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A QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH NORMAL DAMPED RESPONSE FOR THERMO-ELECTRO-ELASTIC-VISCOPLASTIC BODIES

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ABSTRACT. We consider a mathematical problem for quasistatic contact between a thermo-electro–elastic-viscoplastic body and an obstacle. The contact is modeled by a general normal damped response condition with friction law and heat exchange. We present a variational formulation of the problem and prove the existence and uniqueness of the weak solution. The proof is based on the formulation of four intermediate problems for the displacement field, the electric potential field and the temperature field, respectively. We prove the unique solvability of the intermediate problems, then we construct a contraction mapping whose unique fixed point is the solution of the original problem.

1. INTRODUCTION

Phenomena of contact abound in industry and everyday life, especially in motors, engines, and transmissions: the contact of the braking pads with the wheel, the tire with the road and the piston with skirt are just three simple examples. For this reason, considerable literature is devoted to these topics. The early preliminary study of contact problems within the framework of variational inequalities was made in the books [9, 12, 14].

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In this work we consider a general model for the quasistatic process of thermo-electro-elastic-viscoplastic contact between a deformable body and a rigid obstacle. Many crystalline materials like the quartz (also ceramics (BaTiO3, KNbO3, LiNbO3,...) and even the human mandible), exhibit piezoelectric behavior to produce a voltage when they are subjected to mechanical stress. The piezoelectric effect is characterized by the coupling between the mechanical and the electrical properties of the material. This coupling, leads to the appearance of electric field in the presence of a mechanical stress, and conversely, mechanical stress is generated when electric potential is applied. The first effect is used in sensors, and the reverse effect is used in actuators. contact problems have been investigated frictional or frictionless involving piezoelectric materials, see for instance [5, 16, 18] and the references therein.

Contact problems involving the coupling between thermal and mechanical fields are considered in [6, 17]. Dynamic elastic or viscoelastic frictional contact problems, with thermal considerations, can be found in [2, 13], and the references therein. In particular, one-dimensional thermal problems for rods and beams have been investigated in [3, 10, 15]. Recent existence and uniqueness results for various quasistatic contact problems can be found in the extensive review [19] .

The novelty of this paper is the study of a model that describes the interaction between thermal, mechanical and electrical fields, in the normal damped response conditions, with the corresponding frictional condition. Numerous examples of normal damped response contact condition may be found in [1, 8, 11].

The rest of the article is structured as follows. In Section 2 we present contact model and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. We prove in Section 4 the existence and uniqueness of the solution.

2. Problem statement

The physical setting is the following. A body occupies the domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with outer Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. such that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. We denote by ν the unit outer normal on Γ. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$ and the displacement vanishes there.

Surface tractions of density f_2 act on $\Gamma_2 \times (0,T)$ and a volume force of density f_0 is applied in $\Omega \times (0,T)$.

We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0,T)$. On Γ_3 the potential contact surface, the body is in contact with an insulator obstacle, the so-called foundation.

The classical formulation of the mechanical problem of electro elastic-viscoplastic with thermal effects, be stated as follows.

Problem $\boldsymbol{P}$. Find a displacement field $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$, and an electric displacement field $\boldsymbol{D} : \Omega \times (0,T) \to \mathbb{R}^d$ such that

$$\begin{aligned}\boldsymbol{\sigma} = \mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}\right) + \mathcal{B}\varepsilon\left(\boldsymbol{u}\right) - \mathcal{E}^{*}E(\varphi) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}(s)\right)\right. \\ \left. + \mathcal{E}^{*}E(\varphi)(s), \varepsilon\left(\boldsymbol{u}(s)\right)\right) ds - C_e\theta \quad \text{in } \Omega \times (0,T),\end{aligned} \tag{2.1}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}E(\varphi), \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$\dot{\theta} - \operatorname{div} K(\Delta\theta) = r(\dot{\boldsymbol{u}}) + \mathbf{q}, \quad \text{in } \Omega \times (0,T), \tag{2.3}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0, \quad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0, \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

$$\boldsymbol{u} = \mathbf{0}, \quad \text{on } \Gamma_1 \times (0,T), \tag{2.6}$$

$$\sigma\nu = f_2, \quad \text{on } \Gamma_2 \times (0,T), \tag{2.7}$$

$$-\sigma_{\nu 6} = p_\nu\left(\dot{u}_\nu\right), \quad \text{on } \Gamma_3 \times (0,T) \tag{2.8}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \leq p_\tau\left(\dot{u}_\nu\right) \\ \boldsymbol{\sigma}_\tau = -p_\tau\left(\dot{u}_\nu\right) \dfrac{\dot{\boldsymbol{u}}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \quad \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0} \end{cases}, \quad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$-k_{ij}\frac{\partial \theta}{\partial x_i}\nu_j = k_e\left(\theta - \theta_R\right) + h_\tau\left(|\dot{\mathbf{u}}_\tau|\right), \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\varphi = 0, \quad \text{on } \Gamma_a \times (0,T), \tag{2.11}$$

$$\boldsymbol{D}.\nu = q_2, \quad \text{on } \Gamma_b \times (0,T), \tag{2.12}$$

$$\theta = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{2.13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \quad \text{in } \Omega. \tag{2.14}$$

First, equations (2.1)-(2.3) represent the electro-elastic-viscoplastic constitutive law with thermal effects, were $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{G}$ are, respectively, nonlinear operators describing the purely viscous, the elastic and the viscoplastic properties of the material, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represent the third order piesoelectric tensor, $\mathcal{E}^*$ is its transposition and $\boldsymbol{B}$ denotes the electric permittivity tensor, θ represent the temperature, $C_e = (c_{ij})$ represents the thermal expansion tensor, K represent the thermal conductivity tensor, $\mathbf{q}$ represent the density of volume heat source and r is non linear function of velocity.

Equations (2.4) and (2.5) represent the equilibrium equations for the stress and electric displacement fields. Equations (2.6)-(2.7) are the displacement-traction conditions.

frictional contact conditions with normal damped response of the form (2.8) and (2.9) represent an appropriate version of Coulomb's law of friction. This condition states a general dependence of the normal stress σ_ν on the normal velocity $\dot{u}_\nu$, which represents the possible behavior of a layer of lubricant on the contact surface. Here p_ν and p_τ represent given contact functions. Here, again, the tangential shear stress cannot exceed the maximal frictional resistance p_τ. When the strict inequality holds the surface adheres to the foundation and is in the so-called stick state; and when the equality holds then there is relative sliding between the surface and the foundation; this is the so-called slip state.

This type of contact case has been considered by several authors (See for example [20])

(2.10), represent, on Γ, a Fourier boundary condition for the temperature. (2.11) and (2.12) represent the electric boundary conditions. Equation (2.13) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0, T)$. Finally, The functions $\boldsymbol{u}_0$ and θ_0 in (2.14) are the initial data.

3. VARIATIONAL FORMULATION AND PRELIMINARIES

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, $(d = 2, 3)$ while $(.,.)$ and $\|.\|$ represent the inner product and Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ respectively. Let $\Omega \subset \mathbb{S}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We define the function spaces

$$H = L^2(\Omega)^d = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \quad H_1 = \{\boldsymbol{u} = (u_i) \mid \varepsilon(\boldsymbol{u}) \in \mathcal{H}\},$$
$$\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}.$$

Here ε and Div are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}), \quad \operatorname{Div}(\boldsymbol{\sigma}) = \sigma_{ij,j}.$$

Here and below, the indices i and j run between 1 and d. the summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable

The sets H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$
$$(\mathbf{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H_1,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau})_H, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms are denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$. For every element $\boldsymbol{u} \in H_1$, we denote by u_ν and $\boldsymbol{u}_\tau$ the normal and tangential components of

$\boldsymbol{u}$ on Γ given by

$$u_\nu = \boldsymbol{u}.\boldsymbol{\nu}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}.$$

Similarly, for a regular tensor field $\boldsymbol{\sigma} \in \mathcal{H}_1$ we define its normal and tangential components by

$$\sigma_\nu = \sigma\nu \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu},$$

and we recall that the following Green's formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu.\boldsymbol{v} da, \quad \forall \boldsymbol{v} \in H_1,$$

where da is the surface measure element. Now, let $\mathcal{X}$ denote the closed subspace of $H^1(\Omega)$ given by

$$\mathcal{X} = \left\{ \gamma \in H^1(\Omega) \mid \gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\},$$

and we denote by $\mathcal{X}'$ the dual space of $\mathcal{X}$.

Let V denote the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1 \right\}.$$

Since meas $(\Gamma_1) > 0$, the following Korn's inequality holds,

$$\|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}} \geq C_0 \|\boldsymbol{v}\|_{H^1(\Omega)^d}, \quad \forall \boldsymbol{v} \in V,$$

where the constant $C_0 > 0$, depends only on Ω and Γ_1.

On V, we consider the inner product and the associated norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \ \|\boldsymbol{v}\|_V = \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}, \ \boldsymbol{u}, \boldsymbol{v} \in V. \tag{3.1}$$

It follows from Korn's inequality that the norms $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent on V and therefore $(V, (.,.)_V)$ is a real Hilbert space.

We also introduce the spaces

$$W = \left\{ \xi \in H^1(\Omega), \xi = 0 \text{ on } \Gamma_a \right\},$$
$$\mathcal{W} = \left\{ \mathbf{D} \in H \mid \mathrm{div}\, \mathbf{D} \in L^2(\Omega) \right\},$$

where $\mathrm{div}\, \mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \xi)_W = \int_\Omega \nabla\varphi.\nabla\xi dx,$$
$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega \mathbf{D} \cdot \mathbf{E} dx + \int_\Omega \mathrm{div}\, \mathbf{D} \cdot \mathrm{div}\, \mathbf{E} dx.$$

The associated norms will be denoted by $\|.\|_W$ and $\|.\|_{\mathcal{W}}$, respectively.

Since meas $(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds:

$$\|\nabla\zeta\|_H \geq c_F\|\zeta\|_{H^1(\Omega)}, \quad \forall\zeta \in W, \tag{3.2}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. It follows from (3.2) that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W and therefore $(W, \|.\|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace theorem, there exist two positive constants c_0 and $\tilde{c}_0$ such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0\|\boldsymbol{v}\|_V, \quad \forall\boldsymbol{v} \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0\|\psi\|_W, \quad \forall\psi \in W. \tag{3.3}$$

Moreover, when $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds

$$(\boldsymbol{D}, \nabla\zeta)_H + (\operatorname{div}\boldsymbol{D}, \zeta)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D}\cdot\boldsymbol{\nu}\zeta da, \quad \forall\zeta \in H^1(\Omega). \tag{3.4}$$

For any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$ and $k \geq 1$. For $T > 0$ we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\boldsymbol{f}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{f}(t)\|_X,$$

$$\|\boldsymbol{f}\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{f}(t)\|_X + \max_{t\in[0,T]} \|\dot{\boldsymbol{f}}(t)\|_X.$$

Now we introduce assumptions on the data in the study of Problem P. For the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$, we assume

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \qquad \|\mathcal{A}(\boldsymbol{x},\varsigma_1) - \mathcal{A}(\boldsymbol{x},\varsigma_2)\| \leq L_{\mathcal{A}}\|\varsigma_1 - \varsigma_2\|, \text{ for all } \varsigma_1, \varsigma_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ \qquad (\mathcal{A}(\boldsymbol{x},\varsigma_1) - \mathcal{A}(\boldsymbol{x},\varsigma_2)).(\boldsymbol{\varepsilon}_1 - \varsigma_2) \geq m_{\mathcal{A}}\|\varsigma_1 - \varsigma_2\|^2, \\ \quad \text{for all } \varsigma_1, \varsigma_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\varsigma) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \varsigma \in \mathbb{S}^d. \\ (d) \text{ The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\mathbf{0}) \in \mathcal{H}. \end{cases} \tag{3.5}$$

For the elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$, we assume

$$(3.6)\quad \begin{cases} \text{(a) There exists } L_{\mathcal{B}} > 0 \quad \text{such that} \\ \qquad \|\mathcal{B}(\boldsymbol{x}, \varsigma_1) - \mathcal{B}(\boldsymbol{x}, \varsigma_2)\| \leq L_{\mathcal{B}} \|\varsigma_1 - \varsigma_2\|, \text{ for all } \varsigma_1, \varsigma_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \varsigma) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \varsigma \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \mathbf{0}) \in \mathcal{H}. \end{cases}$$

For the visco-plasticity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$, we assume

$$(3.7)\quad \begin{cases} \text{(a) There exists a constant } L_{\mathcal{G}} > 0 \quad \text{such that} \\ \qquad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \varsigma_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varsigma_2)\| \leq L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\varsigma_1 - \varsigma_2\|), \\ \qquad \text{for all } t \in (0,T), \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varsigma_1, \varsigma_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \varsigma) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\sigma}, \varsigma \in \mathbb{S}^d, t \in (0,T), \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \in \mathcal{H}. \end{cases}$$

For the thermal expansion operator $C_e : \Omega \times \mathbb{R} \to \mathbb{R}$, we assume

(3.8)

$$\begin{cases} \text{(a) There exists } L_{C_e} > 0 \text{ such that} \\ \qquad \|C_e(\boldsymbol{x}, \mu_1) - C_e(\boldsymbol{x}, \mu_2)\| \leq L_{C_e} \|\mu_1 - \mu_2\| \text{ for all } \mu_1, \mu_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } C_e = (c_{ij}), c_{ij} = c_{ji} \in L^\infty(\Omega). \\ \text{(c) The mapping } \boldsymbol{x} \mapsto C_e(\boldsymbol{x}, \mu) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \mu \in \mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto C_e(\boldsymbol{x}, 0) \in \mathcal{H}. \end{cases}$$

For the thermal conductivity operator $K : \Omega \times \mathbb{R} \to \mathbb{R}$, we assume

(3.9)

$$\begin{cases} \text{(a) There exists } L_K > 0 \text{ such that} \\ \qquad \|K(\boldsymbol{x}, \mu_1) - K(\boldsymbol{x}, \mu_2)\| \leq L_K \|\mu_1 - \mu_2\|, \text{ for all } \mu_1, \mu_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } k_{ij} = k_{ji} \in L^\infty(\Omega), k_{ij}\alpha_i\alpha_j \leq c_k\alpha_i\alpha_j \text{ for some } c_k > 0, \\ \qquad \text{for all } (\alpha_i) \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto k(\boldsymbol{x}, 0) \text{ belongs to } L^2(\Omega). \end{cases}$$

For the electric permittivity operator $\boldsymbol{B} = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, we assume

$$(3.10)\quad \begin{cases} \text{(a) } \boldsymbol{B}(\boldsymbol{\varepsilon}, E) = (b_{ij}(\boldsymbol{\varepsilon})E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{\varepsilon} \in \Omega. \\ \text{(b) } b_{ij} = b_{ji} \in L^\infty(\Omega), 1 \le i, j \le d. \\ \text{(c) There exists a constant } m_{\boldsymbol{B}} > 0 \text{ such that} \\ \qquad \boldsymbol{B}E.E \ge m_{\boldsymbol{B}} \|E\|^2, \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega. \end{cases}$$

For the piezoelectric operator $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$, we assume

$$(3.11)\quad \begin{cases} (a)\ \mathcal{E} = (f_{ijk}), f_{ijk} \in L^\infty(\Omega), 1 \le i, j, k \le d. \\ (b)\ \mathcal{E}(\mathrm{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{\tau}, \text{ for all } \boldsymbol{\sigma} \in \mathbb{S}^d, \text{ and all } \boldsymbol{\tau} \in \mathbb{R}^d. \end{cases}$$

For the tangential function $p_e : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $e = \nu, \tau$, we assume

$$(3.12)\quad \begin{cases} \text{(a) There exists } L_e > 0 \text{ such that} \\ \qquad \|p_e(\boldsymbol{x}, \mu_1) - p_e(\boldsymbol{x}, \mu_2)\| \le L_e \|\mu_1 - \mu_2\| \\ \qquad \text{for all } \mu_1, \mu_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(d) For any } \mu \in \mathbb{R}, \boldsymbol{x} \mapsto p_e(\boldsymbol{x}, \mu) \text{ is Lebesgue measurable on } \Gamma_3 \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_e(\boldsymbol{x}, 0) \text{ belongs to } L^2(\Gamma_3). \end{cases}$$

We assume that the boundary and initial data θ_R, k_e, $\boldsymbol{u}_0$ and θ_0 the volume of forces f_0 and f_2 and the charges densities q_0, q_2, the heat source density $\mathbf{q}$, satisfy

$$(3.13)\quad \theta_R \in C(0, T; L^2(\Gamma_3)),\ k_e \in L^\infty(\Omega, \mathbb{R}_+),$$

$$(3.14)\quad \boldsymbol{u}_0 \in V, \quad \theta_0 \in \mathcal{X},$$

$$(3.15)\quad f_0 \in C\left(0, T; L^2(\Omega)^d\right),\ f_2 \in C\left(0, T; L^2\left(\Gamma_2\right)^d\right),$$

$$(3.16)\quad q_0 \in C\left(0, T; L^2(\Omega)\right),\ q_2 \in C\left(0, T; L^2\left(\Gamma_b\right)\right),$$

$$(3.17)\quad \mathbf{q} \in C\left(0, T; L^2\left(\Omega\right)\right).$$

Finally, for the function $r : V \to L^2(\Omega)$ satisfies that there exists a constant $L_r > 0$ such that

$$(3.18)\quad \|r(\boldsymbol{u}_1) - r(\boldsymbol{u}_2)\|_{L^2(\Omega)} \le L_r \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V, \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in V.$$

Next. We define six mappings $j : V \times V \to \mathbb{R}$, $f : [0,T] \to V$, $q : [0,T] \to W$, $Q : [0,T] \to \mathcal{X}'$, $K : \mathcal{X} \to \mathcal{X}'$, and $R : V \to \mathcal{X}'$ respectively, by

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu (u_\nu) v_\nu \mathrm{da} + \int_{\Gamma_3} p_\tau (u_\nu) \|\boldsymbol{v}_\tau\| \mathrm{da}, \tag{3.19}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} da, \tag{3.20}$$

$$(q(t), \zeta)_W = \int_\Omega q_0(t) \zeta dx - \int_{\Gamma_b} q_2(t) \zeta da, \tag{3.21}$$

$$(Q(t), y)_{\mathcal{X}' \times \mathcal{X}} = \int_{\Gamma_3} k_e \theta_R(t) y da + \int_\Omega q(t) y dx, \tag{3.22}$$

$$(K\tau, y)_{\mathcal{X}' \times \mathcal{X}} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial y}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau y da, \tag{3.23}$$

$$(R\boldsymbol{v}, y)_{\mathcal{X}' \times \mathcal{X}} = \int_\Omega r(\boldsymbol{v}) y dx + \int_{\Gamma_3} h_\tau (|\boldsymbol{v}_\tau|) y da. \tag{3.24}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\zeta \in W$, $y, \tau \in \mathcal{X}$ and $t \in [0,T]$. Note that

$$\boldsymbol{f} \in C(0, T; V), \quad q \in C(0, T; W). \tag{3.25}$$

By using a standard arguments, we obtain the following variational formulation of the mechanical problem (2.1)-(2.14).

problem $\boldsymbol{PV}$. Find a displacement field $\boldsymbol{u} : (0,T) \to V$, a stress field $\boldsymbol{\sigma} : (0,T) \to \mathcal{H}$, an electric potential $\varphi : (0,T) \to W$, and a temperature $\theta : (0,T) \to \mathcal{X}$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t) \\ &+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)) - \mathcal{E}^*\nabla\varphi(s), \varepsilon(\boldsymbol{u}(s))) ds - C_e\theta(t), \end{aligned} \tag{3.26}$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} + j(\dot{\boldsymbol{u}}(t), \boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)) \geq (\mathbf{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V, \tag{3.27}$$

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}(t)), \nabla\phi)_H = (q(t), \phi)_W, \quad \forall \phi \in W, t \in (0,T) \tag{3.28}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{\mathbf{u}}(t) + Q(t), \quad \text{in } \mathcal{X}', \tag{3.29}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0. \tag{3.30}$$

Our main existence and uniqueness result, which we state now and prove in the next section, is the following

4. EXISTENCE AND UNIQUENESS

Theorem 4.1. *Assume that (3.5)-(3.18) hold, Then there exists a unique solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \theta, \boldsymbol{D})$ *to problem* PV. *Moreover, the solution has the regularity*

$$\boldsymbol{u} \in C^1(0,T;V), \tag{4.1}$$

$$\varphi \in C(0,T;W), \tag{4.2}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}), \tag{4.3}$$

$$\theta \in C\left(0,T;L^2(\Omega)\right) \cap L^2(0,T;\mathcal{X}) \cap W^{1,2}\left(0,T;\mathcal{X}'\right), \tag{4.4}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{4.5}$$

We note that elements $\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, θ, and $\boldsymbol{D}$ which solves Problem PV is a weak solution of the contact Problem P. Theorem 4.1 thus states that the contact Problem P has a unique weak solution, provided that (3.5)-(3.18) hold.

The proof of Theorem 4.1, is carried out is several steps and is based on arguments of evolutionary quasivariational inequalities, differential equations and fixed points.

We denote by C a constant whose value may change from line to line when no confusing can arise.

Let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$, and consider the auxiliary problem.

Problem $\mathcal{P}^1_{\boldsymbol{\eta}}$. Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ such that for all $t \in [0,T]$

$$\begin{aligned}
&\left(\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_\eta(t)\right), \varepsilon\left(\boldsymbol{v}\right) - \varepsilon(\dot{\boldsymbol{u}}_\eta(t))\right)_{\mathcal{H}} + \left(\mathcal{B}\varepsilon\left(\boldsymbol{u}_\eta(t)\right), \varepsilon\left(\boldsymbol{v}\right) - \varepsilon(\dot{\boldsymbol{u}}_\eta(t))\right)_{\mathcal{H}} \\
&+ \left(\boldsymbol{\eta}(t), \varepsilon(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}_\eta(t))\right)_{\mathcal{H}} + j(\dot{\boldsymbol{u}}_\eta(t), \boldsymbol{v}) - j\left(\dot{\boldsymbol{u}}_\eta(t), \dot{\boldsymbol{u}}_\eta(t)\right) \\
&\geq \left(\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)\right)_V, \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T),
\end{aligned} \tag{4.6}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.7}$$

Lemma 4.1. *There exists a unique solution* $\boldsymbol{u}_\eta \in C^1(0,T;V)$ *to the problem (4.6) and (4.7).*

Proof. Let us introduce operators $A : V \to V$ and $B : V \to V$

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{4.8}$$

$$(B\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{B}\varepsilon(\boldsymbol{u})\varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.9}$$

Therefore, (4.6) can be rewritten as follows

$$
\begin{aligned}
&(A\dot{\boldsymbol{u}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V + (B\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V + j(\dot{\boldsymbol{u}}_\eta(t), \boldsymbol{v}) \\
&- j(\dot{\boldsymbol{u}}_\eta(t), \dot{\boldsymbol{u}}_\eta(t)) \geq (\mathbf{f}_\eta(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V\,,
\end{aligned} \tag{4.10}
$$

where

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \boldsymbol{\eta}(t), \quad \text{a.e.t } \in [0, T].$$

Using (4.8)-(4.9) and (3.5)-(3.6) it follows that A and B are Lipschitz continuous operators. Using again (4.8) and (3.5) we deduce that A is a strongly monotone operator on V,

$$
\begin{aligned}
(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V &= (\mathcal{A}(\varepsilon(\boldsymbol{u}_1)) - \mathcal{A}(\varepsilon(\boldsymbol{u}_2)), \varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2))_{\mathcal{H}} \\
&\geq m_{\mathcal{A}} \|\varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2)\|_{\mathcal{H}}^2 \geq C \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2\,.
\end{aligned}
$$

It follows from (3.12) that the functional j defined in (3.19) is continuous and, therefore, it is convex lower semicontinuous function on V.

Finally, note that $\mathbf{f}_\eta \in C([0, T]; V)$ and $\boldsymbol{u}_0 \in V$ and we use classical arguments of functional analysis concerning evolutionary quasivariational inequality [, 20] that there exists a unique solution $\boldsymbol{u}_\eta \in C^1(0, T; V)$ to the problem $\mathcal{P}_\eta^1$ □

In the next step we use the solution $\boldsymbol{u}_\eta$, obtained in Lemma 4.1, to construct the following variational problem for the electrical potential.

Problem $\mathcal{P}_{\boldsymbol{\eta}}^{\mathbf{2}}$. Find an electrical potential $\varphi_\eta : (0, T) \to W$ such that

$$(B\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\zeta)_H = (q(t), \zeta)_W, \ \forall \zeta \in W, t \in (0, T). \tag{4.11}$$

We have the following result

Lemma 4.2. *Problem (4.11) has unique solution φ_η which satisfies the regularity (4.2). Moreover, if $\varphi_{\boldsymbol{\eta}_1}$ and $\varphi_{\boldsymbol{\eta}_2}$ are the solutions of (4.11) corresponding to $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C([0, T]; \mathcal{H})$, then there exists $C > 0$ such that*

$$\left\|\varphi_{\boldsymbol{\eta}_1}(t) - \varphi_{\boldsymbol{\eta}_2}(t)\right\|_W \leq C \left\|\boldsymbol{u}_{\boldsymbol{\eta}_1}(t) - \boldsymbol{u}_{\boldsymbol{\eta}_2}(t)\right\|_V, \quad \forall t \in [0, T]. \tag{4.12}$$

Proof. we consider the form $S : W \times W \to \mathbb{R}$

$$S(\varphi, \phi) = (\mathbf{B}\nabla\varphi, \nabla\phi)_H\,, \quad \forall \varphi, \phi \in W, \tag{4.13}$$

we use (3.2), (3.10), (4.13) and defined $(\varphi, \psi)_W$ to show that the form S is bilinear continuous, symmetric and coercive on W, moreover using (3.21) and the Riesz representation Theorem we may define an element $\xi_\eta : [0,T] \to W$ such that

$$(\xi_\eta(t), \phi)_W = (q(t), \phi)_W + (\mathcal{E}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\phi)_H, \quad \forall \phi \in W, t \in (0,T),$$

we apply the Lax-Milgram Theorem to conclude that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$S(\varphi_\eta(t), \phi) = (\xi_\eta(t), \phi)_W, \quad \forall \phi \in W. \tag{4.14}$$

It follows from (4.14) that φ_η is a solution of the equation (4.11). Let $\varphi_{\eta_i} = \varphi_i$, and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1, 2$. We use (4.11) to obtain

$$\|\varphi_1(t) - \varphi_2(t)\|_W \leq C\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V, \quad \forall t \in [0,T].$$

Now since $\boldsymbol{u}_\eta \in C^1(0,T;V)$, so implies that $\varphi_\eta \in C(0,T;W)$. This completes the proof. □

In the third step, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.1 to consider the following variational problem.

Problem $\mathcal{P}_\eta^3$. Find the temperature field $\theta_\eta : (0,T) \to L^2(\Omega)$

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{\boldsymbol{u}}_\eta(t) + Q(t), \quad \text{in } \mathcal{X}', \quad \text{a.e.t } \in [0,T], \tag{4.15}$$

$$\theta_\eta(0) = \theta_0. \tag{4.16}$$

Lemma 4.3. *There exists a unique solution θ_η to the auxiliary problem $\mathcal{P}_\eta^3$ satisfying (4.4).*

Proof. The result follows from classical first order evolution equation given in Refs. [4, 21]. Here the Gelfand triple is given by

$$\mathcal{X} \subset L^2(\Omega) = \left(L^2(\Omega)\right)' \subset \mathcal{X}'.$$

The operator K is linear and coercive. By Korn's inequality, we have

$$(K\tau, \tau)_{\mathcal{X}' \times \mathcal{X}} \geq C\|\tau\|_{\mathcal{X}}^2.$$

□

In the fourth step, we use $\boldsymbol{u}_\eta$, φ_η and θ_η obtained in Lemmas 4.1, 4.2 and 4.3, respectively to construct the following Cauchy problem for the stress field.

Problem $\mathcal{P}_{\eta}^{4}$. Find the stress field $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ which is a solution of the problem

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{B}\left(\varepsilon\left(\mathbf{u}_\eta(t)\right)\right) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}_\eta(s), \varepsilon\left(\mathbf{u}_\eta(s)\right)\right) ds - C_e\theta_\eta(t), \text{ a.e. } t \in (0,T). \tag{4.17}$$

Lemma 4.4. *$\mathcal{P}_\eta^4$ has a unique solutions $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H})$. Moreover, if $\boldsymbol{\sigma}_{\eta_i}$, $\boldsymbol{u}_{\eta_i}$ and θ_{η_i} represent the solutions of Problems $\mathcal{P}_\eta^4$, $\mathcal{P}_\eta^1$ and $\mathcal{P}_\eta^3$ respectively, for $\boldsymbol{\eta}_i \in C(0,T;\mathcal{H})$, $i = 1,2$, then there exists $C > 0$ such that*

$$\begin{aligned} \|\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\eta_2}(t)\|_{\mathcal{H}}^2 \le C \Big(& \|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V^2 \\ & + \|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|_V^2 \Big). \end{aligned} \tag{4.18}$$

Proof. Let $\boldsymbol{\Sigma}_\eta : C(0,T;\mathcal{H}) \to C(0,T;\mathcal{H})$ be the operator given by

$$\boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}(t) = \mathcal{B}\left(\varepsilon\left(\mathbf{u}_\eta(t)\right)\right) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}_\eta(s), \varepsilon\left(\mathbf{u}_\eta(s)\right)\right) ds - C_e\theta_\eta(t), \tag{4.19}$$

Let $\boldsymbol{\sigma}_i \in C(0,T;\mathcal{H})$, $i = 1,2$ and $t_1 \in (0,T)$. Using hypothesis (3.7) and Holder's inequality, we find

$$\|\boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_1(t_1) - \Sigma_\eta\boldsymbol{\sigma}_2(t_1)\|_{\mathcal{H}}^2 \le L_{\mathcal{G}}^2 T \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds.$$

Integration on the time interval $(0,t_2) \subset (0,T)$, it follows that

$$\int_0^{t_2} \|\boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_1(t_1) - \boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_2(t_1)\|_{\mathcal{H}}^2 dt_1 \le L_{\mathcal{G}}^2 T \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds dt_1.$$

Therefore

$$\|\boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_1(t_2) - \boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_2(t_2)\|_{\mathcal{H}}^2 \le L_{\mathcal{G}}^4 T^2 \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds dt_1.$$

For $t_1, t_2, \ldots, t_p \in (0,T)$, we generalize the procedure above by recurrence on p. We obtain the inequality

$$\begin{aligned} & \|\boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_1(t_p) - \boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_2(t_p)\|_{\mathcal{H}}^2 \\ & \le L_{\mathcal{G}}^{2p} T^p \int_0^{tp} \cdots \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds dt_1 \ldots dt_{p-1}. \end{aligned}$$

Which implies

$$\|\boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_1(t_p) - \boldsymbol{\Sigma}_\eta\boldsymbol{\sigma}_2(t_p)\|_{\mathcal{H}}^2 \le \frac{L_{\mathcal{G}}^{2p} T^{p+1}}{p!} \int_0^T \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\|\mathbf{\Sigma}_\eta\boldsymbol{\sigma}_1-\mathbf{\Sigma}_\eta\boldsymbol{\sigma}_2\|^2_{C(0,T;\mathcal{H})}\leq\frac{L_{\mathcal{G}}^{2p}T^{p+2}}{p!}\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|^2_{C(0,T;\mathcal{H})}.$$

It follows from this inequality that for large p enough, the operator $\mathbf{\Sigma}^p_\eta$ is a contraction on the Banach space $C(0,T;\mathcal{H})$, and therefore there exists a unique element $\boldsymbol{\sigma}\in C(0,T;\mathcal{H})$ such that $\mathbf{\Sigma}_\eta\boldsymbol{\sigma}=\boldsymbol{\sigma}$. Moreover, $\boldsymbol{\sigma}$ is the unique solution of Problem $\mathcal{P}^4_\eta$, and using (4.17), the regularity of $\boldsymbol{u}_\eta$, θ_η, and the properties of the operators $\mathcal{B}$, $\mathcal{G}$ and C_e it follows that $\boldsymbol{\sigma}\in C(0,T;\mathcal{H})$. Consider now $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in C(0,T;\mathcal{H})$ and for $i=1,2$, denote $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\theta_{\eta_i}=\theta_i$, and $\boldsymbol{\sigma}_{\eta_i}=\boldsymbol{\sigma}_i$. We have

$$\begin{aligned}\boldsymbol{\sigma}_i(t)&=\mathcal{B}\left(\varepsilon\left(\boldsymbol{u}_i(t)\right)\right)\\&\quad+\int_0^t\mathcal{G}\left(\boldsymbol{\sigma}_i(s),\varepsilon\left(\boldsymbol{u}_i(s)\right)\right)ds-C_e\theta_i(t),\qquad\text{a.e. } t\in(0,T).\end{aligned}\tag{4.20}$$

and using the properties (3.6), (3.7) and (3.8) of $\mathcal{B}$, $\mathcal{G}$ and C_e we find

$$\begin{aligned}&\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}}\leq C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}\\&+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V\,ds+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,ds\Big),\quad\forall t\in[0,T].\end{aligned}\tag{4.21}$$

We use Gronwall argument in the previous inequality to deduce (4.18), which concludes the proof of Lemma 4.4. □

Finally, as a consequence of these results and using the properties of the operators $\mathcal{G}$, $\mathcal{E}$ and C_e, for $t\in(0,T)$, we consider the element $\Lambda\boldsymbol{\eta}(t)\in\mathcal{H}$ defined by

$$\begin{aligned}(\Lambda\boldsymbol{\eta}(t),\boldsymbol{v})_{\mathcal{H}\times V}&=(\mathcal{E}^*\nabla\varphi_\eta(t),\varepsilon(\boldsymbol{v}))_{\mathcal{H}}+(C_e\theta_\eta(t),\varepsilon(\boldsymbol{v}))_{\mathcal{H}}\\&\quad+\left(\int_0^t\mathcal{G}\left(\boldsymbol{\sigma}_\eta,\boldsymbol{\varepsilon}\left(\boldsymbol{u}_\eta(s)\right)\right)ds,\varepsilon(\boldsymbol{v})\right)_{\mathcal{H}},\forall\boldsymbol{v}\in V,\end{aligned}\tag{4.22}$$

Here, for every $\boldsymbol{\eta}\in C(0,T;\mathcal{H})$. $\boldsymbol{u}_\eta$, φ_η, θ_η and $\boldsymbol{\sigma}_\eta$ represent the displacement field, the electric potential field, the temperature field and the stress field, obtained in Lemmas 4.1, 4.2, 4.3 and 4.4 respectively. We have the following result.

Lemma 4.5. *The mapping Λ has a fixed point $\boldsymbol{\eta}^*\in C(0,T;\mathcal{H})$, such that $\Lambda\boldsymbol{\eta}^*=\boldsymbol{\eta}^*$.*

Proof. Let $t\in(0,T)$ and $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in C(0,T;\mathcal{H})$. We use the notation that $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i}=\dot{\boldsymbol{u}}_i$, $\theta_{\eta_i}=\theta_i$, $\varphi_{\eta_i}=\varphi_i$, and $\boldsymbol{\sigma}_{\eta_i}=\boldsymbol{\sigma}_i$ for $i=1,2$.

Let us start by using (3.3), (3.7), (3.8) and (3.11), we have

$$
\begin{aligned}
&\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}}^2 \le \|\mathcal{E}^*\nabla\varphi_1(t)-\mathcal{E}^*\nabla\varphi_2(t)\|_{\mathcal{H}}^2 \\
&\quad + \|C_e\theta_1(t))-C_e\theta_2(t)\|_{\mathcal{H}}^2 \\
&\quad + \int_0^t \|\mathcal{G}\left(\boldsymbol{\sigma}_1(s),\varepsilon(\boldsymbol{u}_1(s))\right)-\mathcal{G}\left(\boldsymbol{\sigma}_2(s),\varepsilon(\boldsymbol{u}_2(s))\right)\|_{\mathcal{H}}^2\,ds \\
&\le C\Big(\|\varphi_1(t)-\varphi_2(t)\|_W^2+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2 \\
&\quad + \int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\,ds+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\Big).
\end{aligned}
\tag{4.23}
$$

We use estimates (4.18), (4.12) to obtain

$$
\begin{aligned}
\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}}^2 \le C\Big(&\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2 \\
&+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\Big).
\end{aligned}
\tag{4.24}
$$

Using inequality (4.6) for $\boldsymbol{\eta}=\boldsymbol{\eta}_1$, we find

$$
\begin{aligned}
&\left(\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_1(t)\right),\varepsilon\left(\boldsymbol{v}\right)-\varepsilon(\dot{\boldsymbol{u}}_1(t))\right)_{\mathcal{H}}+\left(\mathcal{B}\varepsilon\left(\boldsymbol{u}_1(t)\right),\varepsilon\left(\boldsymbol{v}\right)-\varepsilon(\dot{\boldsymbol{u}}_1(t))\right)_{\mathcal{H}} \\
&+\left(\boldsymbol{\eta}_1(t),\varepsilon(\boldsymbol{v})-\varepsilon(\dot{\boldsymbol{u}}_1(t))\right)_{\mathcal{H}}+j(\dot{\boldsymbol{u}}_1(t),\boldsymbol{v})-j\left(\dot{\boldsymbol{u}}_1(t),\dot{\boldsymbol{u}}_1(t)\right) \\
&\ge\left(\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_1(t)\right)_V,\forall\boldsymbol{v}\in V,\ \text{a.e. } t\in(0,T),
\end{aligned}
\tag{4.25}
$$

for $\boldsymbol{\eta}=\boldsymbol{\eta}_2$, we find

$$
\begin{aligned}
&\left(\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_2(t)\right),\varepsilon\left(\boldsymbol{v}\right)-\varepsilon(\dot{\boldsymbol{u}}_2(t))\right)_{\mathcal{H}}+\left(\mathcal{B}\varepsilon\left(\boldsymbol{u}_2(t)\right),\varepsilon\left(\boldsymbol{v}\right)-\varepsilon(\dot{\boldsymbol{u}}_2(t))\right)_{\mathcal{H}} \\
&+\left(\boldsymbol{\eta}_2(t),\varepsilon(\boldsymbol{v})-\varepsilon(\dot{\boldsymbol{u}}_2(t))\right)_{\mathcal{H}}+j(\dot{\boldsymbol{u}}_2(t),\boldsymbol{v})-j\left(\dot{\boldsymbol{u}}_2(t),\dot{\boldsymbol{u}}_2(t)\right) \\
&\ge\left(\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}_2(t)\right)_V,\forall\boldsymbol{v}\in V,\ \text{a.e. } t\in(0,T),
\end{aligned}
\tag{4.26}
$$

we take $\boldsymbol{v}=\dot{\boldsymbol{u}}_2(t)$ in (4.25) and $\boldsymbol{v}=\dot{\boldsymbol{u}}_1(t)$ in (4.26), add the two inequalities to obtain

$$
\begin{aligned}
&\left(\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_1(t)\right)-\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_2(t)\right),\varepsilon(\dot{\boldsymbol{u}}_1(t))-\varepsilon(\dot{\boldsymbol{u}}_2(t))\right)_{\mathcal{H}} \\
&\le\left(\mathcal{B}\varepsilon\left(\boldsymbol{u}_1(t)\right)-\mathcal{B}\varepsilon\left(\boldsymbol{u}_2(t)\right),\varepsilon\left(\dot{\boldsymbol{u}}_2(t)\right)-\varepsilon(\dot{\boldsymbol{u}}_1(t))\right)_{\mathcal{H}} \\
&\quad+\left(\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\varepsilon(\dot{\boldsymbol{u}}_2(t))-\varepsilon(\dot{\boldsymbol{u}}_1(t))\right)+j\left(\dot{\boldsymbol{u}}_1(t),\dot{\boldsymbol{u}}_2(t)\right) \\
&\quad-j\left(\dot{\boldsymbol{u}}_1(t),\dot{\boldsymbol{u}}_1(t)\right)+j\left(\dot{\boldsymbol{u}}_2(t),\dot{\boldsymbol{u}}_1(t)\right)-j\left(\dot{\boldsymbol{u}}_2(t),\dot{\boldsymbol{u}}_2(t)\right),
\end{aligned}
$$

then we use assumptions (3.5), (3.6) and (3.12) to find

$$\begin{aligned} m_{\mathcal{A}} \|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|_V^2 \leq L_{\mathcal{B}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|_V + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}} \|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|_V \\ + c_0^2(L_\nu + L_\tau) \|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|_V^2 \end{aligned} \tag{4.27}$$

It follows that

$$\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|_V \leq C\left(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}}\right). \tag{4.28}$$

Since $\boldsymbol{u}_i(t) = \int_0^t \dot{\boldsymbol{u}}_i(s)ds + \boldsymbol{u}_0, \forall t \in [0,T]$, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leq \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V \, ds. \tag{4.29}$$

Using (4.28), (4.29) and the Gronwall's inequality, we find

$$\int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V \, ds \leq \int_0^t \|\boldsymbol{\eta}(s) - \boldsymbol{\eta}(s)\|_{\mathcal{H}} ds, \tag{4.30}$$

Then, we find

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leq C \int_0^t \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}} \, ds, \quad t \in [0,T]. \tag{4.31}$$

Let $\theta_{\eta_i} = \theta_i$, and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1, 2$. Let $t \in \mathbb{R}^+$ be fixed. Then, we have

$$\begin{aligned} &\left(\dot{\theta}_1(t) - \dot{\theta}_2(t), \theta_1(t) - \theta_2(t)\right)_{\mathcal{X}' \times \mathcal{X}} + \left(K\theta_1(t) - K\theta_2(t), \theta_1(t) - \theta_2(t)\right)_{\mathcal{X}' \times \mathcal{X}} \\ &= \left(R\dot{\mathbf{u}}_1(t) - R\dot{\mathbf{u}}_2(t), \theta_1(t) - \theta_2(t)\right)_{\mathcal{X}' \times \mathcal{X}}. \end{aligned}$$

We integrate the above equality over $(0,t)$ and we use the strong monotonicity of K and the Lipschitz continuity of $R : V \to \mathcal{X}'$ to deduce that

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 ds \leq C \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V^2 ds,$$

It follows now from (4.30), that

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leq C \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}}^2 \, ds, \quad \forall t \in [0,T].$$

Form the previous inequality and estimates (4.31) and (4.24) it follows now that

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}}^2 \leq C \int_0^T \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}}^2 \, ds. \tag{4.32}$$

Reiterating this inequality m times we obtain

$$\|\Lambda^m \boldsymbol{\eta}_1 - \Lambda^m \boldsymbol{\eta}_2\|^2_{C(0,T;\mathcal{H})} \leq \frac{C^m T^m}{m!} \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|^2_{C(0,T;\mathcal{H})}.$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H})$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

Existence. Let $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H})$ be the fixed point of Λ and

$$\boldsymbol{u} = \boldsymbol{u}_{\eta^*}, \quad \theta = \theta_{\eta^*}, \quad \varphi_{\eta^*} = \varphi, \tag{4.33}$$

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{E}^*\nabla\varphi(t) + \boldsymbol{\sigma}_{\eta^*}, \tag{4.34}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla(\varphi). \tag{4.35}$$

We prove that $(\boldsymbol{u}, \boldsymbol{\sigma}, \theta, \varphi, \boldsymbol{D})$ satisfies (3.26)-(3.30) and (4.1)-(4.5). Indeed, we write (4.17) for $\boldsymbol{\eta}^* = \boldsymbol{\eta}$ and use (4.33)-(4.34) to obtain that (3.26) is satisfied. Now we consider (4.6) for $\boldsymbol{\eta}^* = \boldsymbol{\eta}$ and use (4.33) to find

$$\begin{aligned} &(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)), \varepsilon(\boldsymbol{v} - \dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\boldsymbol{u}(t)), \varepsilon(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} \\ &+ (\boldsymbol{\eta}^*(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\mathcal{H}} + j(\dot{\boldsymbol{u}}(t), \boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \\ &\forall \boldsymbol{v} \in V, t \in [0,T]. \end{aligned} \tag{4.36}$$

The equalities $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$ combined with (4.22), (4.33) and (4.34) show that for all $\boldsymbol{v} \in V$,

$$\begin{aligned} (\boldsymbol{\eta}^*(t), \boldsymbol{v})_{\mathcal{H}\times V} &= (\mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} - (C_e\theta(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \\ &+ \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)) - \mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(s)))ds, \varepsilon(\boldsymbol{v})\right)_{\mathcal{H}}, \end{aligned} \tag{4.37}$$

We substitute (4.37) in (4.36)) and use (3.26) to see that (3.27) is satisfied.

We write now (4.11) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (4.33) to find (3.28). From (4.15) and (4.33) we see that (3.29) is satisfied.

Next, (3.30), The regularities (4.1), (4.2), (4.3) and (4.4) follow from Lemmas 4.1, 4.2, 4.4 and 4.3.

Let now $t_1, t_2 \in [0,T]$, from (3.2), (3.10), (3.11) and (4.35), we conclude that there exists a positive constant $C > 0$ verifying

$$\|\boldsymbol{D}(t_1) - \boldsymbol{D}(t_2)\|_H \leq C\left(\|\varphi(t_1) - \varphi(t_2)\|_W + \|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_V\right).$$

The regularity of $\boldsymbol{u}$ and φ given by (4.1) and (4.2) implies

$$\boldsymbol{D} \in C(0,T;H). \tag{4.38}$$

We choose $\phi \in D(\Omega)^d$ in (3.28) and using (3.21) we find

$$\operatorname{div} \boldsymbol{D}(t) = q_0(t), \quad \forall t \in [0,T], \tag{4.39}$$

Property $\boldsymbol{D} \in C(0,T;\mathcal{W})$ follows from (3.16),(4.38) and (4.39) which concludes the existence part the Theorem 4.1.

Uniqueness. It follows by the unique solvability of the Problems $\mathcal{P}^1_\eta$, $\mathcal{P}^2_\eta$, $\mathcal{P}^3_\eta$ and $\mathcal{P}^4_\eta$ that the quintuple $(\boldsymbol{u}, \boldsymbol{\sigma}, \theta, \varphi, \boldsymbol{D})$ is a unique solution of the problem PV and with the regularity express (4.1)-(4.5). Finally, the uniqueness follows from the uniqueness of the fixed point of the operator Λ, which completes the proof of Theorem

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