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ON THE PROBLEM OF RESTORING A FUNCTION IN THREE DIMENSIONAL SPACE BY THE FAMILY OF CONES

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ABSTRACT. In this paper, we consider the problem of recovering a function in three-dimensional space from a family of cones with a weight function of a special form. Exact solutions of the problem are obtained for the given weight functions. A class of parameters for the problem that has no solution is constructed.

1. INTRODUCTION

Integral geometry problems naturally arise in the study of many mathematical models in such widely applied areas as seismic exploration, interpretation of geophysical and aerospace observations, various processes described by kinetic equations, etc. The apparatus developed here is the mathematical basis for the computational tomography is a promising and rapidly developing area of modern science [3, 4].

In this article, the following class of problems of integral geometry is considered: on the recovery of a function given by integrals over a certain family of cones. These tasks are associated with numerous applications. In order to study the internal structure of the earth's interior, a series of explosions is carried out on

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the earth's surface. For each explosion, the vibration modes of the earth's surface are measured on the instrument system. The purpose of the study is to determine the distribution of physical parameters associated with the laws of propagation of seismic waves within the Earth using the readings of the instruments. The clearest functional in the readings of the instruments is the time of arrival of the seismic wave, it is this that serves as the basis in the practice of interpretation.

Let us give a general statement of the problem of integral geometry [1].

Let $u(x)$ be a sufficiently smooth function defined in R^n and $\{S(y)\}$ be a family of piecewise smooth manifolds in this space, depending on the parameter $y = (y_1, y_2, ..., y_n)$.

Let us consider the integral

$$\int\limits_{S(y)} g(x,y)u(x)ds = f(y), \tag{1.1}$$

where $u(x)$ is a unknown function, $g(x, y)$ is a given weight function, ds is an element of the measure on $S(y)$. It is required to restore the function $u(x)$ by function $f(y)$.

The main questions that arise in the study of this problem are as follows. The first and most fundamental question is, does the assignment of a function $f(y)$ unambiguously define a function $u(x)$? Next, how do you find a function $u(x)$ by function $f(y)$? In this case, the question is important: how to get an analytical formula expressing $u(x)$ in terms of $f(y)$? It should be noted here that this is not possible in all cases. And, finally, a question naturally related to the existence theorem for a solution to the problem: what are the necessary and sufficient conditions for belonging to the class of functions $f(y)$ representable through the integral (1.1)?

Integral geometry problems of Volterra type are those problems that can be reduced to the study of Volterra operator equations in the sense of the definition given in [1]. We also give definitions of weak and strong ill-posedness of an integral geometry problem. The problem of solving equation (1.1) is called weakly ill-posed if for the given problem and its solution it is possible to choose a pair of function spaces in which a finite number of derivatives are involved in determining the norm such that the inversion operator for this pair of spaces is continuous. If such a pair of spaces does not exist, then the problem is strongly ill-posed [1].

The questions of uniqueness of the solution of the plane problem of integral geometry on a family of parabolas with perturbations were considered in [2].

In [5], the problem of integral geometry is studied for a family of spatial curves. The uniqueness theorem is proved for the solution of the considered integral geometry problem. In the works of [6] studied a new class of ray transform inversion problems with incomplete data. By the nature of the instability, this is a highly ill-posed problem. The results obtained in these articles had a beneficial effect on the work [7–13]. Weakly ill-posed problems of integral geometry of Voltaire type with weighted functions having a singularity were studied in [14]. Uniqueness theorems, stability estimates, and formulas for inversion of weakly ill-posed problems of integral geometry with respect to special curves and surfaces with singularities at vertices were obtained in [15]. In [16,17], new classes of the integral geometry problem were studied, new approaches were introduced to the study of problems of recovering a function from weight functions with a singularity.

In this paper, we consider the problem of recovering a function in three-dimensional space from a family of cones with a weight function of a special form. Using integral equation method exact solutions of the problem are obtained for the given weight functions. A class of parameters for the problem that has no solution is constructed.

2. Problem statement and the main results

Let us introduce the notation

$$(x, y, z) \in R^3, (\xi, \eta, \zeta) \in R^3, \lambda \in R^1, \mu \in R^1,$$

$$\Omega = \{(x, y, z) :\, x \in R^1,\, y \in R^1,\, z \in (0, h),\, h < \infty\},$$

$$\bar{\Omega} = \{(x, y, z) :\, x \in R^1,\, y \in R^1,\, z \in [0, h]\}.$$

In the layer $\bar{\Omega}$ we consider the family of cones $S(x, y, z)$, which are uniquely parameterized using the coordinates of their vertices $(x, y, z) \in \bar{\Omega}$ as

$$S(x, y, z) = \left\{(\xi, \eta, \zeta) : (x - \xi)^2 + (y - \eta)^2 = (z - \zeta)^2,\ \xi \in R,\, \eta \in R,\, 0 \leq \zeta \leq z\right\}.$$

Problem A. Determine a function of three variables $u(x, y, z)$, if the integrals of the function $u(x, y, z)$ over a family of cones $S(x, y, z)$ are known for all (x, y, z) of the layer $\bar{\Omega}$:

$$\iint_{S(x,y,z)} g\left(x-\xi, y-\eta\right) u\left(\xi, \eta, \zeta\right) ds = f\left(x, y, z\right) \tag{2.1}$$

where $g(x-\xi, y-\eta)$ is weight function, $f(x, y, z)$ is a given function on $\bar{\Omega}$.

Remark 2.1. *Note that for some g with $g(0,0) = 0$ the solution of the Problem A may not exist (see example-1).*

The next theorems give exact solutions to the problem for some given weight functions.

Theorem 2.1. *Let the function $f \in L_2(\bar{\Omega})$ be continuous in $(x, y, z) \in \bar{\Omega}$ and have continuous partial derivatives up to the second order with respect to z from the class $L_2(\bar{\Omega})$. Let*

$$g(x-\xi, y-\eta) = e^{-i\rho[\lambda(x-\zeta)+\mu(y-\eta)]}. \tag{2.2}$$

Then the function

$$u\left(x, y, z\right) = \frac{\partial^2}{\partial z^2} f\left(x, y, z\right) \tag{2.3}$$

is the unique solution of the Problem A.

Theorem 2.2. *Let the function $f \in L_2(\bar{\Omega})$ be a continuous function in $(x, y, z) \in \bar{\Omega}$ and have continuous partial derivatives with respect to z from the class $L_2(\bar{\Omega})$. Let*

$$g(x-\xi, y-\eta) = \frac{e^{-i\rho[\lambda(x-\zeta)+\mu(y-\eta)]}}{\alpha^2\sqrt{\left(x-\xi\right)^2+\left(y-\eta\right)^2}+\beta^2\left(|x-\xi|+|y-\eta|\right)}, \tag{2.4}$$

where $\alpha^2 + \beta^2 \neq 0$. Then the function

$$u\left(\lambda, \mu, z\right) = \frac{1}{\sqrt{2}\Phi_{\alpha,\beta}} \frac{\partial}{\partial z} f\left(\lambda, \mu, z\right) \tag{2.5}$$

is the unique solution of the Problem A, where

$$\Phi_{\alpha,\beta} = \int_0^{2\pi} \frac{1}{\alpha^2 + \beta^2\left(|\cos\phi| + |\sin\phi|\right)} d\phi.$$

3. Proof of the main results

We first reduce the problem A to investigate the Volterra type equation. In the integral on the left hand side of (2.1) we take the following change of variables

$$\zeta = z - \sqrt{(x-\xi)^2 + (y-\eta)^2}, \quad \zeta'_\xi = -\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}},$$

$$\zeta'_\eta = -\frac{y-\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} ds = \sqrt{1+\left(\zeta'_\xi\right)^2+\left(\zeta'_\eta\right)^2} d\xi d\eta.$$

Then the integral in (2.1) reduces to integration over $d\xi d\eta$ as

$$\text{(3.1)} \quad \sqrt{2}\int\int\limits_{D(x,y,z)} g(x-\xi, y-\eta)u(\xi,\eta, z-\sqrt{(x-\xi)^2+(y-\eta)^2})d\xi d\eta = f(x,y,z).$$

Let's make the following substitutions by variables

$$\xi = x - \rho\cos\phi, \quad \eta = y - \rho\sin\phi.$$

The the integral (3.1) represents as

(3.2)

$$\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z} g\left(\rho\cos\phi, \rho\sin\phi\right) u\left(x-\rho\cos\phi, y-\rho\sin\phi, z-\rho\right)\rho d\rho d\phi = f\left(x,y,z\right).$$

We apply the Fourier transform with respect to the variable x, to both sides of equation (3.2)

$$\text{(3.3)} \quad \hat{f}(\lambda, y, z) = \sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z} e^{i\lambda\rho\cos\phi} g(\rho\cos\phi,\rho\sin\phi)\hat{u}(\lambda, y-\rho\sin\phi, z-\rho)\rho d\rho d\phi,$$

where

$$\hat{f}\left(\lambda,y,z\right) = \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{+\infty} e^{i\lambda x} f(x,y,z)dx,$$

$$\hat{u}\left(\lambda, y-\rho\sin\phi, z-\rho\right) = \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{+\infty} e^{i\lambda x} u\left(x, y-\rho\sin\phi, z-\rho\right)dx.$$

Now we apply the Fourier transform with respect to the variable y, to both sides of equation (3.3)

$$\text{(3.4)} \quad \hat{\hat{f}}\left(\lambda,\mu,z\right) = \sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z} e^{i\rho(\lambda cos\phi+\mu\sin\phi)} g\left(\rho\cos\phi,\rho\sin\phi\right)\hat{\hat{u}}\left(\lambda,\mu,z-\rho\right)\rho d\rho d\phi.$$

where

$$\hat{\hat{f}}(\lambda,\mu,z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{i\mu y}\hat{f}(\lambda,y,z)dy,$$
$$\hat{\hat{u}}(\lambda,\mu,z-\rho)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{i\mu y}\hat{u}(\lambda,y,z-\rho)dy.$$

Thus the equation (2.1) is equivalent to (3.4).

Proof of the Theorem 2.1. Substituting (2.2) into equation (3.4), we obtain

$$\hat{\hat{f}}(\lambda,\mu,z)=\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z}\hat{\hat{u}}(\lambda,\mu,z-\rho)\rho d\rho d\phi. \tag{3.5}$$

In equation (3.5) making change of variable $z-\rho=t$, $d\rho=-dt$ we get

$$\hat{\hat{f}}(\lambda,\mu,z)=\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z}z\hat{\hat{u}}(\lambda,\mu,t)\,dtd\phi-\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z}t\hat{\hat{u}}(\lambda,\mu,t)\,dtd\phi. \tag{3.6}$$

In equation (3.6) we take derivative with respect to the variable z

$$\frac{\partial}{\partial z}\hat{\hat{f}}(\lambda,\mu,z)=\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z}\hat{\hat{u}}(\lambda,\mu,t)\,dtd\phi.$$

In the last equation again taking derivative with respect to the variable z, we obtain

$$\hat{\hat{u}}(\lambda,\mu,z)=\frac{1}{2\sqrt{2}\pi}\frac{\partial^2}{\partial z^2}\hat{\hat{f}}(\lambda,\mu,z). \tag{3.7}$$

Further applying the inverse double Fourier transform with respect to the variables μ and λ to equation (3.7),using the inversion theorem and the convolution theorem, we obtain

$$u(x,y,z)=\frac{1}{2\sqrt{2}\pi}\frac{\partial^2}{\partial z^2}f(x,y,z).$$

□

Proof of the Theorem 2.2. Substituting (2.5) into equation (3.4), we obtain

$$\hat{\hat{f}}(\lambda,\mu,z)=\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z}\frac{\hat{\hat{u}}(\lambda,\mu,z-\rho)}{\alpha^2+\beta^2(|\cos\phi|+|\sin\phi|)}d\rho d\phi. \tag{3.8}$$

Making the change of variable $t = z - \rho$ we get

$$\hat{\hat{f}}(\lambda, \mu, z) = \sqrt{2} \int_0^{2\pi} \int_0^{z} \frac{1}{\alpha^2 + \beta^2 (|\cos\phi| + |\sin\phi|)} \hat{\hat{u}}(\lambda, \mu, t)\, dt d\phi. \tag{3.9}$$

Taking the derivatives with respect to the variable z on both sides of (3.9) we have

$$\begin{aligned} \frac{\partial}{\partial z}\hat{\hat{f}}(\lambda, \mu, z) &= \frac{\partial}{\partial z}\left(\sqrt{2} \int_0^{2\pi} \int_0^{z} \frac{1}{\alpha^2 + \beta^2 (|\cos\phi| + |\sin\phi|)} \hat{\hat{u}}(\lambda, \mu, t)\, dt d\phi\right) \\ &= \sqrt{2} \int_0^{2\pi} \frac{1}{\alpha^2 + \beta^2 (|\cos\phi| + |\sin\phi|)} \hat{\hat{u}}(\lambda, \mu, z)\, d\phi = \sqrt{2}\Phi_{\alpha,\beta}\hat{\hat{u}}(\lambda, \mu, z), \end{aligned}$$

where

$$\Phi_{\alpha,\beta} = \int_0^{2\pi} \frac{1}{\alpha^2 + \beta^2 (|\cos\phi| + |\sin\phi|)} d\phi. \tag{3.10}$$

Thus

$$\hat{\hat{u}}(\lambda, \mu, z) = \frac{1}{\sqrt{2}\Phi_{\alpha,\beta}} \frac{\partial}{\partial z}\hat{\hat{f}}(\lambda, \mu, z). \tag{3.11}$$

To find u we apply the inverse double Fourier transform to equation (3.11) with respect to the variables μ and λ

$$u(x, y, z) = \frac{1}{\sqrt{2}\Phi_{\alpha,\beta}} \frac{\partial}{\partial z} f(x, y, z).$$

□

Example 1. *Let g be a continuous function with $g(0,0) = 0$ and the function $f \in L_2(\bar{\Omega})$ be continuous in $(x, y, z) \in \bar{\Omega}$ and have continuous partial derivatives up to the second order with respect to z from the class $L_2(\bar{\Omega})$ with with $\frac{\partial^2}{\partial z^2} f(x, y, 0) \neq 0$. Then the solution to problem A does not exist. Indeed, in the integral on* (3.4) *making the change of variables $z - \rho = t, \quad d\rho = -dt$ we obtain*

$$\begin{aligned} \hat{\hat{f}}(\lambda, \mu, z) = \sqrt{2} \int_0^{2\pi} \int_0^{z} & e^{i(z-t)(\lambda\cos\phi + \mu\sin(\phi))} g((z-t)\cos\phi, (z-t)\sin\phi) \cdot \\ & \hat{\hat{u}}(\lambda, \mu, t)(z - t) dt d\phi. \end{aligned} \tag{3.12}$$

Taking derivative with respect z we have

$$\frac{\partial}{\partial z}\hat{\hat{f}}\,(\lambda,\mu,z)=\sqrt{2}\int\limits_0^{2\pi}\int\limits_0^{z}e^{i(z-t)(\lambda\cos\phi+\mu\sin(\phi)}g((z-t)\cos\phi,(z-t)\sin\phi)\hat{\hat{u}}\,(\lambda,\mu,t)\,dtd\phi.$$

Again taking derivative with respect z we get

$$\frac{\partial^2}{\partial z^2}\hat{\hat{f}}\,(\lambda,\mu,z)=0.$$

On the other hand $\frac{\partial^2}{\partial z^2}\hat{\hat{f}}\,(\lambda,\mu,z)\neq 0$, since $\frac{\partial^2}{\partial z^2}f\,(x,x,z)\neq 0$. Hence, the problem A has no any solution, since the equation (3.12) is equivalent to (3.4).

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