x + y - \tau t, \Re = -m_1 x - m_2 y - \omega t$. Here, ξ is the traveling wave, $H(\xi)$ is the real amplitude function and $\Re$ is the phase of the envelope. The parameters

m_1 and m_2 represent the wave numbers in the x and y directions respectively, τ is the group velocity of the wave packet and ω is the frequency of the pulse.

By employing transformation equation (3.1) into equation (1.1) and then decomposing the result into real and imaginary parts and simplifying the terms, a pair of relations is obtained. The imaginary part gives

$$\tau = -2aL_1 - \vartheta m_1 - \vartheta m_2 - 2\varrho m_2, \tag{3.2}$$

and the real part gives

$$(\omega - am_1^2 - \vartheta m_1 m_2 - \varrho m_2^2)H - hH^3 + (a + \vartheta + \varrho)H'' = 0. \tag{3.3}$$

Balancing the highest order derivative of the linear term H'' and the nonlinear term H^3, we obtain $N = 1$. Then, the solution of equation (3.3) has the form

$$H(\xi) = \alpha_0 + \alpha_1\psi + \beta_1\psi^{-1}. \tag{3.4}$$

Substituting $H(\xi)$ and its derivatives with equation (2.5) into equation (3.3) and equating all the coefficients of ψ^ℓ, $\ell \in [-3, 3]$ to be zero, Then we obtain a system of algebraic equations. Solving this system using mathematica and consider the various kinds of fundamental solution, we obtain the following cases which leads to different types of wave propagation of our model.

$\psi^{-3}(\xi)$ Coeff.:

$$\frac{1}{2}(4g_0\beta_1(a + \vartheta + \varrho) - 2\beta_1^3 h) = 0,$$

$\psi^{-2}(\xi)$ Coeff.:

$$\frac{1}{2}(3g_1\beta_1(a + \vartheta + \varrho) - 6\alpha_0\beta_1^2 h) = 0,$$

$\psi^{-1}(\xi)$ Coeff.:

$$g_2\beta_1(a + \vartheta + \varrho) - \beta_1\left(am_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega\right) - 3\alpha_0^2\beta_1 h - 3\alpha_1\beta_1^2 h = 0,$$

$\psi^{0}(\xi)$ Coeff.:

$$\frac{1}{2}\Big(g_1\alpha_1(a + \vartheta + \varrho) + g_3\beta_1(a + \vartheta + \varrho) - 2\alpha_0\left(am_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega\right) -12\alpha_1\alpha_0\beta_1 h - 2\alpha_0^3 h\Big) = 0,$$

$\psi^1(\xi)$ Coeff.:

$$\frac{1}{2}\left(2g_2\alpha_1(a+\vartheta+\varrho)-2\alpha_1\left(am_1^2+\vartheta m_2m_1+\varrho m_2^2-w\right)-6\alpha_1^2\beta_1h-6\alpha_1\alpha_0^2h\right)=0,$$

$\psi^2(\xi)$ Coeff.:

$$\frac{1}{2}\left(3g_3\alpha_1(a+\vartheta+\varrho)-6\alpha_0\alpha_1^2h\right)=0,$$

$\psi^3(\xi)$ Coeff.:

$$\frac{1}{2}\left(4g_4\alpha_1(a+\vartheta+\varrho)-2\alpha_1^3h)\right)=0.$$

Solving this system of equations with the help of mathematica program we conclude the following cases:

Case 1: If we set $g_0=g_1=g_3=0.$ then we have

$$g_2=\frac{am_1^2+\vartheta m_2m_1+\varrho m_2^2-w}{a+\vartheta+\varrho},\quad g_4=\frac{\alpha_1^2h}{2(a+\vartheta+\varrho)},\quad \beta_1=0,\quad \alpha_0=0.$$

Then, the corresponding solution of equation (1.1) is

$$q(x,y,t)=\pm\sqrt{\frac{-2(am_1^2+\vartheta m_2m_1+\varrho m_2^2-\omega)}{h}}\times\operatorname{sech}\left[\sqrt{\frac{(am_1^2+\vartheta m_2m_1+\varrho m_2^2-\omega)}{a+\vartheta+\varrho}}(x+y-\tau t)\right]e^{i(-m_1x-m_2y-\omega t)},\tag{3.5}$$

and

$$q(x,y,t)=\pm\sqrt{\frac{-2(am_1^2+\vartheta m_2m_1+\varrho m_2^2-\omega)}{h}}\times\sec\left[\sqrt{-\frac{(am_1^2+\vartheta m_2m_1+\varrho m_2^2-\omega)}{a+\vartheta+\varrho}}(x+y-\tau t)\right]e^{i(-m_1x-m_2y-\omega t)},\tag{3.6}$$

These solutions represent bright soliton and singular periodic solution.

Case 2:

(i) $g_1=g_3=0, g_0=\frac{g_2^2}{4g_4}$. We have

$$\alpha_0=0,\quad g_2=\frac{am_1^2+\vartheta m_2m_1+m_2^2\varrho-w}{a+\vartheta+\varrho},$$

$$g_4 = \frac{a_2^2\left(-m_2 m_1(a+\varrho) + a m_1^2 + m_2^2\varrho - \omega\right)}{2\beta_1^2 h\left(a_2 - m_1 m_2\right)}, \quad \alpha_1 = 0,$$

and

$$\alpha_0 = 0, \quad \beta_1 = 0, \quad g_2 = \frac{a m_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega}{a+\vartheta+\varrho}, \quad g_4 = \frac{\alpha_1^2 h}{2(a+\vartheta+\varrho)}.$$

Then, the corresponding solution of equation (1.1) is

$$\text{(3.7)} \qquad q(x,y,t) = \sqrt{-\frac{a m_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega}{h}} \times \coth\left[\sqrt{\frac{(-a m_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega)}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1 x - m_2 y - \omega t)},$$

$$\text{(3.8)} \qquad q(x,y,t) = \sqrt{\frac{a m_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega}{h}} \times \cot\left[\sqrt{\frac{(a m_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega)}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1 x - m_2 y - \omega t)},$$

and

$$\text{(3.9)} \qquad q(x,y,t) = \sqrt{-\frac{a m_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega}{h}} \times \tanh\left[\sqrt{\frac{(-a m_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega)}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1 x - m_2 y - \omega t)},$$

$$\text{(3.10)} \qquad q(x,y,t) = \sqrt{\frac{a m_1^2 + \vartheta m_2 m_1 + \varrho L_2^2 - \omega}{h}} \times \tan\left[\sqrt{\frac{(a m_1^2 \vartheta m_2 m_1 + \varrho m_2^2 - \omega)}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1 x - m_2 y - \omega t)},$$

These solutions represent singular soliton, dark soliton and singular periodic wave solution.

(ii) $g_1 = g_3 = 0, g_0 = \frac{g_2^2 m^2\left(1-m^2\right)}{g_4\left(2m^2-1\right)^2}$, we have

$$\alpha_0 = 0, \quad \alpha_1 = 0, \quad g_2 = \frac{a m_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega}{a+\vartheta+\varrho},$$

$$g_4 = -\frac{2g_2^2 m^2 (m^2-1)(a+\vartheta+\varrho)}{\beta_1^2 h (1-2m^2)^2}.$$

Then, the corresponding solution of equation (1.1) is

$$q(x,y,t) = \pm\sqrt{-\frac{2(m^2-1)(am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega)}{(1-2m^2)h}} \times \mathrm{nc}\left[\sqrt{\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{(2m^2-1)(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1 x-m_2 y-\omega t)}, \tag{3.11}$$

if we set $m=0$, we have

$$q(x,y,t) = \pm\sqrt{\frac{2(am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega)}{h}} \times \sec\left[\sqrt{-\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1 x-m_2 y-\omega t)}, \tag{3.12}$$

(iii) $g_1 = g_3 = 0, \quad g_0 = \frac{g_2^2(1-m^2)}{g_4(2-m^2)^2}$, we have

$$\alpha_0 = 0, \quad \alpha_1 = 0, \quad g_2 = \frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{a+\vartheta+\varrho},$$

$$g_4 = -\frac{2g_2^2 (m^2-1)(a+\vartheta+\varrho)}{\beta_1^2 h (m^2-2)^2}.$$

Then, the corresponding solution of equation (1.1) is

$$q(x,y,t) = \pm\sqrt{-\frac{2(m^2-1)(am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega)}{m^2(m^2-2)h}} \times \mathrm{nd}\left[\sqrt{\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{(m^2-2)(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1 x-m_2 y-\omega t)}, \tag{3.13}$$

(iv) $g_1 = g_3 = 0, \quad g_0 = \frac{a_2 m^2}{g_4(m^2+1)^2}$. we have

$$\alpha_0 = 0, \quad \alpha_1 = 0, \quad g_2 = \frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{a+\vartheta+\varrho},$$

$$g_4 = -\frac{2a_2^2 m^2 (a+\vartheta+\varrho)}{\beta_1^2 h (m^2+1)^2}.$$

Then, the corresponding solution of equation (1.1) is

$$q(x,y,t) = \pm\sqrt{-\frac{2(am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega)}{(m^2+1)h}} \times \mathrm{ns}\left[\sqrt{-\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{(m^2+1)(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1x-m_2y-\omega t)}, \tag{3.14}$$

if we set $m=0$, we get

$$q(x,y,t) = \pm\sqrt{-\frac{2(am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega)}{h}} \times \csc\left[\sqrt{-\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1x-m_2y-\omega t)}, \tag{3.15}$$

if we set $m=1$, we get

$$q(x,y,t) = \pm\sqrt{-\frac{(am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega)}{h}} \times \coth\left[\sqrt{-\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right] e^{i(-m_1x-m_2y-\omega t)}, \tag{3.16}$$

Case 3: $g_2 = g_4 = 0, \quad g_0 \neq 0, \quad g_1 \neq 0$. we have

$$\alpha_0 = \pm\sqrt{\frac{-am_1^2-\vartheta m_2 m_1-\varrho m_2^2+\omega}{3h}}, \quad \alpha_1 = 0, \quad g_3 = -\frac{4\alpha_0^3 h}{\beta_1(a+\vartheta+\varrho)},$$
$$\beta_1 = \pm\sqrt{\frac{2g_0(a+\vartheta+\varrho)}{h}}, \quad g_1 = \frac{2\alpha_0\beta_1 h}{a+\vartheta+\varrho}.$$

we obtain

$$q(x,y,t) = \pm\left\{\sqrt{\frac{-am_1^2-\vartheta m_2 m_1-\varrho m_2^2+\omega}{3h}} + \frac{\sqrt{\frac{2g_0(a+\vartheta+\varrho)}{h}}}{\wp\left(\frac{\sqrt{g_3}}{2}\xi; -\frac{4g_1}{g_3}, -\frac{4g_0}{g_3}\right)}\right\} \times e^{i(-m_1x-m_2y-\omega t)}. \tag{3.17}$$

This solution represent a Weierstrass elliptic periodic type solution, where g_0, g_1, g_3 are given by (3.20)

Case 4: $g_3 = g_4 = 0, \quad g_0 = \frac{g_1^2}{4g_2}$, we have

$$\alpha_1 = 0,\ \alpha_0 = \pm\sqrt{\frac{-am_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega}{h}},$$
$$\beta_1 = \pm\frac{g_1(a+\vartheta+\varrho)}{2\sqrt{h\left(-am_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega\right)}}$$
$$g_2 = -\frac{2\left(am_1^2 + \vartheta m_2 m_1 + \varrho L_2^2 - \omega\right)}{a+\vartheta+\varrho}.$$

Then, the corresponding solution of equation (1.1) is

$$q(x,y,t) = \pm\sqrt{\frac{-am_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega}{h}} \pm \frac{g_1(a+\vartheta+\varrho)}{2\sqrt{h\left(-am_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega\right)}}$$
$$\left\{\frac{g_1}{\frac{2\left(am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega\right)}{a+\vartheta+\varrho}} + \exp[\pm\sqrt{-\frac{2\left(am_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega\right)}{a+\vartheta+\varrho}}(x+y-\tau t)]\right\}^{-1}$$
$$\times e^{i(-m_1 x - m_2 y - \omega t)}.$$

This solution represent exponential function solution.

Case(5)

(i) $g_0 = g_1 = 0$, we have

$$\alpha_1 = \pm\frac{g_3(a+\vartheta+\varrho)}{2\sqrt{h\left(-am_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega\right)}},$$
$$\alpha_0 = -\sqrt{\frac{-am_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega}{h}}, \quad \beta_1 = 0,$$
$$g_2 = -\frac{2\left(am_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega\right)}{a+\vartheta+\varrho}, \quad g_4 = -\frac{g_3^2(a+\vartheta+\varrho)}{8\left(am_1^2 + \vartheta m_2 m_1 + \varrho m_2^2 - \omega\right)}.$$

Then, the corresponding solution of equation (1.1) is

$$q(x,y,t) = \sqrt{\frac{-am_1^2 - \vartheta m_2 m_1 - \varrho m_2^2 + \omega}{h}} \tag{3.18}$$
$$\times\left\{\frac{\sec^2\left[\sqrt{-\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right]}{\tan\left[\sqrt{-\frac{am_1^2+\vartheta m_2 m_1+\varrho m_2^2-\omega}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right]+1} - 1\right\} e^{i(-m_1 x - m_2 y - \omega t)},$$

and

$$q(x,y,t)=\sqrt{\frac{-am_1^2-\vartheta m_2m_1-\varrho m_2^2+\omega}{h}} \times\left\{\frac{\operatorname{sech}^2\left[\sqrt{-\frac{am_1^2+\vartheta m_2m_1+\varrho m_2^2-\omega}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right]}{\tanh\left[\sqrt{-\frac{am_1^2+\vartheta m_2m_1+\varrho m_2^2-\omega}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right]-1}-1\right\}e^{i(-m_1x-m_2y-\omega t)}, \tag{3.19}$$

(ii) $g_0=g_1=0, g_2=\frac{g_3^2}{4g_4}$, we have

$$\alpha_1=\pm\sqrt{\frac{2g_4(a+\vartheta+\varrho)}{h}},\quad \alpha_0=\pm g_3\sqrt{\frac{a+\vartheta+\varrho}{8g_4h}},\quad \beta_1=0,$$

$$L_2=\frac{-2g_4\vartheta m_1\pm\sqrt{2}\sqrt{-g_4\left(g_3^2\varrho(a+\vartheta+\varrho)-2g_4\left(m_1^2\left(\vartheta^2-4a\varrho\right)+4\varrho\omega\right)\right)}}{4g_4\varrho}.$$

Then, the corresponding solution of equation (1.1) is

$$q(x,y,t)=\pm g_3\sqrt{\frac{a+\vartheta+\varrho}{8g_4h}}+\sqrt{\frac{g_2(a+\vartheta+\varrho)}{2h}} \times\left\{1+\tanh\left[\sqrt{\frac{h\alpha_0^2}{2(a+\vartheta+\varrho)}}(x+y-\tau t)\right]\right\}e^{i(-m_1x-m_2y-\omega t)}. \tag{3.20}$$

This solution represent dark soliton solution.

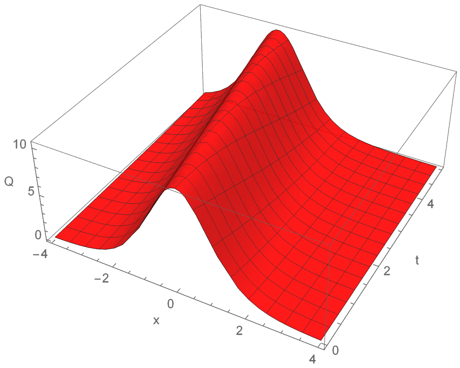
4. Graphic representation of solutions

Fig.1. 3D and 2D graphs of the bright soliton solution (equation 3.5) with parameters $a=0.2,\vartheta=0.1,\varrho=0.5,\omega=-0.5,m_1=0.1,m_2=0.2,\alpha_1=0.5,h=-0.1$.

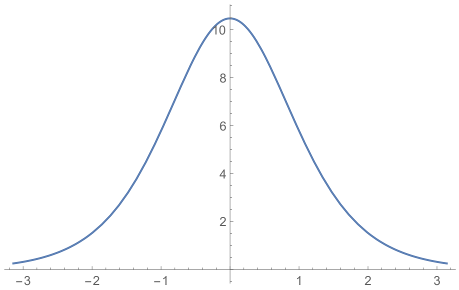
Fig.2. 3D and 2D graphs of the singular periodic solution (equation 3.6) with parameters $a=-0.2,\vartheta=-0.1,\varrho=-0.5,\omega=-0.5,m_1=0.1,m_2=0.2,\alpha_1=0.5,h=-0.1$.

Fig.3. 3D and 2D graphs of the singular soliton solution (equation 3.7) with parameters $a=0.4,\vartheta=0.2,\varrho=0.06,\omega=0.5,m_1=0.1,m_2=0.2,\alpha_1=1,h=2$.

Fig.4. 3D and 2D graphs of the singular periodic solution (equation 3.8) with parameters $a=0.4,\vartheta=0.2,\varrho=0.06,\omega=-0.5,m_1=0.1,m_2=0.2,\alpha_1=1,h=2$.

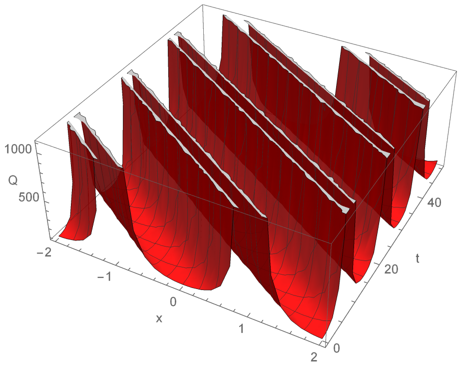


(A) 3D diagram

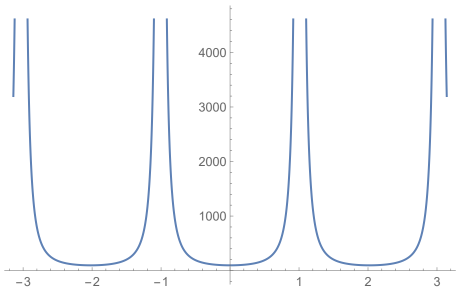


(B) 2D diagram

FIGURE 1

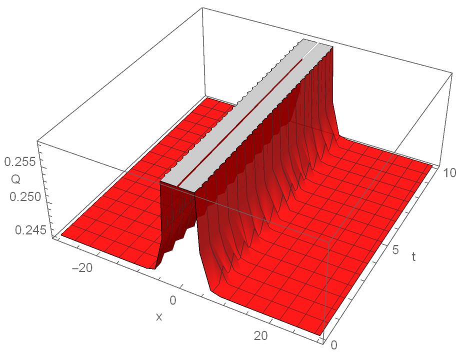


(A) 3D diagram

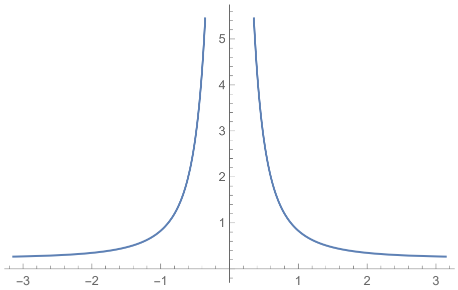


(B) 2D diagram

FIGURE 2

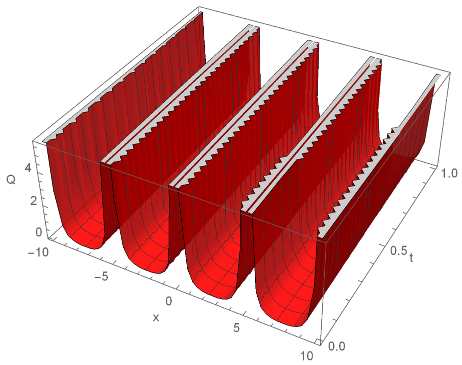


(A) 3D diagram

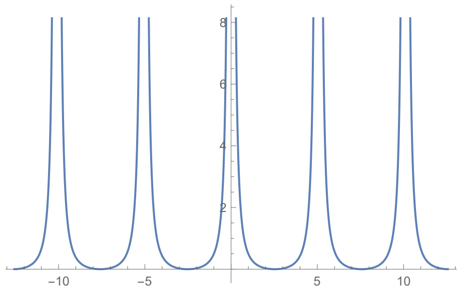


(B) 2D diagram

FIGURE 3

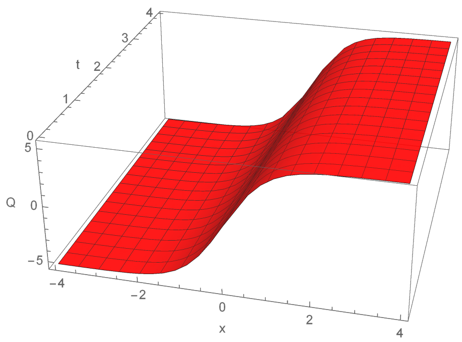


(A) 3D diagram

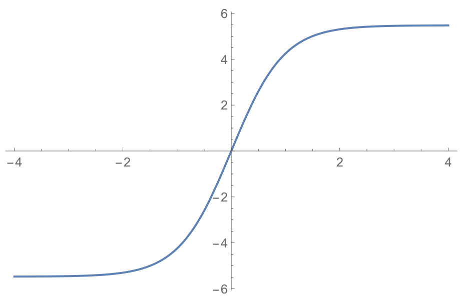


(B) 2D diagram

FIGURE 4



(A) 3D diagram



(B) 2D diagram

FIGURE 5

Fig.5. 3D and 2D graphs of the dark soliton solution (equation 3.9) with parameters $a = -0.8, \vartheta = 0.3, \varrho = -0.9, \omega = -3, m_1 = -0.02, m_2 = 0.03, \alpha_1 = 0.5, h = -0.1$.

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