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INERTIAL RELAXED TSENG METHOD FOR SOLVING VARIATIONAL INEQUALITY PROBLEM IN HILBERT SPACE

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ABSTRACT. The research efforts of this paper is to present a new inertial relaxed Tseng extrapolation method with weaker conditions for approximating the solution of a variational inequality problem, where the underlying operator is only required to be pseudomonotone. The strongly pseudomonotonicity and inverse strongly monotonicity assumptions which the existing literature used are successfully weakened. The strong convergence of the proposed method to a minimum-norm solution of a variational inequality problem are established. Furthermore, we present an application and some numerical experiments to show the efficiency and applicability of our method in comparison with other methods in the literature.

1. INTRODUCTION

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, C be a nonempty closed convex subset of H and $A : H \to H$ be an operator. The classical variational inequality problem for A on C is denoted by $VI(A, C)$ and is defined as follows.

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Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \;\forall\, y \in C. \tag{1.1}$$

The notion of $VI(A, C)$ was introduced independently by Stampacchia [22] and Fichera [10, 11] for modeling problems arising from mechanics and for solving Signorini problem. It is well-known that many problems in economics, mathematical sciences, mathematical physics can be formulated as $VI(A, C)$. The set of solution of (1.1) is denoted by Ω, that is

$$\Omega = \{x \in C : \langle Ax, y - x\rangle \geq 0 \;\forall\, y \in C\}. \tag{1.2}$$

The metric projection (P_C) property is well-known in the literature and it is defined as $x \in \Omega$ if and only if

$$x = P_C(I - \lambda A)x.$$

Due to the fruitful applications of $VI(A, C)$, many useful iterative algorithms have been developed to approximate the solution of (1.1). For example, Xu [31] introduced the iterative process

$$x_{n+1} = P_C(I - \lambda A)x_n. \tag{1.3}$$

It has been established that if A strongly monotone and Lipschitz continuous, then the iterative scheme (1.3) has strong convergence results under some suitable conditions. In addition, if A is inverse strongly monotone, the iterative scheme (1.3) has weak convergence results under some suitable conditions. An attempt to overcome these setbacks was made by Korpelevich [15]. The extragradient type method which is given by (1.4), was introduced. The convergence of the method was established for a monotone and Lipschitz continuous operator A in the finite-dimensional Euclidean spaces.

$$\begin{cases} x_1 \in C \\ y_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = P_C(y_n - \lambda A y_n) \;\forall\, n \in \mathbb{N}. \end{cases} \tag{1.4}$$

Under some suitable conditions, the sequence $\{x_n\}$ was shown to converge to the solution set Ω. Since then, other authors have studied the $VI(A, C)$ in Hilbert spaces using different iterative algorithms, (see [1, 2, 12–14, 21] and the references therein). However, in all of these approaches, the convergence of their methods were obtained under the inversely strongly monotone or strongly pseudomonotone or monotonicity and Lipschitz continuity or pseudomonotonicity and Lipschitz continuity assumption of the underlying operator A. The challenge about these methods is how to calculate the Lipschitz constant of the given monotone or pseudomonotone operator, which is difficult or even sometimes impossible. Thus, making their methods very difficult in applications.

Remark 1.1. *In the light of the above facts, it is natural to ask, if an iterative algorithm can be introduced to approximate the $VI(A,C)$* (1.1) *in which the underlining operator is just pseudomonotone, with the minimum metric projection.*

It is well-known that the $VI(A,C)$ (1.1) can be associated with the dynamical system in [30],

$$\begin{aligned}\frac{du(t)}{dt} &= \rho[-u(t) + P_C(u(t) - \lambda(t)A(u(t)) + \lambda(t)A(u(t)) \\ &\quad - \lambda(t)A(P_C(u(t) - \lambda(t)A(u(t))))]\end{aligned} \tag{1.5}$$

where $\rho, \lambda > 0$. Taking a time step size $h_n > 0, u(t) = u_n$ and $\lambda(t) = \lambda_n$, and using an explicit finite-difference of the system (1.5), we have

$$\begin{aligned}\frac{u_{n+1} - u_n}{h_n} &= \rho[-u_n + P_C(u_n - \lambda_n A(u_n) + \lambda_n A(u_n) \\ &\quad - \lambda_n A(P_C(u_n - \lambda_n A(u_n)))].\end{aligned} \tag{1.6}$$

If $h_n = 1,$ (1.6) becomes

$$\begin{aligned}u_{n+1} &= (1-\rho)u_n + \rho P_C(u_n - \lambda_n A(u_n) + \rho\lambda_n A(u_n) \\ &\quad - \rho\lambda_n A(P_C(u_n - \lambda_n A(u_n))).\end{aligned} \tag{1.7}$$

Now, observe that (1.7) can be written in two steps. That is of the form

$$\begin{cases} v_n = P_C(u_n - \lambda_n A u_n) \\ u_{n+1} = (1-\rho)u_n + \rho v_n + \rho\lambda_n(A(u_n) - A(v_n)) \ \forall\, n \in \mathbb{N}. \end{cases} \tag{1.8}$$

It is easy to see that if $P_C(u_n - \lambda_n A u_n) = (I + \lambda_n B)^{-1}(I - \lambda_n A)$ and $\rho = 1,$ we have that u_{n+1} reduces to the well-known Tseng's forward backward-forward method introduced in [28]. The convergence of the scheme in [28] requires that $0 < \lambda_n < \frac{1}{L},$ where L is the Lipschitz constant of the operator A or λ_n can be computed using a line search procedure with a stopping criterion.

Remark 1.2. *It is unfortunate that the iterative scheme* (1.8) *is not an answer to the proposition in Remark 1.1. The underlying operator A is required to be inversely strongly monotone or pseudomonotone and Lipschitz continuous. The question of constructing an iterative scheme that will approximate the solution of a $VI(A,C)$* (1.1) *in which the underlying operator will just be pseudomontone, with minimum projection still persists.*

The inertial extrapolation method has proven to be an effective way for accelerating the rate of convergence of iterative algorithms. The technique was introduced in 1964 and is based on a discrete version of a second order dissipative dynamical system [4, 5]. The inertial type algorithms use its two previous iterates to obtain its next iterate [3, 16]. For details on the inertia extrapolation, see [7,17,18] and the references therein. In 2018, Dong et al. [9] proposed an inertial type iterative algorithm for approximating the solution of (1.1). The method is of the form:

$$
\begin{cases}
x_0, x_1 \in H, \\
w_n = x_n + \theta_n(x_n - x_{n-1}) \\
y_n = P_C(w_n - \lambda A w_n), \\
d(w_n, y_n) = w_n - y_n - \lambda(Aw_n - Au_n) \\
x_{n+1} = w_n - \zeta \eta_n d(w_n, y_n)
\end{cases}
\tag{1.9}
$$

where $\zeta \in (0,2)$, $\lambda \in (0, \frac{1}{L})$, $\eta_n := \phi(w_n, y_n)$ if $d(w_n, y_n) \neq 0$ and $\eta_n := \phi(w_n, y_n)$ if $d(w_n, y_n) = 0$. They established that the sequence $\{x_n\}$ converges weakly to an element of Ω. An inertial type iterative algorithm for approximating the solution of (1.1) was also proposed by Thong et al. [25], which is of the form,

$$
\begin{cases}
x_0, x_1 \in H, \\
w_n = x_n + \theta_n(x_n - x_{n-1}) \\
u_n = P_C(w_n - \lambda A w_n), \\
x_{n+1} = u_n - \lambda(Au_n - Ax_n),
\end{cases}
\tag{1.10}
$$

where A is monotone and Lipschitz continuous. They established that the sequence $\{x_n\}$ converges weakly to an element of Ω.

Remark 1.3. *Notice that in Algorithm 1.9 and Algorithm 1.10, the underlying operator A is monotone and L-Lipschitz continuous. Also, since strong convergence is* *always more desirable than weak convergence. It is therefore natural to ask if Algorithm 1.9 and Algorithm 1.10 can be further modified to get a strong convergence. In addition, can we further weaken the condition on the operator A?*

In 2020, Thong et al. [27] provide a partial answer to the question raised in Remark 1.3 by introducing a vicosity based iterative algorithm for approximating the solution of a $VI(A, C)$ (1.1). The method is of the form:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ u_n = P_C(w_n - \lambda A w_n), \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) z_n \\ \text{where } z_n = u_n - \lambda(Au_n - Aw_n) \end{cases} \tag{1.11}$$

and A is monotone and Lipschitz continuous. They established that the sequence $\{x_n\}$ converges strongly to an element of Ω.

Remark 1.4. *In Algorithm 1.11, the underlying operator A is monotone and L-Lipschitz continuous. More so, the step size requires the knowledge of the Lipschitz constant of the underlying operator, which is difficult or even sometimes impossible to calculate, thus, making their methods very difficult in applications. Therefore, the answer to the proposition in Remark 1.1 and Remark 1.3 are still lingering.*

Motivated by the work of Thong et al. [25], Thong et al. [27] and (1.8), in this paper, we provide an affirmative answer to the questions raised in Remark 1.1 and Remark 1.3. An iterative algorithm is constructed in view of inertial method variational inequality problems for a pseudomonotone operator in the framework of real Hilbert spaces. Using this algorithm we establish weak and strong convergence results without using the conventional two cases approach. Furthermore, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite and finite dimensional Hilbert spaces. The comparison of our proposed iterative algorithm with existing ones in the literature shows that our proposed schemes approximate faster and the convergence analysis is easy to follow.

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method and highlight its advantages over other existing algorithms. In

Section 4, we establish weak and strong convergence analysis of our method and in Section 5, we give an application to equilibrium problem. More so, in Section 6, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite and finite dimensional Hilbert spaces and lastly in Section 7, we give the conclusion of the paper. The results obtained in this work extend, generalize and improve several results in this direction.

2. Preliminaries

In this section, we recall some known and useful results which are needed in the sequel.

Let H be a real Hilbert space. The set of fixed point of T will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$\langle x, y\rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x - y\|^2). \tag{2.1}$$

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.2}$$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \tag{2.3}$$

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C\| \le \|u - y\| \; \forall y \in C.$$

P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2,$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$,

$$\langle x - P_C x, P_C x - y\rangle \ge 0$$

for all $y \in C$ and

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2$$

for all $x \in H$ and $y \in C$.

Lemma 2.1. *Let $P_C : H \to C$ be a metric projection. Then, we have the following:*

(a) $\langle P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2$ *for all* $x, y \in H$.

(b) $\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2$ *for all* $x \in C$ *and* $y \in H$.
(c) $y = P_C x$ *if and only if* $\langle x - y, y - z \rangle \; \forall \; z \in C$.

Definition 2.1. *Let* $T : H \to H$ *be an operator. Then the operator* T *is called*

(a) L*-Lipschitz continuous if*

$$\|Tx - Ty\| \leq L\|x - y\|,$$

where $L > 0$ *and* $x, y \in H$. *If* $L = 1$, *Then* T *is called nonexpansive. Also, if* $y \in F(T)$ *and* $L = 1$, *Then* T *is called quasi-nonexpansive.*

(b) *monotone if*

$$\langle Tx - Ty, x - y \rangle \geq 0, \; \forall x, y \in H.$$

(c) *pseudomonotone if*

$$\langle Tx, y - x \rangle \geq 0 \Rightarrow \langle Ty, y - x \rangle \geq 0, \; \forall x, y \in H.$$

(d) *firmly nonexpansive if*

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \; \forall \; x, y \in H,$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \; \forall \; x, y \in H,$$

(e) k*-inverse strongly monotone (*k*-ism) if there exists* $k > 0$, *such that*

$$\langle Tx - Ty, x - y \rangle \geq k\|Tx - Ty\|^2, \; \forall \; x, y \in H.$$

(f) *sequentially weakly continuous mapping if for each* $\{x_n\}$ *in* H, *such that* $\{x_n\}$ *converges weakly to a point* $x \in H$, *then,* $\{Tx_n\}$ *converges weakly to* Tx.

It is well-known that for any nonexpansive mapping T, the set of fixed point is closed and convex. Also, T satisfies the following inequality

$$\langle (x - Tx) - (y - Ty), Ty - Tx \rangle \leq \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2, \; \forall \; x, y \in H. \tag{2.4}$$

Thus, for all $x \in H$ and $x^* \in F(T)$, we have that

$$\langle x - Tx, x^* - Tx \rangle \leq \frac{1}{2}\|Tx - x\|^2, \; \forall \; x, y \in H. \tag{2.5}$$

Lemma 2.2. *[19] Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real number in $(0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim\limits_{n\to\infty} a_n = 0$.

3. Proposed Algorithm

In this section, we present our proposed method and discuss some motivations for proposing it. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 3.1. *Suppose that the following conditions hold:*

(1) *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
(2) *$A : H \to H$ is pseudomonotone, sequentially weakly continuous and uniformly continuous on bounded subsets of C.*
(3) *The solution set $\Omega = \{x \in C : \langle Ax, y-x\rangle \geq 0 \;\forall\, y \in C\} \neq \emptyset$.*

Algorithm 3.2. Initialization: *Given $\gamma, \kappa > 0$, $\rho \in (0,1]$ and $\theta_n, \beta_n, \alpha_n, \mu, l, \in (0,1)$, for all $n \in \mathbb{N}$, let $x_0, x_1, \in H$ be arbitrary.*

Iterative step:

Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\theta}{\kappa}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \frac{\theta}{\kappa}, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$.

Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = P_C(w_n - \lambda_n A w_n), \tag{3.2}$$

where λ_n *is chosen to be the largest* $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ *satisfying*

$$\lambda \|Aw_n - Au_n\| \leq \mu \|w_n - u_n\|. \tag{3.3}$$

If $w_n = u_n$*, then stop,* u_n *is a solution of* (1.1).
Step 3. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n((1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n)). \tag{3.4}$$

Stopping criterion: *If* $w_n = u_n = x_n$, *then stop, otherwise, set* $n := n + 1$ *and go back to* **Step 1.**

We highlight the motivation for the proposed algorithm.

Remark 3.1.

(1) A notable advantage of this method (Algorithm 3.2) is that the operator A is pseudomonotone unlike the inversely strongly monotone or strongly pseudomonotonicity assumptions used in other papers (see for example, [12, 14, 20, 21]). No extra projection is required under the setting. The use of the Armijo-line search rule in our algorithm stands as a local approximation of the Lipschitz constant of the operator A. The knowledge of the Lipschitz constant of A is not required.

(2) The proof of the strong convergence of Algorithm 3.2 (that is, proof of Theorem 4.1) does not rely on the usual "Two cases approach (Case 1 and Case 2)" usually used in numerous paper for solving optimization problems (see [13, 21, 24, 26] and the reference therein). The techniques and ideas employed in the strong convergence analysis are new.

(3) In Algorithm 3.2, it is easy to compute step 1 since the value of $\|x_n - x_{n-1}\|$ is a prior knowledge before choosing θ_n. It is easy to see from (3.1) that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$
Recall that, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, which means that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$. Clearly, we have that that $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n$ for all

$n \in \mathbb{N}$, which together with $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, it follows that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

It is worth mentioning that, we can take $\alpha_n = 1/(n+1)^p$ and $\epsilon_n = 1/(n+1)^{1-p}$, where $p \in [0, 1/2)$.

4. Convergence Analysis

Lemma 4.1. *Let A be an operator satisfying the Assumption 3.1. Then, for all $p \in \Omega$, we have the*

$$\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - u_n\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle.$$

Proof. Since $u_n = P_C(w_n - \lambda_n(Aw_n))$ and $p \in \Omega$, then by the characteristics of P_C (Lemma 2.1), we have that

$$\langle w_n - u_n - \lambda_n Aw_n, u_n - p\rangle \geq 0,$$

which is equivalent to

$$2\langle w_n - u_n, u_n - p\rangle - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle - 2\lambda_n\langle Au_n, u_n - p\rangle \geq 0. \tag{4.1}$$

Since $2\langle w_n - u_n, u_n - p\rangle = \|w_n - p\|^2 - \|w_n - u\|^2 - \|u_n - p\|^2$, (4.1) becomes

$$\begin{aligned} \|w_n - p\|^2 - \|w_n - u\|^2 - \|u_n - p\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle \\ -2\lambda_n\langle Au_n, u_n - p\rangle \geq 0. \end{aligned} \tag{4.2}$$

Using the fact that A is pseudomonotone, we have that $\langle Au_n, u_n - p\rangle \geq 0$. It follows that

$$\begin{aligned} \|u_n - p\|^2 &\leq \|w_n - p\|^2 - \|w_n - u\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle - 2\lambda_n\langle Au_n, u_n - p\rangle \\ &\leq \|w_n - p\|^2 - \|w_n - u\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle. \end{aligned}$$

This implies that

$$\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - u_n\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle.$$

□

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Then, under the Assumptions 3.1, we have that $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$ and since $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$. Then from **Step** 2 of Algorithm 3.2, we have

$$\begin{aligned}\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \alpha_n N_1. \end{aligned} \tag{4.3}$$

Now, suppose that $v_n = (1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n)$. Then, using Lemma 4.1, we have that

$$\begin{aligned}&\|v_n - p\|^2 \\ &= \|(1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n) - p\|^2 \\ &= \|(1-\rho)(w_n - p) + \rho(u_n - p) + \rho\lambda_n(Aw_n - Au_n)\|^2 \\ &= (1-\rho)^2\|w_n - p\|^2 + \rho^2\|u_n - p\|^2 + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 \\ &\quad + 2\rho(1-\rho)\langle w_n - p, u_n - p\rangle + 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle \\ &\quad + 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\ &= (1-\rho)^2\|w_n - p\|^2 + \rho^2\|u_n - p\|^2 + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 \\ &\quad + \rho(1-\rho)[\|w_n - p\|^2 + \|u_n - p\|^2 - \|w_n - u_n\|^2] \\ &\quad + 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle + 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\ &= (1-\rho)\|w_n - p\|^2 + \rho\|u_n - p\|^2 - \rho(1-\rho)\|w_n - u_n\| \\ &\quad + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 + 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle \\ &\quad + 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\ &\leq (1-\rho)\|w_n - p\|^2 + \rho[\|w_n - p\|^2 - \|w_n - u_n\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle] \\ &\quad - \rho(1-\rho)\|w_n - u_n\| + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 + 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle \\ &\quad + 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\ &= \|w_n - p\|^2 - \rho(2-\rho)\|w_n - u_n\|^2 + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2\end{aligned}$$

$$
\begin{aligned}
&+ 2\lambda_n\rho^2\langle w_n - u_n, Aw_n - Au_n\rangle \\
&\leq \|w_n - p\|^2 - \rho[2 - \rho - \mu(2(1-\rho) + \mu)]\|w_n - u_n\|^2 \\
&\leq \|w_n - p\|^2, \qquad (4.4)
\end{aligned}
$$

which implies that

$$\|v_n - p\| \leq \|w_n - p\|. \qquad (4.5)$$

Also, using Algorithm 3.2, (4.3) and (4.4), we have that

$$
\begin{aligned}
&\|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(v_n - p)\|^2 \\
&= (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|v_n - p\|^2 y \\
&\quad + 2(1-\alpha_n-\beta_n)\beta_n\langle x_n - p, v_n - p\rangle \\
&\leq (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 \\
&\quad + 2(1-\alpha_n-\beta_n)\beta_n\|x_n - p\|\|v_n - p\| \\
&\leq (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\alpha_n-\beta_n)\beta_n\|x_n - p\|^2 \\
&\quad + (1-\alpha_n-\beta_n)\beta_n\|v_n - p\|^2 \\
&\leq (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\alpha_n-\beta_n)\beta_n\|x_n - p\|^2 \\
&\quad + (1-\alpha_n-\beta_n)\beta_n\|w_n - p\|^2 \\
&= (1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|w_n - p\|^2 \\
&\leq (1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\beta_n(\|x_n - p\| + \alpha_n N_1)^2 \\
&= (1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\alpha_n\|x_n - p\|N_1 + (1-\alpha_n)\beta_n\alpha_n^2 N_1^2 \\
&\leq (1-\alpha_n)(1-\alpha_n)\|x_n - p\|^2 + 2(1-\alpha_n)\alpha_n\|x_n - p\|N_1 + \alpha_n^2 N_1^2 \\
&= [(1-\alpha_n)\|x_n - p\| + \alpha_n N_1]^2, \qquad (4.6)
\end{aligned}
$$

which implies that

$$\|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(v_n - p)\| \leq (1-\alpha_n)\|x_n - p\| + \alpha_n N_1. \qquad (4.7)$$

We then have that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)-\alpha_n p\| \\
&\le \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|+\alpha_n\|p\| \\
&\le (1-\alpha_n)\|x_n-p\|+\alpha_n N_1+\alpha_n\|p\| \\
&= (1-\alpha_n)\|x_n-p\|+\alpha_n(N_1+\|p\|) \\
&\le \max\{\|x_n-p\|, N_1+\|p\|\} \\
&\ \vdots \\
&\le \max\{\|x_1-p\|, N_1+\|p\|\}. \qquad (4.8)
\end{aligned}$$

Thus, $\{x_n\}$ generated by Algorithm 3.2 is bounded. □

Lemma 4.3. *Let Assumption 3.1 hold and let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Assume that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point x^*, and $\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=0$, then, $x^*\in\Omega$.*

Proof. By Lemma 2.1 we obtain

$$\langle w_{n_k}-\lambda_{n_k}A(w_{n_k})-u_{n_k},\ x-u_{n_k}\rangle \le 0,\ \forall\, x\in C,$$

which implies that

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k}-u_{n_k},\ x-u_{n_k}\rangle \le \langle A(w_{n_k}),\ x-u_{n_k}\rangle,\ \forall\, x\in C.$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{\lambda_{n_k}}\langle w_{n_k}-u_{n_k},\ x-u_{n_k}\rangle+\langle A(w_{n_k}),\ u_{n_k}-w_{n_k}\rangle \\
&\le \langle A(w_{n_k}),\ x-w_{n_k}\rangle,\ \forall x\in C. \qquad (4.9)
\end{aligned}$$

Suppose that $x\in C$ is fix and using the fact that $\lim_{k\to\infty}\|w_{n_k}-u_{n_k}\|=0$, we have from (4.9) that

$$0\le \liminf_{k\to\infty}\langle A(w_{n_k}), x-w_{n_k}\rangle\ \forall x\in C. \qquad (4.10)$$

Now, choose a sequence $\{\eta_k\}$ of positive numbers such that $\eta_{k+1}\le\eta_k,\ \forall\ k\in\mathbb{N}$ and $\eta_k\to 0$ as $k\to\infty$. Then, for each η_k, we denote by M_k the smallest positive integer such that

$$\langle A(u_{n_j}), x-u_{n_j}\rangle+\eta_k\ge 0\ \forall j\ge M_k. \qquad (4.11)$$

Since $\{\eta_k\}$ is decreasing, it follows that $\{M_k\}$ is increasing. Now, we set for each $k \in \mathbb{N}$, $n_{M_k} = \frac{A(w_{M_k})}{\|A(w_{M_k})\|^2}$, provided $A(w_{M_k}) \neq 0$. Then it is easy to see that $\langle A(w_{M_k}), n_{M_k}\rangle = 1$ for each $k \in \mathbb{N}$. Using (4.11), we have that

$$\langle A(w_{M_k}), x + \eta_k n_{M_k} - w_{M_k}\rangle \geq 0,$$

by the pseudomonotonicity of A, we have that

$$\langle A(x + \eta_k n_{M_k}), x + \eta_k n_{M_k} - w_{N_k}\rangle \geq 0. \tag{4.12}$$

Since $\{x_{n_k}\}$ converges weakly to x^*, we obtain by our hypothesis that $\{u_{n_k}\}$ and $\{w_{n_k}\}$ also converge weakly to x^*. Thus, by the sequentially weakly continuity of A, we have that $\{A(w_{n_k})\}$ converges weakly to $A(x^*)$. If $A(x^*) = 0$, then $x^* \in \Omega$. On the other hand, if we suppose that $A(x^*) \neq 0$, then by the weakly lower semicontinuity of $\|\cdot\|$, we obtain that

$$0 < \|A(x^*)\| \leq \liminf_{k\to\infty} \|A(w_{n_k})\|.$$

Since $\{w_{M_k}\} \subset \{w_{n_k}\}$, we obtain that

$$0 \leq \limsup_{k\to\infty} \|\eta_k n_{M_k}\| = \limsup_{k\to\infty} \left(\frac{\eta_k}{\|A(w_{n_k})\|}\right) \leq \frac{\limsup\limits_{k\to\infty} \eta_k}{\liminf\limits_{k\to\infty} \|A(w_{n_k})\|} = 0,$$

which implies that, $\lim\limits_{k\to\infty} \|\eta_k n_{M_k}\| = 0$. Thus, letting $k \to \infty$ in (4.12) yields

$$\langle A(x), x - x^*\rangle \geq 0 \ \forall x \in C, \tag{4.13}$$

which implies by Lemma 2.1 that $x^* \in \Omega$. □

Theorem 4.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then, under the Assumptions 3.1, if $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, 0 \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. Then, $\{x_n\}$ converges strongly to $p \in \Omega$, where $\|p\| = \min\{\|x^*\| : x^* \in \Omega\}$.*

Proof. Let $p \in \Omega$. To start with observe that

$$
\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n\|x_{n-1} - p\|\|x_n - p\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n N_1] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2, \qquad (4.14)
\end{aligned}
$$

where $N_2 := 2\|x_n - x^*\| + \alpha_n N_1$. In addition, we have that

$$
\begin{aligned}
&\|(1-\beta_n)x_n + \beta_n v_n - p\|^2 \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(v_n - p)\|^2 \\
&= (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|v_n - p\|^2 + 2(1-\beta_n)\beta_n\langle x_n - p, v_n - p\rangle \\
&\leq (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + 2(1-\beta_n)\beta_n\|x_n - p\|\|v_n - p\| \\
&\leq (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\beta_n)\beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n)\beta_n\|v_n - p\|^2 \\
&\leq (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\beta_n)\beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n)\beta_n\|w_n - p\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2. \qquad (4.15)
\end{aligned}
$$

More so, we have that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|(1-\alpha_n)[(1-\beta_n)x_n + \beta_n v_n - p] - [\beta_n\alpha_n(x_n - v_n) + \alpha_n p]\|^2 \\
&\leq (1-\alpha_n)^2\|(1-\beta_n)x_n + \beta_n v_n - p\|^2 - 2\langle\beta_n\alpha_n(x_n - v_n) + \alpha_n p, x_{n+1} - p\rangle
\end{aligned}
$$

$$\begin{aligned}
&\leq (1-\alpha_n)^2\|(1-\beta_n)x_n + \beta_n v_n - p\|^2 \\
&\quad + 2\langle \beta_n\alpha_n(x_n - v_n), x_{n+1} - p\rangle + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\alpha_n)[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2] \\
&\quad + 2\alpha_n\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\beta_n\|x_n - v_n\|\|x_{n+1} - p\| \\
&\quad + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 n + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[2\beta_n\|x_n - v_n\|\|x_{n+1} - p\| \\
&\quad + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle] \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\delta_n, \qquad (4.16)
\end{aligned}$$

where $\delta_n := 2\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle$. According to Lemma 2.2, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\delta_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \qquad (4.17)$$

To establish that $\limsup_{k\to\infty}\delta_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (7) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\} \\
&= \liminf_{k\to\infty}\{(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|)(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|)\} \geq 0. \qquad (4.18)
\end{aligned}$$

Now, using Algorithm 3.2, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \beta_n v_n - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(v_n - p) - \alpha_n p\|^2 \\
&\leq \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(v_n - p)\|^2 + \alpha_n^2\|p\|^2 \\
&\quad - 2\alpha_n\langle (1-\alpha_n-\beta_n)(x_n - p) + \beta_n(v_n - p), p\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2+\alpha_n M\\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|v_n-p\|^2\\
&\quad -(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M\\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2\\
&\quad -(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M\\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2\\
&\quad -(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M, \qquad (4.19)
\end{aligned}$$

for some $M>0$. It implies from (4.18) that

$$\begin{aligned}
&\limsup_{k\to\infty}[(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|v_{n_k}-x_{n_k}\|^2]\\
&\leq \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+\alpha_{n_k}M]\\
&\leq -\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\leq 0, \qquad (4.20)
\end{aligned}$$

which gives

$$\lim_{k\to\infty}\|v_{n_k}-x_{n_k}\|=0. \qquad (4.21)$$

Also, using Algorithm 3.2 and (4.4), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n-\beta_n)x_n+\beta_n v_n-p\|^2\\
&= \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)-\alpha_n p\|^2\\
&\leq \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2+\alpha_n^2\|p\|^2\\
&\quad -2\alpha_n\langle(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p),p\rangle\\
&\leq \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2+\alpha_n M\\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|v_n-p\|^2\\
&\quad -(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M\\
&\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\|w_n-p\|^2\\
&\quad -\rho[2-\rho-\mu(2(1-\rho)+\mu)]\|w_n-u_n\|^2\\
&\quad -(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M. \qquad (4.22)
\end{aligned}$$

It can be deduced from (4.18) that

$$\begin{aligned}
&\limsup_{k\to\infty}[\rho[2-\rho-\mu(2(1-\rho)+\mu)]\|w_{n_k}-u_{n_k}\|^2] \\
&\leq \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2 \\
&\quad -(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|v_n-x_n\|^2+\alpha_{n_k}M] \\
&\leq -\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\leq 0, \qquad (4.23)
\end{aligned}$$

which gives

$$\lim_{k\to\infty}\|w_{n_k}-u_{n_k}\|=0. \qquad (4.24)$$

Notice that as $k\to\infty$, we have

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}||x_{n_k}-x_{n_k-1}||=\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||\to 0. \qquad (4.25)$$

In addition, we have the following

$$\|w_{n_k}-v_{n_k}\|\leq\|w_{n_k}-x_{n_k}\|+\|x_{n_k}-v_{n_k}\|\to 0 \text{ as } k\to\infty, \qquad (4.26)$$

$$\|u_{n_k}-x_{n_k}\|\leq\|u_{n_k}-v_{n_k}\|+\|v_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.27)$$

From the Algorithm 3.2 and (4.21), observe that

$$\begin{aligned}
\|x_{n_k+1}-v_{n_k}\| &= \|(1-\alpha_n-\beta_n)x_{n_k}+\beta_n v_{n_k}-v_{n_k}\| \\
&\leq (1-\alpha_{n_k}-\beta_{n_k})\|x_{n_k}-v_{n_k}\|+\beta_{n_k}\|v_{n_k}-v_{n_k}\| \\
&\quad +\alpha_{n_k}\|v_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.28)
\end{aligned}$$

Using (4.28) and (4.21), it gives

$$\|x_{n_k+1}-x_{n_k}\|\leq\|x_{n_k+1}-v_{n_k}\|+\|v_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.29)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^*\in H_1$. By (4.21), (4.25) and (4.27), we have that the subsequences $\{w_{n_{k_j}}\}$ of $\{w_{n_k}\}$, $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ and $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$, all converge weakly to x^* respectively. From (4.24) and Lemma 4.3, we have that $x^*\in\Omega$.

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to x^* such that

$$\limsup_{k\to\infty}\langle p,p-x_{n_k}\rangle=\lim_{j\to\infty}\langle p,p-x_{n_{k_j}}\rangle=\langle p,p-x^*\rangle. \qquad (4.30)$$

Hence, since $p=P_\Omega 0$, we have obtain from (4.30) that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \langle p, p - x^*\rangle \le 0, \tag{4.31}$$

which implies that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k+1}\rangle \le 0. \tag{4.32}$$

Using using our assumption, (4.21) and (4.32), we have that $\limsup_{k\to\infty}\delta_{n_k} := 2\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle \le 0$. Thus, the last part of Lemma 2.2 is achieved. Hence, we have that $\lim_{n\to\infty}\|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p \in \Omega$. □

5. Application to Equilibrium Problem

In this section, we apply our results to equilibrium problem.

The equilibrium problem is one of the interesting problems in this area of research. Equilibrium problems are special cases of monotone inclusion problems, saddle point problems, minimization problems, optimization problems, variational inequality problems, Nash equilibria in noncooperative games, and various forms of feasibility problems. Let C be a closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction, the equilibrium problem is defined as finding $x \in C$ such that

$$F(x, y) \ge 0 \;\forall\; y \in C. \tag{5.1}$$

The solution set for x is denoted by $EP(F)$. It is well-known that to approximate the solution of problem (5.1), we assume the bifunction F satisfy the following well-known conditions:

(1) $F(x, x) = 0 \;\forall x \in C$,
(2) F is monotone, that is $F(x, y) + F(y, x) \le 0 \;\forall\; x, y \in C$,
(3) for each $x, y, z \in C$ $\lim_{t\to 0^+} F(\alpha z + (1-\alpha)x, y) \le F(x, y)$,
(4) for each $x \in C, y \to F(x, y)$ is convex and lower semi-continuous.

Lemma 5.1. *[6] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(1)-(4)$. Suppose that $\lambda > 0$ and $x \in H$, thus, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \ge 0, \;\; \forall y \in C. \tag{5.2}$$

In addition, if

$$J_\lambda^F x = \{x \in C : F(z,y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}, \tag{5.3}$$

then the following hold:

(1) J_λ^F *is single valued and firmly nonexpansive,*
(2) $F(J_\lambda^F) = EP(F)$,
(3) $EP(F)$ *is closed and convex.*

We note that J_λ^F is the resolvent of F for $\lambda > 0$.

Lemma 5.2. *[23] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(1) - (4)$. Let B_F be a set valued mapping of H into H defined by*

$$B_F = \begin{cases} \{z \in H : F(x,y) + \langle y - x, z\rangle \geq 0\ \forall y \in C\} & \text{if } x \in C \\ \emptyset, & \text{otherwise}. \end{cases} \tag{5.4}$$

Then $EP(F) = B_F^{-1}(0)$ and B_F is a maximal monotone operator with $Dom(B_F) \subset C$. Furthermore, for any $x \in H$ and $\lambda > 0$, the resolvent J_λ^F of F coincides with the resolvent of B_F, that is

$$J_\lambda^F(x) = (I + B_F)^{-1}(x).$$

Using the above results. Setting $A = 0$, and $J_\lambda^F(x) = (I + B_F)^{-1}(x)$ from Lemma 5.2, we obtain the following algorithm and result.

Assumption 5.1. *Suppose that the following conditions hold:*

(1) *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
(2) *$F : C \times C \to \mathbb{R}$ be a function satisfying conditions $(1) - (4)$.*
(3) *The solution set $\Omega = EP(F) \neq \emptyset$.*

Algorithm 5.2. Initialization: *Given $\gamma, \kappa > 0$, $\rho \in (0,1]$ and $\theta_n, \beta_n, \alpha_n, \mu, l, \in (0,1)$, for all $n \in \mathbb{N}$, let $x_0, x_1, \in H$ be arbitrary.*

Iterative step:

Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$,* *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\theta}{\kappa}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \textit{if } x_n \neq x_{n-1}, \\ \\ \frac{\theta}{\kappa}, & \textit{otherwise}, \end{cases} \tag{5.5}$$

where $\theta > 0$ *and* $\{\epsilon_n\}$ *is a positive sequence such that* $\epsilon_n = \circ(\alpha_n)$.
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = J^F_{\lambda_n} w_n, \tag{5.6}$$

where λ_n *is chosen to be the largest* $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ *satisfying*

$$\lambda\|Aw_n - Au_n\| \leq \mu\|w_n - u_n\|. \tag{5.7}$$

Step 3. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n((1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n)). \tag{5.8}$$

Stopping criterion: *Set* $n := n+1$ *and go back to* **Step 1.**

Theorem 5.3. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 5.2. Then, under the Assumptions 5.1, if* $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $0 \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Then,* $\{x_n\}$ *converges strongly to* $p \in \Omega$, *where* $\|p\| = \min\{\|x^*\| : x^* \in \Omega\}$.

6. Numerical Example

In this section, we present some numerical examples in finite dimensional Hilbert spaces and compare our proposed Algorithm 3.2 with Algorithm 3.1 of [8]. In addition, we compare our proposed Algorithm 3.2 with Algorithm 3.1 of [27].

Example 1. *Let* $H = L^2([0,1])$ *and norm* $\|x\| = (\int_0^1 |x(t)|dt)^{\frac{1}{2}}$ *and the inner product* $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ *for all* $x, y \in L^2([0,1])$. *Define the operator* $A : L^2([0,1]) \to L^2([0,1])$ *by*

$$Ax(t) = \max\{0, x(t)\}. \tag{6.1}$$

Suppose that $C = \{x \in H : \|x\| \leq 1\}$ *is a unit ball, then*

$$P_C(x) = \begin{cases} \frac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} > 1, \\ x, & \text{if } \|x\|_{L^2} \leq 1. \end{cases} \tag{6.2}$$

Choose $\gamma = 0.03, l = 1, \mu = 0.38, \kappa = 1, \theta = 0.001, \alpha_n = \frac{1}{n+1}, \epsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{3n}{3n+5}, \rho = 0.38$. *It is easy to verify that all hypothesis of Theorem 4.1 are satisfied and the set of solutions to the* $VI(A, C)$ (1.1) *is given by* $\Omega = \{0\} \neq \emptyset$. *We use different choices of* x_0, x_1 *and test the convergence of our algorithm with* $\|x_{n+1} - x_n\| < 10^{-5}$ *as stopping criterion. We compare the performance of Algorithm 3.2 with the Algorithm 3.1 of Cholamjiak et al. [8].*

(1) *Case I:* $x_0(t) = \frac{-3te^{2t}}{5}, x_1(t) = e^{-2t}$.
(2) *Case II:* $x_0(t) = \sin 5t, x_1(t) = \cos -3t$.
(3) *Case III:* $x_0(t) = t^3 + 1, x_1(t) = e^{2t}$.
(4) *Case IV:* $x_0(t) = e^{3t}, x_1(t) = -2\sin 2t$.

The computational results are shown in Table 1 and Figure 1.

TABLE 1. Computation result for Example 1.

		Algorithm 3.2	*Algorithm 3.1 of [8]*
Case I	*No of Iter.*	*14*	*12*
	CPU time (sec)	*2.6427*	*4.0313*
Case II	*No of Iter.*	*13*	*12*
	CPU time (sec)	*2.5892*	*5.3755*
Case III	*No of Iter.*	*17*	*17*
	CPU time (sec)	*2.1036*	*8.2650*
Case IV	*No of Iter.*	*17*	*18*
	CPU time (sec)	*3.1534*	*7.1918*

Example 2. *Let* $H = \mathbb{R}^N$, *with the Euclidean norm on* $\mathbb{R}^N$. *Suppose that* $C = \{x \in H : \|x\| \leq 1\}$ *is the unit ball, define the operator* $A : C \to \mathbb{R}^N$ *by*

$$Ax(t) = x. \tag{6.3}$$

We have

$$P_C(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| > 1, \\ x, & \text{if } \|x\| \leq 1. \end{cases} \tag{6.4}$$

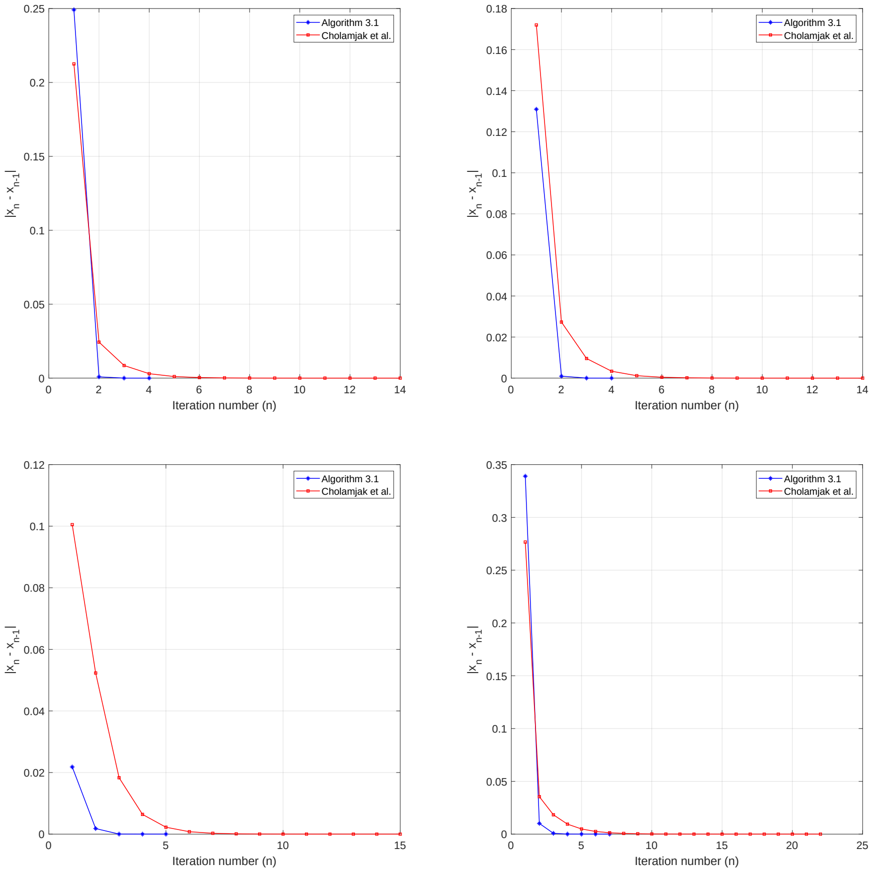

FIGURE 1. Example 1, Top Left: Case I; Top Right: Case II; Bottom Left: case III; Bottom Right: Case IV.

With these given C *and* A*, the set of solutions to the* $VI(A, C)$ (1.1) *is known to be* $\Omega = \{0\} \neq \emptyset$*. Choose* $\gamma = 0.05, l = 4, \mu = 0.38, \kappa = 0.01, \theta = 0.001, \alpha_n = \frac{1}{\sqrt{n+1}}, \epsilon_n = \frac{1}{(n+1)}, \beta_n = \frac{1}{2} + \frac{2}{(2n+4)}, \rho = 0.38$ *It is easy to verify that all hypothesis of Theorem 4.1 are satisfied. We use different choices of* x_0, x_1 *and test the convergence of our algorithm with* $\|x_{n+1} - x_n\| < 10^{-6}$ *as stopping criterion. We compare the performance of Algorithm 3.2 with the Algorithm 3.1 of Thong et al. [27].*

(1) *Case I:* $N = 5$.
(2) *Case II:* $N = 10$.
(3) *Case III:* $N = 30$.
(4) *Case IV:* $N = 50$.

The computational results are shown in Table 2 and Figure 2.

TABLE 2. Computation result for Example 2.

		Algorithm 3.2	*Algorithm 3.1 of [27]*
Case I	*No of Iter.*	*11*	*27*
	CPU time (sec)	*0.0022*	*0.0042*
Case II	*No of Iter.*	*12*	*28*
	CPU time (sec)	*0.0016*	*0.0027*
Case III	*No of Iter.*	*12*	*29*
	CPU time (sec)	*0.0014*	*0.0049*
Case IV	*No of Iter.*	*12*	*29*
	CPU time (sec)	*0.0014*	*0.0041*

7. CONCLUSION

In this work, we introduce a new inertial relaxed Tseng extrapolation method for approximating the solution of a variational inequality problem in which the underlying operator is pseudomonotone in the framework of Hilbert space. The main advantage of this method is the fact that the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to the minimum-norm of the solution set Ω. In addition, the proposed iterative algorithm is combination of both the inertial extrapolation step and relaxation parameter, which is known to help speed up the rate of convergence. Furthermore, we present some examples and numerical experiments to show the efficiency and applicability of our method in the framework of infinite and finite dimensional Hilbert spaces. The results obtained in this work extends, generalizes and improves several results in this direction.

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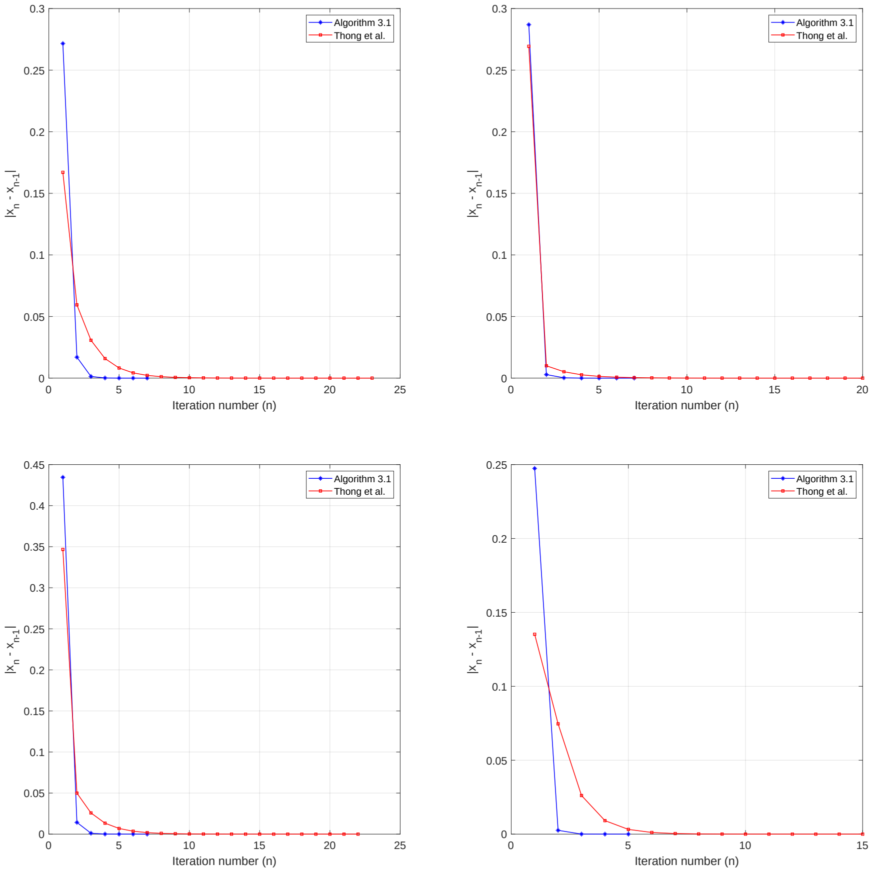

FIGURE 2. Example 2, Top Left: Case I; Top Right: Case II; Bottom Left: case III; Bottom Right: Case IV.

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