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ON THE SPECTRUM OF THE TWO-PARTICLE SCHRÖDINGER OPERATOR WITH POINT POTENTIAL: ONE DIMENTIONAL CASE

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ABSTRACT. In the paper a one-dimensional two-particle quantum system interacted by two identical point interactions is considered. The corresponding Schrödinger operator (energy operator) h_ε depending on ε, is constructed as a self-adjoint extension of the symmetric Laplace operator. The main results of the work are based to the study of the operator h_ε. First the essential spectrum is described. The existence of unique negative eigenvalue of the Schrödinger operator is proved. Further, this eigenvalue and corresponding eigenfunction are found.

1. INTRODUCTION

The problems of the point interaction of two and three identical quantum particles interacted by point potentials (also called contact potentials and also occasionally singular potentials) have been studied in various physical works. In the works of F.A. Berezin and L.D. Faddeev [1] and R.A. Minlos and L.D. Faddeev [2], [3] first proposed a rigorous mathematical description of the point interaction of two and three particles, respectively.

In [2], [3] the Hamiltonian of the system under consideration was investigated using the theory of self-adjoint extensions of symmetric operators and was introduced as some self-adjoint extension of the symmetric Laplace operator defined on

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the domain of functions of three variables $x_1, x_2, x_3; x_j \in \mathbb{R}, j = 1, 2, 3$ vanishing if any two arguments $x_j = x_k, j \neq k,\ k = 1, 2, 3$ coincide.

The proposed extension is called the Skornyakov-Ter-Martirosyan expansion. In [4], using the results of [1], [2] the Hamiltonian of three particles (two fermions and one particle of a different nature) with the same masses interacting with point potential was studied and it was shown that the Skornyakov-Ter-Martirosyan extensions are self-adjoint and semi-bounded.

In [5], the results in [1]- [4] were generalized to the case of three arbitrary particles and it was shown that the corresponding Hamiltonian has a discrete spectrum unbounded below. Note that the advantage of one-dimensional models with point perturbations is clear as they are useful for the study of a variety of qualitative properties. For instance, you can see [6]- [11] for one body problems with delta potential. In this article, following the basic scheme used in [2]- [5], we study the problem of the point interaction of two arbitrary particles for one-dimensional space. The Laplace operator with domain on variables $x_1, x_2 \in \mathbb{R}$, vanishing as $x_1 = x_2$ is considered. In the impuls representation of the Hamiltonian after reduction of the variables we establish the Skornyakov-Ter-Martirosyan extension h_ε as a self-adjoint operator on his domain. The essential spectrum of h_ε coincides with the interval $[0; \infty)$. It is proved that the operator h_ε has no any eigenvalue as $\varepsilon \leq 0$ and if the parameter extansion is positive, i.e. $\varepsilon > 0$, then h_ε has a unique negative eigenvalue.

2. Preliminaries and Selection of Extension

The Hamiltonian (energy operator) of the two-particle system under consideration is defined as some extension $\tilde{H}$ of the symmetric operator $\tilde{H}_0$ acting in the Hilbert space $L_2(\mathbb{R}^2) \equiv L_2$ of the form

$$\left(\tilde{H}_0\phi\right)(x_1, x_2) = (-\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2})\phi(x_1, x_2),$$

where the domain of $\tilde{H}$ is considered as the manifold

$$D(\tilde{H}_0) = \{\phi \in L_2 : (\Delta_{x_1} + \Delta_{x_2})\phi \in L_2, \phi(x, x) = 0\},$$

where Δ_{x_i}-is the Laplace operator in the x_i variable $x_i \in \mathbb{R}$, m_i-is the mass of the i-th particle, $i = 1, 2$.

After the action of the corresponding Fourier transform, the operator $\tilde{H}_0$ transfers to the operator

$$(H_0 f)(p_1,p_2) = (\frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2) f(p_1,p_2),$$

defined on the set $D(H_0) \subset L_2$ of functions $f(p_1,p_2)$, satisfying the following conditions:

$$\int\limits_{\mathbb{R}^2} (p_1^4 + p_2^4)|f(p_1,p_2)|^2 dp_1 dp_2 < \infty, \int\limits_{\Gamma_p} f(p_1,p_2)\, d\nu_p = 0.$$

Here $\Gamma_p = \{(p_1,p_2) \in \mathbb{R}^2 : p_1 + p_2 = p\}$, $p \in \mathbb{R}$- is a family of lines with the natural Lebesgue measure $d\nu_p$.

Make a change of variables

$$P = p_1 + p_2,\ p = \frac{m_2}{M}p_1 - \frac{m_1}{M}p_2,\ M = m_1 + m_2$$

ensure a natural isomorphism between the spaces $L_2(\mathbb{R}) \otimes L_2(\Gamma_p)$ and $L_2(\mathbb{R}^2)$.

The last space can be identified with the space $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$, while the operator H_0 is written as the tensor sum of the following operators

$$H_0 = \left(\frac{1}{2M}P^2 + \frac{1}{2m}h_0\right) \otimes I,$$

where $I-$ is the identity operator, $m = m_1 m_2/(m_1+m_2)$, $(1/2M)P^2$ is the operator of multiplication by the number $P^2/(2M)$ in the space $L_2(\mathbb{R})$, and h_0 is a closed non-negative symmetric operator acting in $L_2(\mathbb{R})$ by

$$h_0 f(p) = p^2 f(p)$$

and its domain $D(h_0)$ consists of functions satisfying the conditions:

$$\int p^4 |f(p)|^2 dp < \infty; \int f(p) dp = 0 \tag{2.1}$$

Further, the integral without indicating limits is understood as integration over $\mathbb{R}$.

The symbol $\Re_z$ denotes the defeciency subspace of the operator h_0, i.e.

$$\Re_z = \{g \in L_2(R) :\ ((h_0 - zI)f, g) = 0,\ f \in D(h_0)\}.$$

Lemma 2.1. For any $z \in \mathbf{\Pi_0} = \mathbb{C}^1 \backslash [0, \infty)$ the defeciency subspace $\Re_z \subset L_2(\mathbb{R})$ of h_0 consists of functions of the form

$$g(p) = \frac{c}{p^2 - \bar{z}},\ c \in \mathbb{C}^1.$$

Proof. Let $g \in \Re_z$. Then for any $f \in D(h_0)$ the relation

$$((h_0 - zI)f, g) = \int (p^2 - z) f(p) \overline{g(p)} dp = \int f(p) \overline{(p^2 - \bar{z}) g(p)} dp = 0$$

holds.

From the last relation and conditions (1) it follows that

$$(p^2 - \bar{z}) g(p) = c$$

or

$$g(p) = \frac{c}{p^2 - \bar{z}}.$$

The lemma is proved. □

It follows from the lemma that for any $z \in \Pi_0$ the defeciency subspace $\Re_z$ of the operator h_0, is determined by

$$\Re_z = \{ g \in L_2(R) :\ ((h_0 - zI)f, g) = 0,\ f \in D(h_0) \}.$$

There fore, h_0 is a symmetric operator with defective indices (1,1). Using the general extension theory [4], we find that the operator h_0 has a one-parameter family of self-adjoint extensions.

Since the operator h_0 is non-negative, as in [2]- [5], we use the theory of extensions of semibounded operators. The defeciency subspace $\Re_{-1}$ of the operator h_0 consists of functions of the form

$$u_{-1}(p) = \frac{c}{p^2 + 1}, c \in \mathbb{C}^1.$$

Moreover, following the schemes of [2]- [4], the adjoint operator h_0^* is described using the following lemma.

Lemma 2.2. The domain of definition $D(h_0^*)$ of h_0^* consists of functions of the form

$$g(p) = f(p) + \frac{c_1}{p^2 + 1} + \frac{c_2}{(p^2 + 1)^2}, \tag{2.2}$$

where $f \in D(h_0)$, $c_1 \in \mathbb{C}^1$. The operator h_0^* acts on an function g of the form (2.2) by the formula

$$(h_0^* g)(p) = p^2 g(p) - c_1,$$

where the constant c_1 is taken from the decomposition (2.2) of the function g.

Further we select the extensions of the operator h_0. We define the set

$$D(h_\varepsilon), D(h_0) \subset D(h_\varepsilon) \subset D(h_0^*),$$

as follows:

$$D(h_\varepsilon) = \left\{ g \in D(h_0^*) : g(p) = f(p) + \frac{c}{p^2+1} + \frac{(\varepsilon - 2)c}{(p^2+1)^2},\ f \in D(h_0) \right\}. \tag{2.3}$$

The restriction of the operator h_0 to the domain $D(h_\varepsilon)$ is denoted by h_ε and it has the form

$$h_\varepsilon g(p) = p^2 g(p) - c.$$

By definition of h_ε , it is an extension of the operator h_0.

Theorem 2.1. For any $\varepsilon \in \mathbb{R}$, the extension h_ε is a self-adjoint operator.

Proof. It is easy to verify that for any $g_1, g_2 \in D(h_\varepsilon)$ the relation $(h_\varepsilon g_1, g_2) = (g_1, h_\varepsilon g_2)$ holds, i.e. h_ε is a symmetric operator. Now we show that the defective indices of the operator h_ε are equal to $(0, 0)$.

Let $\psi \in \Re_{-1}(h_\varepsilon)$. Then the function $\psi(p)$ has the form

$$\psi(p) = \frac{b}{p^2+1},\ b \in \mathbf{C}^1.$$

For any $g \in D(h_\varepsilon)$ the equality $((h_\varepsilon + I)g, \psi) = 0$ holds. According (2.3) the last equality can be written as

$$((h_\varepsilon + I)g, \psi) = \int ((p^2+1)(f(p) + \frac{c}{p^2+1} + \frac{(\varepsilon-2)c}{(p^2+1)^2}) - c)\overline{\psi(p)} dp =$$

$$\int ((p^2+1) f(p) \overline{\psi(p)} dp + (\varepsilon - 2)c \int \frac{\overline{b}}{(p^2+1)^2} dp = 0.$$

Since

$$\int ((p^2+1) f(p) \overline{\psi(p)} dp = 0$$

and $(\varepsilon - 2)c \neq 0$, we have $b = 0$. Hence $\psi(p) = 0$. This proves that the defective indices of the operator h_ε are equal to $(0, 0)$. □

3. Spectral properties of the operator h_ε

The main results of the paper are the following theorems.

Theorem 3.1. For any $\varepsilon \in R$ the essential spectrum of h_ε coincides with interval $[0, \infty)$. If $\varepsilon \geq 0$ then h_ε has no any negative eigenvalue, and for any $\varepsilon < 0$ the operator h_ε has a unique simple eigenvalue $z = -\frac{4}{\varepsilon^2}$ and the corresponding eigenfunction has the form $g_\varepsilon(p) = \dfrac{1}{p^2 + \frac{4}{\varepsilon^2}}$.

Proof. First we show that the essential spectrum of h_ε equals to $[0; \infty)$. For each $z \geq 0$ consider the sequence of cut-off layers:

$$G_n(z) = \left\{ p \in R : \sqrt{z} + \frac{1}{n+1} < |p| < \sqrt{z} + \frac{1}{n} \right\}, n = 1, 2, 3, \ldots.$$

We split the each layer $G_n(z)$ into two half-layers as

$$G_n^+(z) = \{p \in G_n(z) : p \geq 0\}$$

and

$$G_n^-(z) = \{p \in G_n(z) : p < 0\}.$$

By construction, the volume of these parts are equal and

$$\mu(G_n^+(z)) = \mu(G_n^-(z)) = \frac{1}{2}\mu(G_n(z)).$$

One can see that

$$V_n = \mu\left(G_n(z)\right) = \frac{2}{n\left(n+1\right)}.$$

Let $f_n^{(z)},\ n = 1, 2, 3, \ldots$ be a sequence of the test functions

$$f_n^{(z)}(p) = \begin{cases} \dfrac{1}{\sqrt{V_n(z)}}, & p \in G_n^+(z) \\ -\dfrac{1}{\sqrt{V_n(z)}}, & p \in G_n^-(z) \\ 0, & p \in \mathbb{R} \setminus G_n(z). \end{cases}$$

Then it is easy to verify that $f_n^{(z)} \in L_2(R)$, $\left\| f_n^{(z)} \right\| = 1$ and $(f_n^{(z)}, f_m^{(z)}) = 0$ as $n \neq m$.

One can see that

$$\int f_n^{(z)}(p)dp = 0,\ n = 1, 2, 3, \ldots,$$

i.e., $f_n^{(z)} \in D(h_0)$. Note that

$$\|(h_\varepsilon - zI)f_n^{(z)}\|^2 = \int\limits_{G_n(z)} \frac{1}{V_n(z)}|(p^2-z)|^2 dp = \frac{2}{V_n} \int\limits_{\sqrt{z}+\frac{1}{n+1}}^{\sqrt{z}+\frac{1}{n}} \left(p^2-z\right)^2 dp$$

or

$$\|(h_\varepsilon - zI)f_n^{(z)}\|^2 = \frac{2}{V_n} \int\limits_{\sqrt{z}+\frac{1}{n+1}}^{\sqrt{z}+\frac{1}{n}} \left(p^2-z\right)^2 dp. \tag{3.1}$$

Since

$$|p| < \sqrt{z} + \frac{1}{n},\ p^2 - z < \frac{1}{n}(2\sqrt{z} + \frac{1}{n}).$$

This gives

$$(p^2-z)^2 < (2\sqrt{z} + \frac{1}{n})^2 \frac{1}{n^2}.$$

Hense by (3.1) we have

$$\|(h_\varepsilon - zI)f_n^{(z)}\|^2 < (2\sqrt{z} + \frac{1}{n})^2 \frac{1}{n^2}.$$

This shows that

$$\lim_{n\to\infty} \left\|(h_\varepsilon - zI)f_n^{(z)}\right\| = 0.$$

This means that if $z \geq 0$, then $z \in \sigma_{ess}(h_\varepsilon)$ there fore $[0;\infty) \subset \sigma_{ess}(h_\varepsilon)$. In order to show the reverse inclusion $\sigma_{ess}(h_\varepsilon) \subset [0;\infty)$, we construct the resolvent operator of h_ε.

Let

$$(h_\varepsilon - zI)g = \psi.$$

Then

$$(p^2 - z)g(p) - c = \psi(p).$$

If $z < 0$, then $p^2 - z \neq 0$. Hence

$$g(p) = \frac{\psi(p)}{p^2 - z} + \frac{c}{p^2 - z}. \tag{3.2}$$

Since $g \in D(h_\varepsilon)$ it represents as

$$g(p) = f(p) + \frac{c}{p^2+1} + \frac{(\varepsilon - 2)c}{(p^2+1)^2} \tag{3.3}$$

for some $f \in D(h_\varepsilon)$. Comparing (3.3) and (3.2) we obtain the equation for c:

$$f(p) + (\frac{1}{p^2+1} - \frac{1}{p^2-z})c + \frac{(\varepsilon-2)c}{(p^2+1)^2} = \frac{\psi(p)}{p^2-z}, \tag{3.4}$$

where $f \in D(h_0)$. Integrating both sides of (3.4), taking into account (2.1) and the identities

$$\int \frac{dp}{p^2-z}dp = \frac{\pi}{\sqrt{-z}}, \quad z < 0, \tag{3.5}$$

and

$$\int \frac{dp}{(p^2+1)^2} = \frac{\pi}{2}, \tag{3.6}$$

we have

$$(\varepsilon\sqrt{-z} - 2)\pi c = 2\sqrt{-z}\int \frac{\psi(p)}{p^2-z}dp$$

or

$$c = \frac{2\sqrt{-z}}{\pi(\varepsilon\sqrt{-z}-2)}\int \frac{\psi(p)}{p^2-z}dp.$$

This gives

$$g(p) = \frac{\psi(p)}{p^2-z} - \frac{2\sqrt{-z}}{\pi(\varepsilon\sqrt{-z}-2)} \cdot \frac{1}{p^2-z}\int \frac{\psi(q)}{q^2-z}dq.$$

This if $z \in \Pi_0$ and $\varepsilon\sqrt{-z} - 2 \neq 0$, then the resolvent of the operator h_ε acts in $L_2(R)$ as

$$(R_z(h_\varepsilon)g)(p) = \frac{g(p)}{p^2-z} - \frac{2\sqrt{-z}}{\pi(\varepsilon\sqrt{-z}-2)} \cdot \frac{1}{p^2-z}\int \frac{g(q)}{q^2-z}dq.$$

This shows that the resolvent of the operator h_ε is bounded operator for $\varepsilon\sqrt{-z} - 2 \neq 0$ and $z < 0$. It means that $\sigma_{ess}(h_\varepsilon) \subset [0;\infty)$. It follows directly from here that $\sigma_{ess}(h_\varepsilon) = [0;\infty)$. Now we consider an eigenvalue problem for h_ε. From equation $(h_\varepsilon - zI)g(p) = 0$ we obtain that. If $\varepsilon \leq 0$ and $z \in \Pi_0$, then $\varepsilon\sqrt{-z} - 2 \neq 0$. By (3.4) the resolvent of the operator h_ε is definedon $D(h_\varepsilon)$. Hence h_ε has no any negative eigenvalue.

Let $\varepsilon > 0$. Then from the equality $\varepsilon\sqrt{-z} - 2 = 0$ we have $z = -\frac{4}{\varepsilon^2}$. The equation

$$(h_\varepsilon - zI)g(p) = 0$$

gives

$$g(p) = \frac{c}{p^2 - z}. \tag{3.7}$$

We show that $g \in D(h_\varepsilon)$. To this g should be represented of the form (3.3) for some $f \in D(h_0)$. Assume that g represents as (3.3). Comparing (3.3) with (3.7) we obtain

$$f(p) + \frac{c}{p^2+1} + \frac{(\varepsilon-2)c}{(p^2+1)^2} = \frac{c}{p^2-z},$$

i.e,

$$f(p) = \frac{c(1-\frac{4}{\varepsilon^2})}{(p^2+\frac{4}{\varepsilon^2})(p^2+1)} - \frac{c(\varepsilon-2)}{(p^2+1)^2}.$$

Taking into account the identities (3.5) and (3.6) one can see that $\int f(p)dp = 0$. This gives $f \in D(h_0)$. Theorem 3.1 is proved. □

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