ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.12, 3533–3548
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.12.2

SOME ASPECTS OF SOLUTIONS OF SPACE-TIME FRACTIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH OSGOOD CONDITION

Henok Desalegn[1], Tadesse Abdi, and Jebessa B. Mijena

ABSTRACT. In this paper we discuss the following problem with additive noise,

$$\begin{cases} \frac{\partial^\beta u(t,x)}{\partial t} = -(-\triangle)^{\frac{\alpha}{2}} u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x),\ t>0, \\ u(0,x) = u_0(x), \end{cases}$$

where $\alpha \in (0,2)$ and $\beta \in (0,1)$, the fractional time derivative is in the sense of Caputo, $-(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian, σ is a positive parameter, $\dot{W}$ is a space-time white noise, $u_0(x)$ is assumed to be non-negative, continuous and bounded. We study first the equation on $[0,\ 1]$ with homogeneous Drichlet boundary condition and show that the solution of the equation blows up in finite time if and only if b satisfies the Osgood condition,

$$\int_c^\infty \frac{ds}{b(s)} < \infty$$

for some constant $c > 0$. We then consider the same equation on the whole line and show that the above Osgood condition is satisfied whenever the solution of the equation blows up.

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 35B44, 35R60, 60H15.
Key words and phrases. Osgood condition, Fractional stochastic partial differential equation, blow up, space-time white noise.
Submitted: 12.11.2021; *Accepted:* 27.11.2021; *Published:* 02.12.2021.

1. Introduction

Several scholars studied intensively to construct mathematical formula and analyze the existence of a unique solution for a class of ordinary and partial differential equations using analytical and numerical methods. These equations have been successfully used to model a variety of real world problems. However, there are real world problems which showcase some randomness that could not be caught by these differential equations. Thus the concept of stochastic differential equations (SDEs) have been suggested and used extensively in the last decades to solve such type of problems. One of the studied stochastic differential equations is stochastic partial differential equation(SPDE) which is used as a mathematical tool to model many physical and biological systems subject to the influence of noise that appears either intrinsically or extrinsically, see Davar Khoshnevisan [12] and list of references therein for the details. Even so, some problems didn't follow randomness, so different operators such as fractional differential operator were proposed to solve them. Recently, many scholars have paid attention to research problems involving space-time fractional stochastic partial differential equations which is applied in many disciplines. For instance, Mijena and Nane [7] introduced the space-time fractional stochastic partial differential equation with white noise and proved the existence and uniqueness of the integral solution in the sense of Walsh [8], while Foondun and Nane [15] showed the existence of solution of space-time fractional stochastic partial differential equation with colored noise. In [17] it was discussed on blow-up results for space-time fractional stochastic partial differential equations and very recently Foondun [16] showed that the presence of the fractional time derivative induces a significant change in the qualitative behavior of the solution to a class of fractional time stochastic equations.

Hiroshi Fujita [9] looked at the following nonlinear heat equation,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \triangle u(t,x) + u(t,x)^{1+\eta}, \ x \in \mathbb{R}^d, \ t > 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $u_0(x)$ is nonnegative, continuous and bounded. He showed that when $0 < \eta d < 2$ there is no nontrivial global solution no matter how small the nontrivial initial condition u_0 is. While for $\eta d > 2$ there is a nontrivial global solution when u_0 is small enough. The exponent $\eta_c d = 2$ is often called the Fujita exponent, after

Fujita, Kantaro Hayakawa [3] proved that there is no nontrivial global solution in the critical case $\eta_c d = 2$. These results inspired a lot of generalizations (see [13, 14]) and while (1.1) is considered on the interval $[0,1]$ with homogeneous Dirichlet boundary condition a different result is obtained in [1]; if $\eta > 0$ we can construct nontrivial global solution by taking u_0 small enough and when u_0 is large there is no global solution for any $\eta > 0$.

Bonder and Groisman [4] considered the following stochastic heat equation with additive noise,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \triangle u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x),\ x \in [0,1],\ t > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

with homogeneous Dirichlet boundary condition. Here $\dot{W}$ is a space-time white noise, σ is a positive parameter, b is locally Lipschitz real function. If b is globally Lipschitz there exist a global solution to (1.2) almost surely (a.s.) as it is shown in [8, 11] and $u_0(x)$ is non-negative and continuous. They proved in their paper that, the solution to (1.2) blows up in finite time whenever b is non-negative, convex and satisfy the Osgood condition for some constant $c > 0$

$$\int_c^\infty \frac{ds}{b(s)} < \infty. \tag{1.3}$$

In [5] they investigated whether the Osgood condition is optimal. They showed the complement of [4] if $|b(x)| = O\big(|x| \log |x|\big)$ as $|x| \to \infty$, then there exists a global solution to equation (1.2). Foondun and Nualart [2] showed that the Osgood condition (1.3) is also a necessary condition for non-existence of solution to (1.2) which means together with the result of [4] the Osgood condition is necessary and sufficient for blow up of the solution of equation (1.2) in a finite time. Also in [2] they considered equation (1.2) in whole line

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \triangle u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x),\ x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.4}$$

with the assumption that $b : \mathbb{R} \to \mathbb{R}^+$ is nonnegative and non-decreasing on $(0,\infty)$. Then they showed that if b satisfies the Osgood condition (1.3), then a.s. there is no global solution to (1.4).

The main objective of these paper is to investigate the behavior of the solution for a class of equations which involve a fractional Laplacian and fractional time derivative in the sense of Caputo with additive noise for bounded and unbounded domains.

Let us consider the following space-time fractional stochastic heat equation with additive noise,

$$\begin{cases} \frac{\partial^\beta u(t,x)}{\partial t} = -(-\triangle)^{\frac{\alpha}{2}} u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x), \ x \in [0,1], \ t > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.5}$$

along with homogeneous Dirichlet boundary condition. Here, $\alpha \in (0,2)$ and $\beta \in (0,1)$, $u_0(x)$ is non-negative continuous and bounded, the fractional time derivative is in the sense of Caputo and $-(-\triangle)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian. Using Walsh theory, equation (1.5) could be interpreted through the integral equation,

$$\begin{aligned} u(t,x) = &\int_0^1 G(t,x,y)u_0(y)dy + \int_0^t \int_0^1 G(t-s,x,y)b(u(s,y))dyds \\ &+ \sigma \int_0^t \int_0^1 G(t-s,x,y)W(dyds), \end{aligned}$$

where, $G(t,x,y)$ is heat kernel associated with the fractional Laplacian defined on $[0,1]$.

The organization of the paper is as follows: Section 2 contains preliminary information needed to proof the result, Section 3 contains the main result of the paper with the proof and Section 4 contains a concise summary of the paper.

2. Preliminaries

This section contains some background information needed to proof our results. We introduce the Osgood condition for deterministic integral equation from [10] and comparison theorem from [6] and (Theorem 3.1, [6]) with the proof since the main idea of our proof relies on it..

Consider

$$\begin{cases} \frac{dy(t)}{dt} = b(y(t)), \ t > 0 \\ y(0) = a, \ a \geq 0. \end{cases} \tag{2.1}$$

Assumption **(A1)** $b : \mathbb{R} \to \mathbb{R}^+$ is positive, non-decreasing on $(0, \infty)$ and locally Lipschitz Continuous.

A function satisfying (2.1) must also satisfy the following equivalent integral equation,

$$y(t) = a + \int_0^t b(y(s))ds,\ t \geq 0 \tag{2.2}$$

and (2.2) has a unique solution upto its blow up time(by Picard-Lindelöf theorem). The blow up time of y is defined as

$$T = \sup\{t > 0 : |y(t)| < \infty\}.$$

We observe from equation (2.1) that,

$$\int_0^t \frac{y'(s)}{b(y(s))} ds = \int_0^t ds = t.$$

Thus, by change of variable we can write,

$$\int_{v(0)}^{v(t)} \frac{dv}{b(v)} = \int_a^{v(t)} \frac{dv}{b(v)} = t.$$

If we set,

$$D(x) = \int_a^x \frac{dv}{b(v)},$$

then $D(v(t)) = t$. This leads to

$$v(t) = D^{-1}(t),\ 0 < t < D(\infty).$$

Here the explosion time T is

$$T = D(\infty) = \int_a^\infty \frac{ds}{b(s)}.$$

Therefore, the solution $y(t)$ blows up in finite time if and only if

$$T = D(\infty) = \int_a^\infty \frac{ds}{b(s)} < \infty.$$

Definition 2.1. *[18] For a function $u(t, x)$, the Caputo fractional derivative is defined as:*

$$\frac{\partial^\beta u(t,x)}{\partial t} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial}{\partial r} u(r,x) \frac{dr}{(t-r)^\beta},$$

for $0 < \beta < 1$, $t > 0$.

Definition 2.2. *[19] For a function $u(t,x)$ the Fractional Laplacian of order $\frac{\alpha}{2}$ can be defined by the Fourier transform $(\mathcal{F})$*

$$(\mathcal{F}(-\triangle)^{\frac{\alpha}{2}}u)(\xi) = |\xi|^{\alpha}\mathcal{F}(u)(\xi),$$

for $0 < \alpha < 2$.

Theorem 2.1. *([6], Comparison theorem) Assume that b satisfies assumption **(A1)**, $a_1 > a_2$ and $T > 0$. If u and v are two measurable functions on $[0,T]$ such that,*

$$v(t) \geq a_1 + \int_0^t b(v(s))ds, \ t \in [0,T]$$

and

$$u(t) = a_2 + \int_0^t b(u(s))ds, \ t \in [0,T],$$

then $v \geq u$ on $[0,T]$. Similarly, if $a_1 < a_2$ and

$$v(t) \leq a_1 + \int_0^t b(v(s))ds, \ t \in [0,T],$$

then $v \leq u$ on $[0,T]$.

Assumption **(A2)** $g : [0,\infty] \to \mathbb{R}$ is a continuous function such that

$$\limsup_{t\to\infty} \inf_{0\leq h\leq 1} g(t+h) = \infty.$$

Theorem 2.2. *([6]) Let $a \geq 0$ and suppose that Assumptions **(A1)** and **(A2)** hold. Then the solution of the integral equation,*

$$X_t = a + \int_0^t b(X_s)ds + g(t) \tag{2.3}$$

blows up in finite time if and only if b satisfies the Osgood condition (1.3).

Proof. Necessity: Suppose that the solution X_t blows up at finite time T. If we set,

$$M = \sup_{0\leq t\leq T} |g(t)|,$$

then from equation (2.3), $t \in [0,T]$ and b is non-negative we have,

$$\begin{aligned} X_t &= a + \int_0^t b(X_s)ds + g(t), \\ &\leq a + M + \int_0^t b(X_s)ds. \end{aligned}$$

Let

$$Y_t = a + M + 1 + \int_0^t b(Y_s)ds,$$

then by Comparison theorem above $X_t \leq Y_t$. Since X_t blows up at time T, Y_t should also blows up at time T. Thus by Osgood criteria b satisfies the Osgood condition (1.3).

Sufficiency: Suppose that X_t of equation (2.3) does not blow up in finite time. So using Assumption **(A2)** we can find a sequence $\{t_n : n \in \mathbf{N}\}$ such that $t_n \to \infty$ and $\inf\limits_{0\leq h\leq 1} g(t_n + h) \to \infty$ as $n \to \infty$.

Using Assumption **(A1)**

$$\begin{aligned}
X_{t+t_n} &= a + \int_0^{t+t_n} b(X_s)ds + g(t+t_n) \\
&\geq a + \int_{t_n}^{t+t_n} b(X_s)ds + g(t+t_n) \\
&\geq a + \int_0^t b(X_{s+t_n})ds + g(t+t_n),\ t \in [0,1] \\
&\geq \frac{1}{2}(a + \inf_{0\leq h\leq 1} g(h+t_n)) + \int_0^t b(X_{s+t_n})ds.
\end{aligned}$$

This means that

$$X_{t+t_n} \geq Z_t$$

where,

$$Z_t = \tfrac{1}{2}(a + \inf_{0\leq h\leq 1} g(h+t_n)) + \int_0^t b(Z_s)ds.$$

Since we are assuming that X_t doesn't blow up in finite time, Z_t cannot blow up in the interval $[0,1]$ (by comparison theorem). In other words, the time of blow up of Z_t has to be greater than 1, i.e,

$$\int_m^\infty \frac{ds}{b(s)} > 1,$$

where

$$m = \tfrac{1}{2}(a + \inf_{0\leq h\leq 1} g(h+t_n)).$$

Since

$$\inf_{0\leq h\leq 1} g(h+t_n) \to \infty \text{ as } n\to\infty,$$

we have

$$\tfrac{1}{2}(a+\inf_{0\leq h\leq 1} g(h+t_n)) \to \infty.$$

Hence,

$$\int_{.}^{\infty} \frac{ds}{b(s)} = \infty.$$

This complete the proof. □

3. Main results

In what follows, we will present the main result of our paper:

Theorem 3.1. *Suppose that the Assumption **(A1)** holds. If the solution to* (1.5) *blows up in finite time a.s. then* b *satisfies the Osgood condition* (1.3).

Proof. Our method, like in [2], rely on the integral formulation of the solution and comparison theorem which is the basic tool for this paper. Set

$$T = \sup\{t>0 : \sup_{x\in[0,1]} |u(t,x)| < \infty\}.$$

Suppose that the solution $u(t,x)$ of (1.5) blows up in finite time T. From the integral formulation of the solution,

$$\begin{aligned} u(t,x) = &\int_0^1 G(t,x,y)u_0(y)dy + \int_0^t\int_0^1 G(t-s,x,y)b(u(s,y))dyds \\ &+\sigma\int_0^t\int_0^1 G(t-s,x,y)W(dyds) \end{aligned}$$

The stochastic term in the above integral formulation is continuous. Thus the quantity given below is finite a.s.

$$K = \sup_{x\in[0,1],t\in(0,T]} \Big|\int_0^t\int_0^1 G(t-s,x,y)W(dyds)\Big|. \tag{3.1}$$

Since b and the initial condition $u_0(x)$ are non-negative we have,

$$u(t,x) \geq \sigma\int_0^t\int_0^1 G(t-s,x,y)W(dyds).$$

Thus,

$$-u(t,x) \leq -\sigma \int_0^t \int_0^1 G(t-s,x,y)W(dyds).$$

Consequently,

$$\sup_{x\in[0,1],t\in(0,T]} \Big(- u(t,x)\Big) \leq \sup_{x\in[0,1],t\in(0,T]} \Big(- \sigma \int_0^t \int_0^1 G(t-s,x,y)W(dyds)\Big) = \sigma K.$$

This shows that,

$$\Big(- \sup_{x\in[0,1],t\in(0,T]} (-u(t,x))\Big) \geq -\sigma K,$$

and hence,

$$\inf_{x\in[0,1],t\in(0,T]} u(t,x) \geq -\sigma K.$$

Since $u_0(x)$ is bounded we have,

$$\Big| \int_0^1 G(t,x,y)u_0(y)dy\Big| \leq P, \tag{3.2}$$

for some positive constant P.

Define: $A = \{s \in (0,t), y \in (0,1) : -\sigma K \leq u(s,y) \leq 0\}$ and $B = \{s \in (0,t), y \in (0,1) : u(s,y) > 0\}$. But then,

$$\int_0^t \int_0^1 G(t-s,x,y)b(u(s,y))dyds = \underbrace{\iint_A G(t-s,x,y)b(u(s,y))dyds}_{J_1} + \underbrace{\iint_B G(t-s,x,y)b(u(s,y))dyds}_{J_2}.$$

Since $b(u(s,y))$ is assumed to be continuous on $[-\sigma K, 0]$ then b is bounded. So $J_1 \leq d$ where, d is a finite quantity almost surely. Again since b is non-negative and non-decreasing on $(0,\infty)$,

$$J_2 \leq \int_0^t b(y_s)ds,$$

where $y_t = \sup\limits_{x\in[0,1]} u(t,x)$. Thus,

$$J_1 + J_2 \leq d + \int_0^t b(y_s)ds. \tag{3.3}$$

Taking the supremum of equation (1) and combining all the estimates (3.1-3.3) we obtain

$$y_t \leq P + d + \sigma K + \int_0^t b(y_s)ds.$$

Let

$$z_t = (P + d + \sigma K + 1) + \int_0^t b(z_s)ds.$$

By Comparison theorem $y_t \leq z_t$ and since y_t blows up at time T, z_t should also blows up with time T. From the proof of Theorem(2.2) and Osgood criteria for integral equation (2.2) b satisfies the Osgood condition (1.3). □

Theorem 3.2. *Suppose that the Assumption **(A1)** holds. If b satisfies the Osgood condition* (1.3)*, then the solution to* (1.5) *blows up in finite time almost surely.*

Proof. Suppose that the solution $u(t,x)$ of (1.5) doesn't blow up in finite time T. Let $\{t_n\}$ be a sequence of positive numbers such that $t_n \to \infty$ as $n \to \infty$. From the integral formulation of the solution we obtain:

$$\begin{aligned}
&u(t+t_n,x)\\
&= \int_0^1 G(t+t_n,x,y)u_0(y)dy + \int_0^{t+t_n}\int_0^1 G(t+t_n-s,x,y)b(u(s,y))dyds\\
&\quad + \sigma\int_0^{t+t_n}\int_0^1 G(t+t_n-s,x,y)W(dyds)\\
&\geq \int_0^1 G(t+t_n,x,y)u_0(y)dy + \int_0^t\int_0^1 G(t-s,x,y)b(u(s+t_n,y))dyds\\
&\quad + \sigma\int_0^{t+t_n}\int_0^1 G(t+t_n-s,x,y)W(dyds).
\end{aligned}$$

Put

$$g(t_n+t,x) = \int_0^{t+t_n}\int_0^1 G(t+t_n-s,x,y)W(dyds),$$

for $t \in [0,1]$ and $x \in [0,1]$.

Assumption **(A3)** Let $g(t_n+t,x)$ be as defined above such that

$$\inf_{h\in[0,1],x\in[0,1]} g(h+t_n,x) \to \infty \text{ as } n\to\infty.$$

Since b is non-decreasing on $(0,\infty)$,

$$\begin{aligned}
&\int_0^t\int_0^1 G(t-s,x,y)b(u(s+t_n,y))dyds \\
&\geq \int_0^t b\Big(\inf_{y\in[0,1]} u(s+t_n,y)\Big)\int_0^1 G(t-s,x,y)dyds \\
&\geq \int_0^t b\Big(\inf_{y\in[0,1]} u(s+t_n,y)\Big)ds = \int_0^t b(y_s)ds,
\end{aligned}$$

where, $y_t = \inf_{y\in[0,1]} u(t+t_n,y)$. Thus

$$u(t+t_n,x) \geq \int_0^1 G(t+t_n,x,y)u_0(y)dy + \sigma g(t_n+t,x) + \int_0^t b(y_s)ds. \tag{3.4}$$

Taking the infimum of both sides of (3.4) we obtain,

$$y_t \geq \inf_{h\in[0,1],x\in[0,1]}\Big\{\int_0^1 G(h+t_n,x,y)u_0(y)dy + \sigma g(h+t_n,x)\Big\} + \int_0^t b(y_s)ds.$$

This means that $y_t \geq p_t$, where

$$p_t = \inf_{\in[0,1],x\in[0,1]}\Big\{\int_0^1 G(h+t_n,x,y)u_0(y)dy + \sigma g(h+t_n,x)\Big\} + \int_0^t b(p_s)ds.$$

Since we are assuming that $u(t,x)$ doesn't blow up in finite time, p_t can't blow up in the interval $[0,1]$ (by comparison theorem). This means that the time of blow up for p_t has to be greater than 1, that is,

$$\int_q^\infty \frac{ds}{b(s)} > 1,$$

where

$$q = \inf_{h\in[0,1],x\in[0,1]}\{\int_0^1 G(h+t_n,x,y)u_0(y)dy + \sigma g(h+t_n,x)\}.$$

Since we can find a sequence $t_n\to\infty$ such that

$$\inf_{h\in[0,1],x\in[0,1]} g(h+t_n,x) \to \infty$$

as $n\to\infty$ (Assumption **(A3)**).

It follows that

$$\int_{\cdot}^{\infty} \frac{ds}{b(s)} = \infty.$$

Thus, the proof is complete. □

We now consider equation (1.5) on the whole line,

$$\begin{cases} \partial_t^{\beta} u(t,x) = -(-\triangle)^{\frac{\alpha}{2}} u(t,x) + b(u(t,x)) + \sigma \dot{W}(t,x), \; x \in \mathbb{R}, \; t > 0, \\ u(0,x) = u_0(x). \end{cases} \tag{3.5}$$

As before we can interpret equation (3.5) using Walsh theory through the integral equation,

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} G(t,x,y) u_0(y) dy + \int_0^t \int_{\mathbb{R}} G(t-s,x,y) b(u(s,y)) dy ds \\ &+ \sigma \int_0^t \int_{\mathbb{R}} G(t-s,x,y) W(dyds), \end{aligned} \tag{3.6}$$

where $G(t,x,y)$ is the heat kernel associated with Fractional Laplacian defined on the whole line.

Theorem 3.3. *Suppose that the Assumption **(A1)** holds. If the solution of* (3.5) *blows up in finite time a.s., then* b *satisfies the Osgood condition* (1.3).

Proof. For the proof we will apply a similar procedure like that of Theorem 3.1 except we replace the bounded domain $[0,1]$ with $\mathbb{R}$. Set

$$T_e = \sup\{t > 0 : \sup_{x \in \mathbb{R}} |u(t,x)| < \infty\}.$$

Suppose that the solution $u(t,x)$ of equation (3.6) blows up in finite time T_e,

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}} G(t,x,y) u_0(y) dy + \int_0^t \int_{\mathbb{R}} G(t-s,x,y) b(u(s,y)) dy ds \\ &+ \sigma \int_0^t \int_{\mathbb{R}} G(t-s,x,y) W(dyds). \end{aligned}$$

The stochastic term in the above integral formulation is continuous,

$$M = \sup_{x \in \mathbb{R}, t \in [0,T_e]} \Big| \int_0^t \int_{\mathbb{R}} G(t-s,x,y) W(dyds) \Big|. \tag{3.7}$$

Thus, the above quantity is finite a.s.. Since b and the initial condition $u_0(x)$ are non-negative,

$$u(t,x) \geq \sigma \int_0^t \int_{\mathbb{R}} G(t-s,x,y)W(dyds),$$

and hence,

$$-u(t,x) \leq -\sigma \int_0^t \int_{\mathbb{R}} G(t-s,x,y)W(dyds).$$

Thus,

$$\sup_{x\in\mathbb{R},t\in[0,T_e]} \Big(-u(t,x)\Big) \leq \sup_{x\in\mathbb{R},t\in[0,T_e]} \Big(-\sigma \int_0^t \int_{\mathbb{R}} G(t-s,x,y)W(dyds)\Big) = \sigma M.$$

This leads to,

$$\Big(-\sup_{x\in\mathbb{R},t\in[0,T_e]} (-u(t,x))\Big) \geq -\sigma M.$$

Consequently we have,

$$\inf_{x\in\mathbb{R},t\in[0,T_e]} u(t,x) \geq -\sigma M.$$

Again, from boundedness of $u_0(x)$, we have

$$\Big|\int_{\mathbb{R}} G(t,x,y)u_0(y)dy\Big| \leq C \tag{3.8}$$

for some positive constant C.

If we set $A = \{s \in (0,t), y \in (-\infty,\infty) : -\sigma K \leq u(s,y) \leq 0\}$ and $B = \{s \in (0,t), y \in (-\infty,\infty) : u(s,y) > 0\}$, then

$$\int_0^t \int_{\mathbb{R}} G(t-s,x,y)b(u(s,y))dyds = \underbrace{\iint_A G(t-s,x,y)b(u(s,y))dyds}_{I_1} + \underbrace{\iint_B G(t-s,x,y)b(u(s,y))dyds}_{I_2}.$$

Since $b(u(s,y))$ is assumed to be continuous on $[-\sigma M, 0]$ then b is bounded. So $I_1 \leq R$ where, R is a finite quantity almost surely. Again b is non-negative and non-decreasing on $(0,\infty)$. Hence

$$I_2 \leq \int_0^t b(y_s)ds,$$

where $y_t = \sup_{x\in\mathbb{R}} u(t,x)$. Thus,

$$I_1 + I_2 \leq R + \int_0^t b(y_s)ds. \tag{3.9}$$

Putting all these estimate (3.7 -3.9) together we obtain:

$$y_t \leq C + R + \sigma M + \int_0^t b(y_s)ds.$$

Let

$$v_t = (C + R + \sigma M + 1) + \int_0^t b(v_s)ds.$$

By the Comparison theorem $y_t \leq v_t$, and since y_t blows up at time T_e, v_t should also blow up with time T_e. From the proof of Theorem(2.2) and Osgood criteria for integral equation (2.2) b satisfies the Osgood condition (1.3). □

4. Conclusion

In this study, we observed that the behavior of the solution of stochastic Caputo type time fractional differential equation (1.5) blows up in finite time T if and only if b satisfies the Osgood condition (1.3) in the finite interval $[0,1](x \in [0,1])$. We also considered the same equation (1.5) on the whole line ($x \in R$) and obtained that b satisfies the Osgood condition (1.3) whenever the solution of the equation blows up in finite time T_e. For the values $\beta = 1$ and $\alpha = 2$ the equation in (1.5) reduces to the equations in (1.2) and (1.4) maintaining the necessity and sufficiency of the Osgood condition for blow up feature of solutions on the interval [0,1] in finite time and its necessity for blow up of solutions on the whole line in finite time. The work in this paper is essentially an extension of the works in [4] and [2] into fractional setting and hence by virtue of fractional derivatives it captures non-local behaviors.

ACKNOWLEDGMENT

The first author gratefully acknowledge Addis Ababa University Department of Mathematics, International Science Program(ISP), Uppsala University(Sweden) and Simons foundation for their financial support.

REFERENCES

[1] Q. PAVOL, S. PHILIPPE: *Superlinear parabolic problems*, Springer, 2019.

[2] F. MOHAMMUD, N. EULALIA: *The Osgood condition for stochastic partial differential equations*, Bernoulli Society for Mathematical Statistics and Probability, **27** (2021), 295–311.

[3] H. KANTARO: *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proceedings of the Japan Academy, **49** (1973), 503–505.

[4] B.J. FERNÁNDEZ, G. PABLO: *Time–space white noise eliminates global solutions in reaction–diffusion equations*, Physica D: Nonlinear Phenomena, **238**(2) (2009), 209–215.

[5] D. ROBERT C., K. DAVAR, M. CARL, N. DAVID, X. YIMIN: *A minicourse on stochastic partial differential equations*, Springer, 2009.

[6] J.A. LEÓN, J. VILLA: *An Osgood criterion for integral equations with applications to stochastic differential equations with an additive noise,* Statistics & probability letters, **81**(4) (2011), 470–477.

[7] J.B. MIJENAA, E. NANE: *Space–time fractional stochastic partial differential equations*, Stochastic Processes and their Applications, **125**(9) (2015), 3301–3326.

[8] J.B. WALSH: *An introduction to stochastic partial differential equations*, Springer, (1986), 265–439.

[9] H. FUJITA: *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, **13** (1966), 109–124.

[10] R.P. AGARWAL, P. AGARWAL RAVI, V. LAKSHMIKANTHAM: *Uniqueness and nonuniqueness criteria for ordinary differential equations*, World Scientific, 1993.

[11] E. PARDOUX: *Stochastic Partial Differential Equations, Lectures given in Fudan University, Shanghai*, Published by Marseille, France, **2** (2007).

[12] D. KHOSHNEVISAN: *Analysis of stochastic partial differential equations,* **119**, American Mathematical Soc., 2014.

[13] K. DENG, H.A. LEVINE: *The role of critical exponents in blow-up theorems: the sequel*, Journal of Mathematical Analysis and Applications, **243**(1) (2000), 85–126.

[14] L.A. HOWARD: *The role of critical exponents in blowup theorems*, Siam Review, **32**(2) (1990), 262–288.

[15] F. MOHAMMUD, N. ERKAN: *Asymptotic properties of some space-time fractional stochastic equations*, Springer, **287**(1) (2017), 493–519.

[16] F. MOHAMMUD: *Remarks on a fractional-time stochastic equation*, Proceedings of the American Mathematical Society, **149**(5) (2021), 2235–2247.

[17] S.A. ASOGWA, M.B. JEBESSA , N. ERKAN: *Blow-up results for space-time fractional stochastic partial differential equation*, Potential Analysis, **53**(2) (2020), 357–386.

[18] C. MICHELE: *Linear models of dissipation whose Q is almost frequency independent-II*, Geophysical Journal International, **13**(5) (1967), 529–539.

[19] M. KWAŚNICKI: *Ten equivalent definitions of the fractional Laplace operator*, Fractional Calculus and Applied Analysis, **20**(1) (2017), 7–51.

DEPARTMENT OF MATHEMATICS
ADDIS ABABA UNIVERSITY
ADDIS ABABA
ETHIOPIA.
Email address: henokdesalegn@aau.edu.et

DEPARTMENT OF MATHEMATICS
ADDIS ABABA UNIVERSITY
ADDIS ABABA
ETHIOPIA.
Email address: tadesse.abdi@aau.edu.et

DEPARTMENT OF MATHEMATICS
GEORGIA COLLEGE AND STATE UNIVERSITY
MILLEDGEVILLE
USA
Email address: jebessa.mijena@gcsu.edu