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VARIOUS KINDS OF MATRICES IN CYCLOTOMIC GRAPHS

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ABSTRACT. The component matrix, Laplacian matrix, Distance matrix, Peripheral distance matrix, Distance Laplacian of the cyclotomic graphs and some properties are found. The D-energy, D_p-energy, D^L-energy and some indices of the cyclotomic graphs are determined. For the real symmetric matrices, matrices that attain the maximum L, L_s and the minimum S are calculated. The Hausdorff distance and optimal matching distance of the cyclotomic graphs are evaluated.

1. INTRODUCTION

A graph is denoted by G=G(V, E) , where V is its vertex set and E its edge set. The order of G is the number of its vertices and its size is the number of its edges. Let G be a graph, possessing n vertices and m edges. We say that G is an (n, m)-graph. A simple graph has no loops or multiple edges. Let G be a simple graph with n labeled vertices. Let $\overline{G}$ be the complement of G. The adjacency matrix of a simple undirected graph G is the symmetric matrix A_G whose rows and columns are both indexed by identical orderings of V_G such that

$$A(G) = A_G(v_i, v_j) = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases} .$$

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The degree d_i of the vertex $i \in V$ is the number of vertices adjacent to i , i.e., the sum of the i - th row (column) of the adjacency matrix of G. The eigenvalues of A(G) are$\lambda_1, \lambda_2, \ldots \lambda_n$ and can be ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. For a monic characteristic polynomial $\chi_{A(G)}(x) \in Z[x]$ of degree n, we define its associated reciprocal polynomial to be $R_A(z) = z^n \chi_{A(G)}(z + (1/z))$. $R_{A(G)}(z)$ is reciprocal which is a monic reciprocal polynomial of degree 2n.The Mahler measure M(A(G)) of A(G) is defined as $M(R_{A(G)}(z))$. A Symmetric matrix A(G) is called Cyclotomic if its associated reciprocal polynomial $R_{A(G)}$ has integer coefficients and Mahler measure $M(R_{A(G)}) = 1$. The adjacency graph of the cyclotomic matrix is called the cyclotomic graph. The inertia of a square matrix M with with real eigenvalues is the triplet $(n_+(M), n_0(M), n_-(M))$, where $n_+(M), n_-(M)$ denote the number of positive and negative eigenvalues of M, respectively and $n_0(M)$ is the multiplicity of 0 as an eigenvalue. A connected acyclic graph is called a tree.

The degree matrix of the graph is a diagonal matrix where the rows and columns are indexed by the set of vertices and each diagonal entry gives the degree of the corresponding vertex. A spanning tree of a graph G is a connected acyclic subgraph containing all the vertices of G. A symmetric real matrix M is called positive definite if $x^T M x > 0$ for every nonzero real column vector x in R^n. The Randic index of a graph is invented in 1976 by Milan Randic. It is good correlation ability for many physical and biochem properties. Indulal et al. [4] calculated the D-spectra of some graphs and their D-energies and constructed a pair of equienergetic bipartite graphs on $24t, t \geq 3$. Zhou et al. [9] determined lower and upper bounds for the distance spectral radius of graphs. Aouchiche et al. [2] introduced a Laplacian and signless Laplacian for the distance matrix of a connected graph. Aouchiche et al. [1] reported on the results related to the distance matrix of a graph and its spectral properties. Milan et al. [7] proved the largest distance Laplacian eigenvalue of a path is simple and the corresponding eigenvector has the similar property of Fiedler vector. The graphs are classified through Perron number in [8]. Kishori et al. [5] found bounds of Peripheral distance energy in terms of peripheral Wiener index for the graphs of $diam(G) \leq 2$. Milica Andelic et al. [6] computed the distance energy of particular types of graph and found a sequence of infinite families of distance equienergetic graphs. Cong-Trinh [3] studied the Wielandt-Mirsky conjecture for matrix polynomials.

2. Preliminaries

Definition 2.1. *A set of all square matrices which are the adjacency matrix of the simple graphs of order 2. They are denoted by* $\{[a_{ij}]_{2\times 2}/a_{ij} \in \{0,1\}\}$. *The adjacency graph of these matrices are denoted by* $\mathbf{A}_r$ *, where* $r = 0, 1$ *and are shown in Figure 1.*

Definition 2.2. *A set of all square matrices which are the adjacency matrix of the simple graphs of order 3. They are denoted by* $\{[a_{ij}]_{3\times 3}/a_{ij} \in \{0,1\}\}$. *The adjacency graph of these matrices are denoted by* $\mathbf{B}_r$ *, where* $r = 0, 1, \ldots, 7$ *and are shown Figures 2, 4, 11.* $\mathbf{B}_{i-j}$ *represents the graphs from* $\mathbf{B}_i$ *to* $\mathbf{B}_j$ *and* $i > j$. $\mathbf{B}_0$ *is the totally disconnected graph of order 3. Suppose we have seven graphs of order 3, we arrange the graphs in ascending or descending order with respect to eigenvalues of the graph and give the labels* $1, 2, \ldots$ *in the suffix of* $\mathbf{B}_r$. *Suffix r represents nothing but it means that rth graph of order 3.*

Definition 2.3. *A set of all square matrices which are the adjacency matrix A(G) of the simple graphs of order 4. They are denoted by* $\{[a_{ij}]_{4\times 4}/a_{ij} \in \{0,1\}\}$. *The adjacency graph of these matrices are denoted by* $\mathbf{C}_r$ *, where* $r = 0, 1, \ldots, 61$ *and are shown in Figure 3, 5-10, 12-14.* $\mathbf{C}_{i-j}$ *represents the graphs from* $\mathbf{C}_i$ *to* $\mathbf{C}_j$ *and* $i > j$.

Definition 2.4. $\mathbf{D}_r^j$*,* $r = 1, \ldots, 32$ *denote the Simple graphs of order 5 and are shown in Figures 15-27. The number of j values for the graph* $\mathbf{D}_r^j$ *for the corresponding value* $r = 1, \ldots, 32$ *depends upon how many graphs we get, when we label the numbers 1,2,3,4,5 for the five vertices.*

Definition 2.5. $\mathbf{E}_r^j$*,* $r = 1, \ldots, n$ *denote the Simple graphs of order 6 and are shown in Figures 28-39. The number of j values for the graph* $\mathbf{E}_r^j$ *for the corresponding value* $r = 1, \ldots, n$ *depends upon how many graphs we get, when we label the numbers 1,2,3,4,5,6 for the five vertices.*

Properties 2.1. *The properties of the graphs are*

a) $\mathbf{A}_1$, $\mathbf{B}_{4-7}$, $\mathbf{C}_{26-42}$, $\mathbf{D}_{1-4}^j$, $\mathbf{E}_{1-3}^j$ *are connected cyclotomic graphs.*
b) $\mathbf{C}_{43-61}$, $\mathbf{D}_{5-20}^j$, $\mathbf{E}_{4-69}^j$ *are connected non-cyclotomic graphs.*
c) $\mathbf{B}_{1-3}$, $\mathbf{C}_{1-25}$, $\mathbf{D}_{21-29}^j$, $\mathbf{E}_{70-78}^j$ *are disconnected cyclotomic graphs.*

FIGURE 1. $\mathbf{A}_1$

FIGURE 2. $\mathbf{B}_{4-6}$

FIGURE 3. $\mathbf{C}_{30-41}$

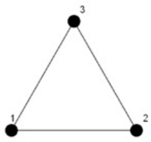

FIGURE 4. $\mathbf{B}_7$

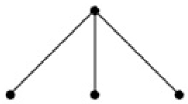

FIGURE 5. $\mathbf{C}_{26-29}$

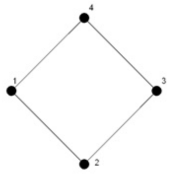

FIGURE 6. $\mathbf{C}_{42}$

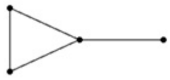

FIGURE 7. $\mathbf{C}_{43-54}$

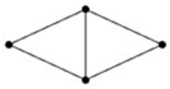

FIGURE 8. $\mathbf{C}_{55-60}$

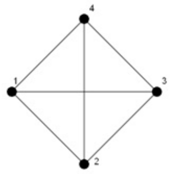

FIGURE 9. $\mathbf{C}_{61}$

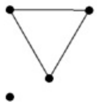

FIGURE 10. $\mathbf{C}_{22-25}$

FIGURE 11. $\mathbf{B}_{1-3}$

FIGURE 12. $\mathbf{C}_{1-6}$

FIGURE 13. $\mathbf{C}_{7-18}$

FIGURE 14. $\mathbf{C}_{19-21}$

d) $\mathbf{D}^j_{30-32}$, $\mathbf{E}^j_{79-97}$ *are disconnected non-cyclotomic graphs.*

e) $\mathbf{A}_1$ *is a 2-path graph,* $\mathbf{B}_7$ *is a triangle graph,* $\mathbf{C}_{42}$ *is a square graph,* $\mathbf{C}_{55-60}$ *is a diamond graph,* $\mathbf{D}^j_1$ *is a 5-cycle graph,* $\mathbf{B}_{4-6}$ *is a 3-path graph,* $\mathbf{C}_{30-41}$ *is a 4-path graph ,* $\mathbf{C}_{26-29}$ *is a claw graph,* $\mathbf{C}_{43-54}$ *is a paw graph.*

f) The line graph L(G) of G is the graph whose vertices correspond to the edges of G with two vertices of L(G) being adjacent iff the corresponding edges in G have a vertex in common. $L(\mathbf{B}_7) \cong \mathbf{B}_7$, $L(\mathbf{C}_{42}) \cong \mathbf{C}_{42}$, $L(\mathbf{D}^j_1) \cong \mathbf{D}^j_1$, $L(\mathbf{B}_{4-6}) \cong \mathbf{A}_1$, $L(\mathbf{C}_{30-41}) \cong \mathbf{B}_{4-6}$, $L(\mathbf{D}^j_4) \cong \mathbf{C}_{30-41}$.

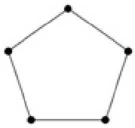

FIGURE 15. $\mathbf{D}_1^j$

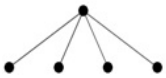

FIGURE 16. $\mathbf{D}_2^j$

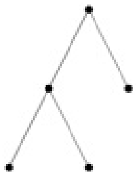

FIGURE 17. $\mathbf{D}_3^j$

FIGURE 18. $\mathbf{D}_4^j$

FIGURE 19. $\mathbf{D}_{21}^j$

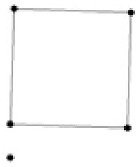

FIGURE 20. $\mathbf{D}_{22}^j$

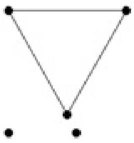

FIGURE 21. $\mathbf{D}_{23}^j$

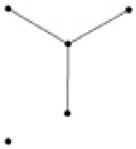

FIGURE 22. $\mathbf{D}_{24}^j$

FIGURE 23. $\mathbf{D}_{25}^j$

FIGURE 24. $\mathbf{D}_{26}^j$

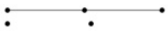

FIGURE 25. $\mathbf{D}_{27}^j$

FIGURE 26. $\mathbf{D}_{28}^j$

FIGURE 27. $\mathbf{D}_{29}^j$

TABLE 1. Values of $n_+(M), n_0(M), n_-(M)$

Graph	$\mathbf{A}_1$	$\mathbf{B}_{4-6}$	$\mathbf{B}_7$	$\mathbf{C}_{26-29}$	$\mathbf{C}_{30-41}$	$\mathbf{C}_{42}$	$\mathbf{D}_1^j$	$\mathbf{D}_2^j$	$\mathbf{E}_3^j$	$\mathbf{D}_4^j$	$\mathbf{E}_1^j$	$\mathbf{E}_2^j$	$\mathbf{E}_3^j$
$n_+(M)$	1	1	2	1	2	1	3	1	2	2	2	3	2
$n_0(M)$	0	1	0	2	0	2	0	3	1	1	2	0	2
$n_-(M)$	1	1	1	1	2	1	2	1	2	2	2	3	2

3. EIGENVALUES OF REAL SYMMETRIC MATRICES

Definition 3.1. *Let $S_n\{0,1\}$ denote the set of $n \times n$ real symmetric matrices whose entries are in the set $\{0,1\}$. For an $n \times n$ real symmetric matrix A(G), we denote the eigenvalues of A(G) in decreasing order by $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$. The spread of an $n \times n$ real symmetric matrix A(G) is $s(A) = \lambda_1(A) - \lambda_n(A)$.*

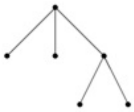

FIGURE 28. $\mathbf{E}_1^j$

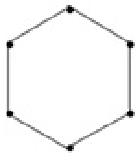

FIGURE 29. $\mathbf{E}_2^j$

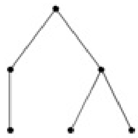

FIGURE 30. $\mathbf{E}_3^j$

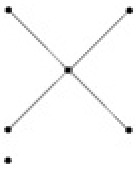

FIGURE 31. $\mathbf{E}_{70}^j$

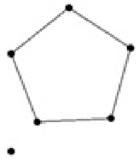

FIGURE 32. $\mathbf{E}_{71}^j$

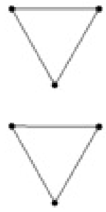

FIGURE 33. $\mathbf{E}_{72}^j$

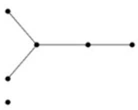

FIGURE 34. $\mathbf{E}_{73}^j$

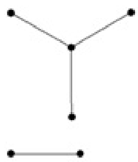

FIGURE 35. $\mathbf{E}_{74}^j$

FIGURE 36. $\mathbf{E}_{75}^j$

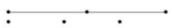

FIGURE 37. $\mathbf{E}_{76}^j$

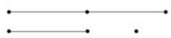

FIGURE 38. $\mathbf{E}_{77}^j$

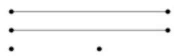

FIGURE 39. $\mathbf{E}_{78}^j$

Definition 3.2. *For a given j with* $2 \leq j \leq n-1, L = max\{\lambda_j(A) : A \in S_n\{0,1\}\}$, $S = min\{\lambda_j(A) : A \in S_n\{0,1\}\}, L_s = max\{s(A) : A \in S_n\{0,1\}\}$.

Properties 3.1. *The properties of the spread of matrix are*

a) *The spread of an adjacency matrix for the cyclotomic graph of order 2 is 2.*
b) *The spread of an A(G) for the cyclotomic graphs of order 3 are 2,* $2\sqrt{2}$ *and 3.*
c) *The spread of an A(G) for the cyclotomic graphs of order 4 are 2,* $2\sqrt{2}$ *, 2, 3,* $2\sqrt{3}$,$1+\sqrt{5}$, *4.*
d) *The spread of an A(G) for the cyclotomic graphs of order 5 are* $(1/2)(5+\sqrt{5})$,*4,* $2\sqrt{2+\sqrt{2}}$, $2\sqrt{3}$, *3, 4, 3,* $2\sqrt{3}$,$1+\sqrt{5}$,$2\sqrt{2}$,$2\sqrt{2}$, *2, 2.*
e) *The spread of an A(G) for the cyclotomic graphs of order 6 are 4, 4,*$2\sqrt{(1/2)(5+\sqrt{5})}$.

Properties 3.2. *The properties of Cyclotomic matrix of the graph of order up to 6 are* *a)* $-4 \leq Det(\chi_{A(G)}(\lambda)) \leq 4$.

b) $\chi_{A(G)}(\lambda)$*is not a positive definite matrix.*

c) $1 \leq \left\|\chi_{A(G)}(\lambda)\right\| \leq 2$.

d) $tr(\chi_{A(G)}(\lambda)) = 0$.

TABLE 2. Matrices that attain extremum

Matrices	Adjacency matrices
Matrices that attain maximum L	A($\mathbf{B}_7$),A($\mathbf{C}_{42}$),A($\mathbf{C}_{22}$), A($\mathbf{D}_1^j$), A($\mathbf{D}_2^j$), A($\mathbf{D}_{21}^j$), A($\mathbf{D}_{22}^j$), A($\mathbf{D}_{23}^j$), A($\mathbf{E}_1^j$), A($\mathbf{E}_2^j$)
Matrices that attain minimum S	A($\mathbf{B}_{4-6}$), A($\mathbf{C}_{42}$), A($\mathbf{D}_2^j$), A($\mathbf{D}_{22}^j$),A($\mathbf{E}_1^j$), A($\mathbf{E}_2^j$)
Matrices that attain maximum L_s	A($\mathbf{B}_7$), A($\mathbf{C}_{42}$), A($\mathbf{D}_2^j$), A($\mathbf{D}_{22}^j$), A($\mathbf{E}_1^j$), A($\mathbf{E}_2^j$)

Definition 3.3. *A vertex subset is called independent if its elements are pairwise nonadjacent. Two vertices are co-neighbour vertices if they share the same neighbours. If S is a pairwise co-neighbour vertices of a graph G, then S is an independent set of G. A cluster of order k of G is a set S of k pairwise co-neighbour vertices. If each vertex of a cluster has the same transmission, then it is called the transmission of the cluster. For example,* $\{1,3\}$ *and* $\{1,4\}$ *are independent set of* $\mathbf{C}_{42}$. *A cluster of order 2 of* $\mathbf{C}_{42}$ *is* $\{\{1,3\},\{2,4\}\}$ *and it is transmission of the cluster.*

4. THE COMPANION MATRIX OF THE CYCLOTOMIC GRAPH

Definition 4.1. *Let* $\chi_{A(G)}(\lambda)$ *be the characteristic polynomial of A(G). The companion matrix of the monic polynomial* $\chi_{A(G)}(\lambda) = c_0 + c_1\lambda + \ldots + c_{n-1}\lambda^{n-1} + x^n$ *is the square matrix defined as*

$$C(\chi_{A(G)}(\lambda)) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}.$$

TABLE 3. The companion matrices

Companion matrix	Eigenvalues	Norm	Rank	Trace	Discriminant
$C(\chi_{A(\mathbf{A}_1)}(x))$	-1, 1	1	2	0	4
$C(\chi_{A(\mathbf{B}_7)}(x))$	-1, -1, 2	$\sqrt{7+3\sqrt{5}}$	3	0	0
$C(\chi_{A(\mathbf{C}_{42})}(x))$	-2, 2,0,0	$\sqrt{17}$	3	0	0
$C(\chi_{A(\mathbf{D}^j_{21})}(x))$	-1, -1, 0, 1, 2	4	4	1	0
$C(\chi_{A(\mathbf{E}^j_2)}(x))$	-2, -1, -1 , 1, 1, 2	$\sqrt{67+3\sqrt{497}}$	6	0	0

Properties 4.1. *The properties of Companion matrix of the characteristic polynomial of the cyclotomic graph of order up to 6.*

a) $-4 \leq Det(C(\chi_{A(G)}(\lambda))) \leq 4$.

b) $C(\chi_{A(G)}(\lambda))$ *is not a positive definite matrix.*

c) $2 \leq \rho(C(\chi_{A(G)}(\lambda))) \leq 6$.

d) $1 \leq \left\|C(\chi_{A(G)}(\lambda))\right\| \leq \sqrt{147+\sqrt{21593}}$.

e) $tr(C(\chi_{A(G)}(\lambda))) = 0$ *or* 1.

f) $0 \leq \triangle(\chi_{A(G)}(\lambda))) \leq 8192$.

5. LAPLACIAN MATRIX

Definition 5.1. *Let G be a simple graph with n vertices. Let the Laplacian matrix L is defined as* $L = Deg(G) - A(G)$ *, where* $Deg(G)$ *is the degree matrix and A(G) is the adjacency matrix of the graph. The elements of L are*

$$_{i,j} = \begin{cases} deg(v_i), & \textit{if } i = j \ , \\ -1, & \textit{if } i \neq j \textit{ and } v_i \textit{ adjacent to } v_j \ , \\ 0, & \textit{if otherwise} \ . \end{cases}$$

Let $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n = 0$ *denote the Laplacian eigenvalues of G. The Laplacian matrix is used to enumerate the number of trees. The determinant of the Laplacian matrix counts the number of spanning trees.*

Properties 5.1. *The properties of Laplacian matrix are*

a) The number of spanning trees of the connected cyclotomic graphs $\mathbf{A}_1$*,* $\mathbf{B}_{4-6}$*,* $\mathbf{B}_7$*,* $\mathbf{C}_{26-29}$*,* $\mathbf{C}_{30-41}$*,* $\mathbf{C}_{42}$*,* $\mathbf{D}^j_2$*,* $\mathbf{D}^j_4$*,* $\mathbf{E}^j_2$ *are 1, 1, 3, 1, 1, 4, 1, 1, 6.*

b) *It is difficult to find the eigenvalues of Laplacian matrix for the graphs* $\mathbf{D}_1^j$, $\mathbf{D}_3^j$, $\mathbf{E}_3^j$, $\mathbf{E}_{73}^j$.
c) $L(G) + L(\overline{G}) = L(K_n) = nI_n - J_n$, *where* J_n *is the matrix of order n with each entry 1.*
d) *If* $\triangle$ *denotes the maximum vertex degree, then* $\beta_1 \geq \triangle + 1$.
e) *The second smallest Laplacian eigenvalue* β_{n-1} *is called the algebraic connectivity.* $\beta_{n-1} = 0$ *iff the graph is disconnected.*
f) *An eigenvector corresponding to* β_{n-1} *is called a Fiedler vector.*
g) *The maximum value of a Fiedler vector for a graph of n vertices is n (for a complete graph).*

Definition 5.2. *Let T = (V, E) be a tree and* f *be a Fiedler vector of it. Let* $f(v)$ *denotes the component of* f *for a vertex* $v \in V$. *If* $f(v) \neq 0$ *for all* $v \in V$, *then T contains exactly one edge* uw *such that* $f(u) > 0$ *and* $f(w) < 0$. *This edge* uw *is called characteristic edge.*

Definition 5.3. *The median of a graph G is the set* $\{v \in V | Tr(v) \leq Tr(x)$, *for all* $x \in V\}$. *The median of a tree is either a single vertex or two adjacent vertices.*

Properties 5.2. *The properties of Laplacian matrix of the characteristic polynomial of the cyclotomic graph of order up to 6.*

a) $-1 \leq Det(L(G)) \leq 0$.
b) $L(G)$ *is not a positive definite matrix.*
c) $1 \leq \rho(L(G)) \leq 6$
d) $\sqrt{2+\sqrt{2}} \leq \|L(G)\| \leq 5$.
e) $2 \leq tr(L(G)) \leq 12$.

6. DISTANCE MATRIX

The Energy of a graph is a concept defined by Gutman in 1978 and originated from theoretical chemistry. The Energy of the graph is the sum of the absolute values of its eigenvalues. It is denoted by $E(G)$. Two graphs are called equienergetic if they have equal energies. If $\lambda_{i_1} > \lambda_{i_2} > \ldots > \lambda_{i_h}$ are the distinct eigenvalues of

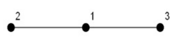

FIGURE 40. $\mathbf{B}_4$

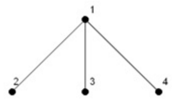

FIGURE 41. $\mathbf{C}_{26}$

FIGURE 42. $\mathbf{C}_{30}$

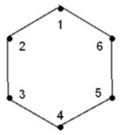

FIGURE 43. $\mathbf{E}_2^j$

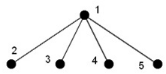

FIGURE 44. $\mathbf{D}_2^j$

FIGURE 45. $\mathbf{D}_4^j$

TABLE 4. Fiedler vector of the Laplacian matrices

Graph	β_{n-1}	Fiedler vector	Characteristic edge	Median
$\mathbf{A}_1$	2	$\{-1, 1\}$	12	$\{1\}$
$\mathbf{B}_4$	1	$\{0, -1, 1\}$	-	$\{2\}$
$\mathbf{B}_7$	3	$\{-1, 1, 0\}$	-	$\{1, 2, 3\}$
$\mathbf{C}_{26}$	1	$\{0, -1, 1, 0\}$	-	$\{1\}$
$\mathbf{C}_{30}$	$2-\sqrt{2}$	$\{1 - \sqrt{2}, -1 + \sqrt{2}, -1, 1\}$	21	$\{1, 2\}$
$\mathbf{C}_{42}$	2	$\{-1, 0, 1, 0\}$	-	$\{1, 2, 3, 4\}$
$\mathbf{D}_2^j$	1	$\{0, -1, 1, 0, 0\}$	-	$\{1\}$
$\mathbf{D}_4^j$	$(3-\sqrt{5})/2$	$\{-1, (1 - \sqrt{5})/2, 0, (-1 + \sqrt{5})/2, 1\}$	-	$\{3\}$
$\mathbf{E}_2^j$	1	$\{-1, -1, 0, 1, 1, 0\}$	-	$\{1, 2, 3, 4, 5, 6\}$

the graph, then the spectrum is

$$spec(G) = \begin{bmatrix} \lambda_{i_1} & \lambda_{i_2} & \cdots & \lambda_{i_h} \\ n_1 & n_2 & \cdots & n_h \end{bmatrix}$$

where n_j indicates the algebraic multiplicity of the eigenvalue λ_{i_j}. Two graphs G and H such that $spec(G) = spec(H)$ are called cospectral graph.

Definition 6.1. *Suppose G is a connected graph with set of vertices* $V(G) = \{v_1, v_2, \ldots, v_n\}$ *and* d_{ij} *represent the shortest path length between vertices* v_i *and* v_j. *The distance matrix of G is defined as an* $n \times n$ *matrix whose* (i, j)*-th entry is* d_{ij}. *It is denoted by D(G). Its eigenvalues can be ordered as* $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k$.

Definition 6.2. *The D-eigenvalues of a connected graph G are the eigenvalues of its distance matrix D, and form the D-spectrum of G. The D-energy of the graph G is the sum of the absolute values of its D-eigenvalues. It is denoted by $E_{D(G)}$. Two graphs are called D-equienergetic if they have equal D-energies.*

Properties 6.1. *The properties of Distance matrix are*

a) *The determinant of the distance matrix of a tree is a function of the number of vertices only.*
b) *If $G = K_n$, the complete graph on n vertices, then $A(K_n) = D(K_n)$ and $E_{D(G)} = E(G) = 2(n-1)$.*
c) *Denote by $\boldsymbol{J}_n$ the all $1's$ $n \times n$ matrix and by $\boldsymbol{I}_n$ the identity matrix of order n. If the diameter of G is atmost two, then $D(G) = 2\boldsymbol{J}_n - 2\boldsymbol{I}_n - A(G) = \boldsymbol{J}_n - \boldsymbol{I}_n + A(\overline{G})$.*
d) *The inertia of the distance matrix is $(1, 0, n-1)$ for all trees on $n \geq 2$ vertices.*
e) *If the characteristic polynomial of the distance matrix D of a tree on n vertices is $P_D(T)(t)$, $det(D) = (-1)^n P_{D(T)}(0)$.*
f) *D(G) is real, symmetric and has trace equal to zero. It is Hermitian.*
g) *The largest eigenvalue μ_1 is called the distance spectral radius or distance index.*

Definition 6.3. *Wiener index of the graph W(G) is the sum of the distances between all unordered pair of vertices of G. It is half the sum of the entries of the distance matrix.*

Definition 6.4. *The Randic index is a degree based topological index. It is defined as*

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.$$

Definition 6.5. *The first Zagreb Index is defined as $M_1(G) = \sum_{uv \in E(G)} (d(u) + d(v))$. The second Zagreb Index is defined as $M_2(G) = \sum_{uv \in E(G)} (d(u) \times d(v))$. The Harmonic Index is defined as $H(G) = \sum_{uv \in E(G)} \dfrac{2}{d(u) + d(v)}$.*

Definition 6.6. *A graph G having energy greater than the complete graph on the same number of vertices is called hyperenergetic.*

Theorem 6.1. *Let G be a (n, m)- graph of diameter 2 and μ_1 be its greatest D-eigenvalues. Then $\mu_1 \geq (2n^2 - 2m - 2n)/n$.*

TABLE 5. The Energy of the connected cyclotomic graphs

G	Energy	D-energy	W	R(G)	M_1	M_2	M_2
$\mathbf{A}_1$	2	2	1	1	1	1	1
$\mathbf{B}_{4-6}$	$2\sqrt{2}$	$2\sqrt{3}+2$	4	$\sqrt{2}$	6	4	$4/3$
$\mathbf{B}_7$	4	4	3	1	8	8	1
$\mathbf{C}_{26-29}$	$2\sqrt{3}$	$2\sqrt{7}+4$	9	$\sqrt{3}$	12	9	$3/2$
$\mathbf{C}_{30-41}$	$2\sqrt{5}$	$4+2\sqrt{10}$	10	$(2\sqrt{2}+1)/2$	10	8	$11/6$
$\mathbf{C}_{42}$	4	8	8	2	16	16	2
$\mathbf{D}_1^j$	$2+2\sqrt{5}$	12	15	$5/2$	20	20	$5/2$
$\mathbf{D}_2^j$	4	$2\sqrt{13}+6$	16	1	20	16	$8/5$
$\mathbf{D}_3^j$	$2\sqrt{2+\sqrt{2}}+2\sqrt{2-\sqrt{2}}$	$2\sqrt{13}+6$	16	$(3\sqrt{2}+1)/2$	13	10	$5/2$
$\mathbf{D}_4^j$	$2\sqrt{3}+2$	-	20	$\sqrt{2}+1$	14	12	$7/3$
$\mathbf{E}_1^j$	6	-	27	$(1+2(\sqrt{2}+\sqrt{3}))/\sqrt{6}$	19	16	$2/3$
$\mathbf{E}_2^j$	8	18	27	3	24	24	3
$\mathbf{E}_3^j$	$2\sqrt{(5+\sqrt{5})/2}+2\sqrt{(5-\sqrt{5})/2}$	-	30	$(3+\sqrt{2})/\sqrt{2}$	17	14	3

TABLE 6. The Energy of the disconnected cyclotomic graphs

G	$\mathbf{B}_{1-3}$	$\mathbf{C}_{1-6}$	$\mathbf{C}_{7-18}$	$\mathbf{C}_{19-21}$	$\mathbf{C}_{22-25}$	$\mathbf{D}_{21}^j$	$\mathbf{D}_{22}^j$	$\mathbf{D}_{23}^j$	$\mathbf{D}_{24}^j$	$\mathbf{D}_{25}^j$	$\mathbf{D}_{26}^j$	$\mathbf{D}_{27}^j$
Energy	2	2	$2\sqrt{2}$	4	4	5	4	4	$2\sqrt{3}$	$2\sqrt{5}$	$2\sqrt{2}+2$	$2\sqrt{2}$

TABLE 7. The Energy of the disconnected cyclotomic graphs

G	$\mathbf{D}_{28}^j$	$\mathbf{D}_{29}^j$	$\mathbf{E}_{70}^j$	$\mathbf{E}_{71}^j$	$\mathbf{E}_{72}^j$	$\mathbf{E}_{73}^j$	$\mathbf{E}_{74}^j$	$\mathbf{E}_{75}^j$	$\mathbf{E}_{76}^j$	$\mathbf{E}_{77}^j$
Energy	4	2	4	10	12	$2(\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}})$	$2+2\sqrt{3}$	$2\sqrt{3}$	$2\sqrt{2}$	6

TABLE 8. Equienergetic and cospectral graphs

Graphs	Types
$\mathbf{B}_7$, $\mathbf{C}_{42}$	Equienergetic graphs
$\mathbf{B}_{4-6}$, $\mathbf{C}_{26-29}$, $\mathbf{C}_{30-41}$	Equienergetic and cospectral graphs
$\mathbf{B}_{4-6}$, $\mathbf{C}_{26-29}$,$\mathbf{C}_{30-41}$	D-equienergetic and D-cospectral graphs
$\mathbf{B}_{1-3}$, $\mathbf{C}_{1-6}$, $\mathbf{C}_{7-18}$, $\mathbf{C}_{19-21}$, $\mathbf{C}_{26-29}$	Equienergetic and cospectral graphs

Proof. Let G be a connected graph of diameter 2 and let its vertices be labeled as $v_1, \ldots, v_n$. Let d_i denote the degree of v_i. Then i-th row of D consists of d_i one ′ s and $n - d_i - 1$ two ′ s. Let $x[1, 1, 1, \ldots, 1]$, the all one vector. Then by the Raileigh Principle $\mu_1 \geq (xDx^T/xx^T) = (1/n)\sum_{i=1}^{n}(2n - d_i - 2) = (2n^2 - 2m - 2n)/n$. □

Theorem 6.2. *If G is a connected* (n, m)*- graph, then*

$$\sqrt{2\sum_{1\leq i\leq j\leq n}(d_{ij})^2} \leq E_D(G) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n}(d_{ij})^2}.$$

Proof. By using Cauchy-Schwartz inequality and assuming $a_i = 1, b_i = |\mu_i|$, we get

$$(\sum_{i=1}^{n}|\mu_i|)^2 \leq n\sum_{i=1}^{n}\mu_i^2 \Rightarrow E_D(G)^2 \leq 2n\sum_{1\leq i\leq j\leq n}(d_{ij})^2,$$

$$E_D(G)^2 \leq (\sum_{i=1}^{n}|\mu_i|)^2 \geq \sum_{i=1}^{n}|\mu_i|^2 = 2\sum_{1\leq i\leq j\leq n}(d_{ij})^2.$$

□

7. Peripheral distance matrix

Let G be connected nontrivial graph. Let u and v be two vertices of a graph G. The distance $d(u, v)$ between the vertices u and v is the length of a shortest path connecting u and v. The eccentricity $e(v)$ of a vertex v in a graph G is the distance between v and a vertex farthest from v in G. The diameter $diam(G)$ of G is the maximum eccentricity of G. A vertex v with $e(v) = diam(G)$ is called a peripheral vertex of G. The set of peripheral vertices of G is called as periphery and denoted by P(G).

Definition 7.1. *Peripheral distance matrix (D_p-matrix) of G is defined as*

$$D_p = D_p(G) = [d_{ij}],$$

where d_{ij} is the distance between two peripheral vertices v_i and v_j in G. The eigenvalues of D_p-matrix are said to be D_p-eigenvalues of G. The peripheral distance energy (D_p-energy)of G is defined as the sum of the absolute values of D_p-eigenvalues of D_p-matrix of G.

Theorem 7.1. *Let G be a graph of order n with k peripheral vertices and let $\mu_1, \mu_2, \ldots, \mu_k$ be its peripheral distance eigenvalues. Then*

a) $\sum_{i=1}^{k} \mu_i = 0$,

b) $\sum_{i=1}^{k} \mu_i^2 = 2\sum_{1\le i\le j\le k}(d_{ij})^2$.

Theorem 7.2. *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \le 2$. Then $\sum_{i=1}^{k} \mu_i^2 = 6\binom{n}{2} + 2\binom{k}{2} - 6m$.*

Definition 7.2. *The sum of the distances between all pairs of peripheral vertices is a peripheral Wiener index of a graph G,*

$$PWI(G) = \sum_{1\le i\le j\le k} d(v_i, v_j),$$

where G is an (n, m)- graph with k peripheral vertices and $v_i, v_j \in P(G)$.

Theorem 7.3. *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \le 2$. Then $PWI(G) = \binom{n}{2} + \binom{k}{2} - m$.*

TABLE 9. D_p- eigenvalues and D_p - energy

Graph	$\mathbf{A}_1$	$\mathbf{B}_{4-6}$	$\mathbf{B}_7$	$\mathbf{C}_{26-29}$	$\mathbf{C}_{30-41}$	$\mathbf{C}_{42}$
D_p- eigenvalues	-1,1	-2,2	2,-1,-1	4,-2,-2	-3, 3	4,-2,-2,0
D_p - energy	2	4	4	8	6	8

TABLE 10. D_p- eigenvalues and D_p - energy

Graph	$\mathbf{D}_1^j$	$\mathbf{D}_2^j$	$\mathbf{D}_3^j$	$\mathbf{D}_4^j$
D_p- eigenvalues	$\{6, (-3 \pm \sqrt{5})/2, (-3 \pm \sqrt{5})/2\}$	6,-2,-2,-2	$\{1 \pm \sqrt{19}, -2\}$	-4,4
D_p - energy	2	4	4	8

TABLE 11. D_p- eigenvalues and D_p - energy

Graph	$\mathbf{E}_1^j$	$\mathbf{E}_2^j$	$\mathbf{E}_3^j$
D_p- eigenvalues	8, -4, -2, -2	9,-4,-4,-1,0,0	$\{1 \pm \sqrt{33}, -2\}$
D_p - energy	16	18	$2\sqrt{33} + 2$

8. DISTANCE LAPLACIAN

Definition 8.1. *The transmission $Tr(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices of G. The distance Laplacian of a connected graph G is defined as the matrix $D^L(G) = Tr(G) - D(G)$, where $Tr(v)$ denotes the diagonal matrix with $Tr(v_i)$ as the i-th diagonal entry. The eigenvalues of D^L -matrix are said to be D^L -eigenvalues of G. The Distance Laplacian energy (D^L-energy)of G is defined as the sum of the absolute values of D^L-eigenvalues of D^L -matrix of G. Let $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_{n-1} \geq \delta_n$ denote the eigenvalues of D^L. D^L is a positive semidefinite matrix and $\delta_n = 0$. (Aouchiche et al. [2])*

TABLE 12. D^L- eigenvalues and D^L - energy

Graph	$\mathbf{A}_1$	$\mathbf{B}_{4-6}$	$\mathbf{B}_7$	$\mathbf{C}_{26-29}$	$\mathbf{C}_{30-41}$	$\mathbf{C}_{42}$
D^L- eigenvalues	2,0	5,3,0	3,3,0	7,7,4,0	$\{7 \pm \sqrt{5}, 6, 0\}$	6,6,4,0
D^L - energy	2	8	6	18	20	16

TABLE 13. D^L- eigenvalues and D^L - energy

Graph	$\mathbf{D}_1^j$	$\mathbf{D}_2^j$	$\mathbf{D}_3^j$	$\mathbf{D}_4^j$	$\mathbf{E}_1^j$	$\mathbf{E}_2^j$	$\mathbf{E}_3^j$
D^L- eigenvalues	-	9,9,9,5,0	-	$\{(23 \pm \sqrt{41})/2, 0, 7, 0\}$	-	13,13,10,9,9,0	-
D^L - energy	30	32	-	40	-	54	-

9. SPECTRAL VARIATION OF GRAPHS

Definition 9.1. *Let A have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda n$ and $\tilde{A}$ have eigenvalues $\tilde{\lambda_1}, \tilde{\lambda_2}$, $\ldots, \tilde{\lambda n}$. Then the spectral variation of $\tilde{A}$ with respect to A is*

$$sv_A(\tilde{A}) = max_i min_j \left|\tilde{\lambda}_i - \lambda_j\right| .$$

Definition 9.2. *The Hausdorff distance between the eigenvalues of A and $\tilde{A}$ is*

$$hd(A, \tilde{A}) = max\{sv_A(\tilde{A}), sv_{\tilde{A}}(A)\}.$$

Definition 9.3. *The optimal matching distance between the eigenvalues of A and $\tilde{A}$ is*

$$md(A, \tilde{A}) = min_\pi\{max_i \left|\tilde{\lambda}_{\pi(i)} - \lambda_i\right|\}$$

where π is taken over all permutations of $\{1, 2, \ldots, n\}$.

TABLE 14. The spectral variation, Hausdorff distance, Optimal matching distance

$A, \tilde{A}$	$sv_A(\tilde{A})$	$sv_{\tilde{A}}(A)$	$hd(A, \tilde{A})$	$md(A, \tilde{A})$
$\mathbf{B}_1, \mathbf{B}_4$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$
$\mathbf{B}_1, \mathbf{B}_7$	1	1	1	1
$\mathbf{B}_4, \mathbf{B}_7$	$2-\sqrt{2}$	1	1	1
$\mathbf{C}_{26}, \mathbf{C}_{30}$	$(\sqrt{5}-1)/2$	$(\sqrt{5}-1)/2$	$(\sqrt{5}-1)/2$	$(\sqrt{5}-1)/2$
$\mathbf{C}_{26}, \mathbf{C}_{42}$	$2-\sqrt{3}$	$2-\sqrt{3}$	$2-\sqrt{3}$	$2-\sqrt{3}$
$\mathbf{C}_{30}, \mathbf{C}_{42}$	$(\sqrt{5}-1)/2$	$(\sqrt{5}-1)/2$	$(\sqrt{5}-1)/2$	$(\sqrt{5}-1)/2$
$\mathbf{D}_1^j, \mathbf{D}_2^j$	$(\sqrt{5}-1)/2$	$(\sqrt{5}+1)/2$	$(\sqrt{5}+1)/2$	$(\sqrt{5}-1)/2$
$\mathbf{D}_1^j, \mathbf{D}_3^j$	$((1+\sqrt{5})/2)-\sqrt{2-\sqrt{2}}$	$2-\sqrt{2+\sqrt{2}}$	$((1+\sqrt{5})/2)-\sqrt{2-\sqrt{2}}$	$\sqrt{2-\sqrt{2}}+(\sqrt{5}-1)/2$
$\mathbf{D}_1^j, \mathbf{D}_4^j$	$(\sqrt{5}-1)/2$	$1-((\sqrt{5}-1)/2)$	$(\sqrt{5}-1)/2$	$\sqrt{3}-((\sqrt{5}-1)/2)$
$\mathbf{D}_2^j, \mathbf{D}_3^j$	$\sqrt{2-\sqrt{2}}$	$2-\sqrt{2+\sqrt{2}}$	$\sqrt{2-\sqrt{2}}$	$\sqrt{2+\sqrt{2}}$
$\mathbf{D}_2^j, \mathbf{D}_4^j$	1	$2-\sqrt{3}$	1	2
$\mathbf{D}_3^j, \mathbf{D}_4^j$	$1-\sqrt{2-\sqrt{2}}$	$1-\sqrt{2-\sqrt{2}}$	$1-\sqrt{2-\sqrt{2}}$	$\sqrt{3}-\sqrt{2-\sqrt{2}}$
$\mathbf{E}_1^j, \mathbf{E}_2^j$	0	1	1	2
$\mathbf{E}_1^j, \mathbf{E}_3^j$	$\sqrt{(5-\sqrt{5})/2}-1$	$\sqrt{(5-\sqrt{5})/2}-1$	$\sqrt{(5-\sqrt{5})/2}-1$	$\sqrt{(5-\sqrt{5})/2}$
$\mathbf{E}_2^j, \mathbf{E}_3^j$	$\sqrt{(5-\sqrt{5})/2}-1$	$\sqrt{(5-\sqrt{5})/2}-1$	$\sqrt{(5-\sqrt{5})/2}-1$	$\sqrt{(5-\sqrt{5})/2}+1$

10. POLYNOMIAL MATRIX

Definition 10.1. *A polynomial matrix is a matrix whose elements are univariate or multivariate variables. It can be written as* $P = \sum_{m=0}^{k} B(m)x^m$ *, where* $B(m)$ *denotes the matrix of constant coefficients. Let A be a polynomial matrix, then the matrix* $\lambda I - A$ *is the characteristic matrix of the matrix A. Its determinant* $|\lambda I - A|$ *is the characteristic polynomial of the matrix A.*

Theorem 10.1. *If the polynomial matrix* $P_1 = A(\mathbf{B}_1) + A(\mathbf{B}_4)x$*, then the solution of the characteristic polynomial of the matrix* P_1 *is* $\lambda = 0, \lambda = \pm\sqrt{1+2x+2x^2}$.

Theorem 10.2. *If the polynomial matrix* $P_2 = A(\mathbf{B}_1) + A(\mathbf{B}_7)x$*, then the solution of the characteristic polynomial of the matrix* P_2 *is* $\lambda = -1-x^2, \lambda = \{1+x^2 \pm \sqrt{1+2x^2+9x^4})/2$.

Theorem 10.3. *If the polynomial matrix $P_3 = A(\mathbf{B}_1) + A(\mathbf{B}_4)x + A(\mathbf{B}_7)x^2$, then the solution of the characteristic polynomial of the matrix P_3 is $\lambda = (Y/3Z) - (Z/3^{2/3}), \lambda = -[((1+i\sqrt{3})Y/6Z)+((1-i\sqrt{3})Z/(2\times 3^{2/3}))]$, $\lambda = -[((1-i\sqrt{3})Y/6Z)+((1+i\sqrt{3})Z/(2\times 3^{2/3}))]$, where $Y = -3-6x-12x^2-12x^3-9x^4$ and $Z = 3^{1/3}\big(-9x^3-18x^4-18x^5-9x^6+\sqrt{3}(-1-6x-24x^2-68x^3-153x^4-276x^5-385x^6-396x^7-291x^8-136x^9-36x^{10})^{1/2}\big)^{1/3}$.*

Theorem 10.4. *If the polynomial matrix $P_4 = A(\mathbf{B}_4) + A(\mathbf{B}_7)x$, then the solution of the characteristic polynomial of the matrix P_4 is $\lambda = -a, \lambda = (x\pm\sqrt{9x^2+16x+8})/2$.*

11. Applications

The applications of graph theory on computer networks, Medical Analysis, solve shortest path problems between cities, scheduling exams and assign channels to television stations, sports scheduling, mobile towering and traffic signals. The application of a distance matrix is the distance between cities by road, to help with planning travel and haulage. In data analysis, distance matrices are used as a data format when performing hierarchical and multidimensional scaling. A distance matrix are used in hierarchical clustering , phylogenetic analysis and determination of protein structures from X-ray crystallography. The applications of Euclidean metrics are in crystallography, psychometrics, machine learning, acoustics, wireless sensor networks, ultrasound tomography, room reconstruction echoes, microphone position calibration. The applications of Hausdorff distance is on comparison of DNA and three dimensional protein structures. It is used to measure the similarity of two dimensional curves, shape matching, image retrieval. Optimal matching distance are used in marine traffic tracking. In chemistry, the experimental heats from the formation of conjugated hydrocarbons are closely related to the total π-electron energy.

12. Conclusion

Hence conclude that the component matrix, Laplacian matrix, Distance matrix, Peripheral distance matrix, Distance Laplacian of the cyclotomic graphs and some properties found. The D-energy, D_p-energy, D^L-energy and some indices of the cyclotomic graphs determined. For the real symmetric matrices, matrices that attain the maximum L, Ls and the minimum S are calculated. The Hausdorff distance and optimal matching distance of the cyclotomic graphs evaluated. Further development of work is on the cyclotomic graphs of order greater than 6 and for non-cyclotomic graphs.

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