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NUMERICAL QUENCHING FOR A NONLINEAR DIFFUSION EQUATION WITH SINGULAR BOUNDARY FLUX

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ABSTRACT. In this paper, we study the semidiscrete approximation of the solution of a nonlinear diffusion equation with nonlinear source and singular boundary flux. We find some conditions under which the solution of the semidiscrete form quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

1. INTRODUCTION

In this paper, we consider the nonlinear diffusion equation with nonlinear source and singular boundary flux

$$\frac{\partial A(u)}{\partial t} = u_{xx} + (1-u)^{-\alpha}, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -B(u(1,t)), \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \tag{1.3}$$

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where $A(s)$ is an appropriately smooth function which satisfies

$$A(0) = 0,\, A(1) = 1,\, A'(s) > 0,\, A''(s) \leq 0\ \forall s > 0,$$

$B(s)$ satisfies

$$B(s) > 0,\, B'(s) < 0,\, B''(s) \geq 0, \text{ for } s > 0,\ \lim_{s \to 0^+} B(s) = +\infty$$

and $u_0 : [0, 1] \longrightarrow (0, 1)$ is nonincreasing and satisfies some compatibility conditions and α is a positive constant.

Definition 1.1. *We say that the solution u of* (1.1)–(1.3) *quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_q)$, but*

$$\lim_{t \to T_q^-} \|u(.,t)\|_\infty = 1,$$

where $\|u(.,t)\|_\infty = \max\limits_{0 \leq x \leq 1} |u(x,t)|$. The time T_q is called quenching time of the solution u.

When $A(u) = u^m$, the problem (1.1)–(1.3) is known as the classical porous medium equation which shows a number of physical phenomenon in the nature such as the flow of an isentropic gas through a porous medium [7], [8] and heat transfer or diffusion.

In recent years, the theoretical study of quenching phenomenon for nonlinear diffusion equations has been the subject of investigations of many authors. Especially for singular and degenerate parabolic equations (see [3], [4], [9], [10], [12], [13] and references therein). Local in time existence and uniqueness of the solution have been proved (see [2], [11] and references therein). In [12], the author suppose that u_0 satisfies:

$(H_1) : u_0''(x) + (1 - u_0(x))^{-\alpha} \geq 0$

$(H_2) : u_0'(x) \leq -xB(u_0(x)), 0 \leq x \leq 1.$

He shows that the solution u of (1.1)–(1.3) quenches in finite time T_q and $x = 0$ is the unique quenching point. He also shows that the time u_t blow-up at the quenching point and he gives a lower bound of the quenching time.

Our aim is to study the numerical quenching phenomenon by semidiscretization of the solution u. For this, we will be inspired by the work of certain authors who have investigated in the numerical study using the semidiscrete form (see [1], [5], [6]).

Now we assume that u_0 satisfies (H_1) and compatibility conditions. Then we rewrite problem (1.1)–(1.3) into the following form:

$$u_t = \gamma(u)u_{xx} + \gamma(u)(1-u)^{-\alpha}, \quad 0 < x < 1, \quad t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -B(u(1,t)), \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{1.6}$$

where $\gamma(u) = \dfrac{1}{A'(u)}$. We organise this paper as follows.

In the next section, we give some lemmas which will be used throughout the paper. In the fourth section, under some hypotheses, we show that the solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time. In the fifth section, we give a result about the convergence of the semidiscrete quenching time to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. The Semidiscrete Problem

Let I be a nonnegative integer, we set $h = \dfrac{1}{I}$, and we define the grid, $x_i = ih$, $i = 0, \ldots, I.$. We approximate the solution u of the problem (1.4)–(1.6) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$. For $t \in (0, T_q^h)$, we have

$$\frac{dU_i(t)}{dt} = \gamma(U_i(t))\delta^2 U_i(t) + \gamma(U_i(t))(1-U_i(t))^{-\alpha}, \quad 1 \le i \le I-1, \tag{2.1}$$

$$\frac{dU_0(t)}{dt} = \gamma(U_0(t))\delta^2 U_0(t) + \gamma(U_0(t))(1-U_0(t))^{-\alpha}, \tag{2.2}$$

$$\frac{dU_I(t)}{dt} = \gamma(U_I(t))\delta^2 U_I(t) + \gamma(U_I(t))(1-U_I(t))^{-\alpha} - \frac{2\gamma(U_I(t))B(U_I(t))}{h}, \tag{2.3}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \tag{2.4}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \le i \le I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2},$$
$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$
$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}, \quad 0 \le i \le I-1,$$
$$\delta^+ \varphi_i \le 0, \quad 0 \le i \le I-1,$$

$\gamma(U_i(t))$ is an approximation of $\gamma(u(x_i, t))$, $0 \le i \le I$. Here, $[0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$ where

$$\|U_h(t)\|_\infty = \max_{0 \le i \le I} |U_i(t)|.$$

When the time T_q^h is finite, then we say that the solution U_h of (2.1)–(2.4) quenches in a finite time, and the time T_q^h is called the quenching time of the solution U_h.

3. Properties of the Semidiscrete Problem

In this section, we give some important results which will be used later.

Lemma 3.1. *Let* $b_h(t) \in C^0([0,T), \mathbb{R}^{I+1})$, $f_h(t) \in C^0([0,T), \mathbb{R}_+^{I+1})$ *and* $V_h(t) \in C^1([0,T), \mathbb{R}^{I+1})$ *such that*

$$\frac{dV_i(t)}{dt} - f_i(t)\delta^2 V_i(t) + b_i(t) V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in [0, T),$$

$$V_i(0) \ge 0, \quad 0 \le i \le I.$$

Then $V_i(t) \ge 0$, $0 \le i \le I$, $t \in [0, T)$.

Proof. Let $T_0 < T$. Define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that $b_i(t) - \lambda > 0$ for $t \in [0, T_0]$, $0 \le i \le I$. Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$. For all $i \in \{0, \dots, I\}$, $Z_i(t)$ is continuous on the compact $[0, T_0]$; there exists $i_0 \in \{0, \dots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$.

We observe that:

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I, \tag{3.1}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, 1 \le i_0 \le I-1, \tag{3.2}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0, \quad i_0 = 0, \tag{3.3}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0, \quad i_0 = I. \tag{3.4}$$

Moreover, by a straightforward computation, we get

$$\frac{dZ_{i_0}(t_0)}{dt} - f_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + (b_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{3.5}$$

Using (3.1)–(3.4), we deduce from (3.5) that $(b_{i_0}(t_0)-\lambda)Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$. We deduce that $V_h(t) \geq 0$, $\forall t \in [0, T_0]$ and the proof is complete. □

Another form of the maximum principle for semidiscrete equations is the comparison lemma below.

Lemma 3.2. *Let $V_h(t)$, $W_h(t) \in C^1([0,T), \mathbb{R}^{I+1})$ and $g \in C^0(\mathbb{R}, \mathbb{R})$ such that $\forall t \in [0,T)$ and $0 \leq i \leq I$,*

$$\frac{dV_i(t)}{dt} - \gamma(V_i(t))\delta^2 V_i(t) + g(V_i(t)) < \frac{dW_i(t)}{dt} - \gamma(W_i(t))\delta^2 W_i(t) + g(W_i(t)), \tag{3.6}$$

$$V_i(0) < W_i(0). \tag{3.7}$$

Then $V_i(t) < W_i(t)$, $0 \leq i \leq I$, $t \in [0,T]$.

Proof. Let $Z_h(t)$ a vector such that $Z_i(t) = W_i(t) - V_i(t)$ and let t_0, be the first $t > 0$ such that $Z_{i_0}(t) > 0$, $\forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. We observe that:

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad 0 \leq i_0 \leq I,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0, \quad i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0, \quad i_0 = I.$$

This implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\gamma(V_{i_0}(t_0))\delta^2 Z_{i_0}(t_0)-\gamma'(\theta_{i_0}(t_0))Z_{i_0}(t_0)\delta^2 W_{i_0}(t_0)+g(W_{i_0}(t_0))-g(V_{i_0}(t_0))\leq 0.$$

Here θ_{i_0} is an intermediate value between V_{i_0} and W_{i_0}. This inequality contradicts (3.6) which ends the proof. □

Lemma 3.3. *Let U_h be the solution of* (2.1)–(2.4). *Then we have for $t \in [0, T_q^h)$ and $0 \leq i \leq I-1$,*

$$U_i(t) > U_{i+1}(t).$$

Proof. Introduce the vector $Z_h(t)$ such that $Z_i(t) = U_i(t) - U_{i+1}(t)$ for $t \in (0, T_q^h)$, $i \in \{0, \ldots, I-1\}$. Let t_0, be the first $t > 0$ such that $Z_{i_0}(t) > 0$, $\forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I-1\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0}\frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \leq 0, \quad 0 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \leq i_0 \leq I-2,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_1(t_0) - 3Z_0(t_0)}{h^2} > 0, \quad i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{I-2}(t_0) - 3Z_{I-1}(t_0)}{h^2} > 0, \quad i_0 = I-1.$$

Moreover, by a straightforward computation, we get

$$\frac{dZ_{i_0}(t_0)}{dt} - \gamma(U_{i_0}(t_0))\delta^2 Z_{i_0}(t_0) - \gamma'(\zeta_{i_0}(t_0))Z_{i_0}(t_0)\delta^2 U_{i_0+1}(t_0)$$
$$- \gamma'(\zeta_{i_0}(t_0))Z_{i_0}(t_0)(1-U_{i_0}(t_0))^{-\alpha} - \alpha\gamma(U_{i_0+1}(t_0))(1-\beta_{i_0}(t_0))^{-\alpha-1}Z_{i_0}(t_0) < 0,$$

$0 \leq i_0 \leq I-2$,

$$\frac{dZ_{I-1}(t_0)}{dt} - \gamma(U_{I-1}(t_0))\delta^2 Z_{I-1}(t_0) - \gamma'(\zeta_{I-1}(t_0))Z_{I-1}(t_0)\delta^2 U_I(t_0)$$
$$- \gamma'(\zeta_{I-1}(t_0))Z_{I-1}(t_0)(1-U_{I-1}(t_0))^{-\alpha} - \alpha\gamma(U_I(t_0))(1-\beta_{I-1}(t_0))^{-\alpha-1}Z_{I-1}(t_0)$$
$$- \frac{2\gamma(U_I(t_0))}{h}B(U_I(t_0)) < 0.$$

But these inequalities contradict (2.1)–(2.3) and this proof is complete. □

Lemma 3.4. *Let U_h be the solution of* (2.1)–(2.4). *Then we have*

$$\frac{dU_i(t)}{dt} > 0, \quad 0 \le i \le I, \quad t \in (0, T_q^h).$$

Proof. Consider the vector $Z_h(t)$ such that $Z_i(t) = \dfrac{dU_i(t)}{dt}$, $t \in (0, T_q^h)$, $i \in \{0, \ldots, I\}$. Let t_0, be the first $t \in (0, T_h^q)$ such that $Z_{i_0}(t) > 0$, $\forall t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \le i_0 \le I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} > 0, \quad i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} > 0, \quad i_0 = I.$$

Moreover, by a straightforward computation, we get

$$\begin{aligned}&\frac{dZ_{i_0}(t_0)}{dt} - \gamma(U_{i_0}(t_0))\delta^2 Z_{i_0}(t_0) - \gamma'(U_{i_0}(t_0))Z_{i_0}(t_0)\delta^2 U_{i_0}(t_0)\\ &- \alpha\gamma(U_{i_0}(t_0))(1 - U_{i_0}(t_0))^{-\alpha-1} Z_{i_0}(t_0) - \gamma'(U_{i_0}(t_0))(1 - U_{i_0}(t_0))^{-\alpha} Z_{i_0}(t_0) < 0,\end{aligned}$$

$0 \le i_0 \le I-1$,

$$\begin{aligned}&\frac{dZ_I(t_0)}{dt} - \gamma(U_I(t_0))\delta^2 Z_I(t_0) - \gamma'(U_I(t_0))\delta^2 U_I(t_0) Z_I(t_0)\\ &- \alpha\gamma(U_I(t_0))(1 - U_I(t_0))^{-\alpha-1} Z_I(t_0) - \gamma'(U_I(t_0))(1 - U_I(t_0))^{-\alpha} Z_I(t_0)\\ &+ \frac{2\gamma(U_I(t))}{h} B'(U_I(t)) Z_I(t_0) + \frac{2\gamma'(U_I(t_0))}{h} B(U_I(t)) Z_I(t_0) < 0.\end{aligned}$$

But these inequalities contradict (2.1)–(2.3) and this proof is complete. □

4. Semidiscrete quenching time

In this section, we show that under some assumptions, the solution U_h of (2.1)–(2.4) quenches in a finite time and estimate its semidiscrete quenching time.

Lemma 4.1. *Let $U_h \in \mathbb{R}^{I+1}$ such that $\|U_h\|_\infty < 1$ and let p be a positive constant. Then, we have*

$$\delta^2(1-U_i)^{-p} \geq p(1-U_i)^{-p-1}\delta^2 U_i,\ 0 \leq i \leq I.$$

Proof. Let us introduce $f(s) = (1-s)^{-p}$. We observe that f is a convex function for nonnegative values of s. Apply Taylor's expansion to obtain

$$\delta^2 f(U_0) = f'(U_0)\delta^2 U_0 + \frac{(U_1 - U_0)^2}{h^2} f''(\theta_0),$$

$$\delta^2 f(U_i) = f'(U_i)\delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{2h^2} f''(\theta_i) + \frac{(U_{i-1} - U_i)^2}{2h^2} f''(\eta_i), 1 \leq i \leq I-1,$$

$$\delta^2 f(U_I) = f'(U_I)\delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{h^2} f''(\eta_I),$$

where θ_i is an intermediate between U_i and U_{i+1} and η_i the one between U_{i-1} and U_i. We use the fact that $\|U_h\|_\infty < 1$ to complete the proof. □

Theorem 4.1. *Let U_h be a solution of* (2.1)–(2.4)*, and assume that there exist a nonnegative constant $\tau \in (0,1]$ such that the initial data at* (2.4) *satisfies*

$$\gamma(\varphi_i)\delta^2\varphi_i + \gamma(\varphi_i)(1-\varphi_i)^{-\alpha} \geq \tau(1-\varphi_i)^{-\alpha}, \quad 0 \leq i \leq I-1, \tag{4.1}$$

$$\gamma(\varphi_I)\delta^2\varphi_I + \gamma(\varphi_I)(1-\varphi_I)^{-\alpha} - \frac{2\gamma(\varphi_I)}{h} B(\varphi_I) \geq \tau(1-\varphi_I)^{-\alpha}. \tag{4.2}$$

Then, the solution U_h quenches in a finite time T_q^h and we have the following estimate

$$T_q^h \leq \frac{(1 - \|\varphi_h\|_\infty)^{\alpha+1}}{\tau(\alpha+1)}.$$

Proof. Let $[0, T_q^h)$ be the maximal time interval on which $\|U_h\|_\infty < 1$. We consider the function $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} - \tau(1-U_i(t))^{-\alpha}, \quad 0 \leq i \leq I. \tag{4.3}$$

By a straightforward computation we get

$$\frac{dJ_i(t)}{dt} - \gamma(U_i(t))\delta^2 J_i(t) = \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \gamma(U_i(t))\delta^2 U_i(t)\right)$$
$$+ \gamma'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t) - \tau\alpha(1-U_i(t))^{-\alpha-1}\frac{dU_i(t)}{dt} + \tau\gamma(U_i(t))\delta^2(1-U_i(t))^{-\alpha},$$

$0 \le i \le I$. From Lemma 4.1, we have $\tau\delta^2(1-U_i(t))^{-\alpha} \ge \alpha\tau(1-U_i(t))^{-\alpha-1}\delta^2 U_i(t)$, $0 \le i \le I$, which implies that

$$\frac{dJ_i(t)}{dt} - \gamma(U_i(t))\delta^2 J_i(t)$$
$$\ge \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \gamma(U_i(t))\delta^2 U_i(t)\right) + \gamma'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t)$$
$$- \tau\alpha(1-U_i(t))^{-\alpha-1}\frac{dU_i(t)}{dt} + \tau\alpha\gamma(U_i(t))(1-U_i(t))^{-\alpha-1}\delta^2 U_i(t),$$
$$\frac{dJ_i(t)}{dt} - \gamma(U_i(t))\delta^2 J_i(t) \ge \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \gamma(U_i(t))\delta^2 U_i(t)\right)$$
$$- \tau\alpha(1-U_i(t))^{-\alpha-1}(\frac{dU_i(t)}{dt} - \gamma(U_i(t))\delta^2 U_i(t)) + \gamma'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t), 0 \le i \le I.$$

We get

$$\frac{dJ_i(t)}{dt} - \gamma(U_i(t))\delta^2 J_i(t) \ge \frac{d}{dt}\gamma(U_i(t))(1-U_i(t))^{-\alpha}$$
$$- \tau\alpha(1-U_i(t))^{-\alpha-1}\gamma(U_i(t))(1-U_i(t))^{-\alpha} + \gamma'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t), 0 \le i \le I-1,$$
$$\frac{dJ_I(t)}{dt} - \gamma(U_I(t))\delta^2 J_I(t) \ge \frac{d}{dt}\left(\gamma(U_I(t))(1-U_I(t))^{-\alpha} - \frac{2\gamma(U_I(t))}{h}B(U_I(t))\right)$$
$$- \tau\alpha(1-U_I(t))^{-\alpha-1}(\gamma(U_I(t))(1-U_I(t))^{-\alpha} - \frac{2\gamma(U_I(t))}{h}B(U_I(t)))$$
$$+ \gamma'(U_I(t))\frac{dU_I(t)}{dt}\delta^2 U_I(t),$$

$$\frac{dJ_i(t)}{dt} - \gamma(U_i(t))\delta^2 J_i(t) \ge \alpha\gamma(U_i(t))(1-U_i(t))^{-\alpha-1}\left(\frac{dU_i(t)}{dt} - \tau(1-U_i(t))^{-\alpha}\right)$$
$$+ \gamma'(U_i(t))\frac{dU_i(t)}{dt}(\delta^2 U_i(t) + (1-U_i(t))^{-\alpha}), 0 \le i \le I-1,$$

$$\frac{dJ_I(t)}{dt} - \gamma(U_I(t))\delta^2 J_I(t) \geq \alpha\gamma(U_I(t))(1-U_I(t))^{-\alpha-1}\left(\frac{dU_I(t)}{dt} - \tau(1-U_I(t))^{-\alpha}\right)$$
$$+\gamma'(U_I(t))\frac{dU_I(t)}{dt}(\delta^2 U_I(t) + (1-U_I(t))^{-\alpha} - \frac{2}{h}B(U_I(t)))$$
$$+\frac{2}{h}\gamma(U_I(t))(\alpha\tau(1-U_I(t))^{-\alpha-1}B(U_I(t)) - B'(U_I(t))\frac{dU_I(t)}{dt}).$$

Finally, we get

$$\frac{dJ_i(t)}{dt} - \gamma(U_i(t))\delta^2 J_i(t) \geq \alpha\gamma(U_i(t))(1-U_i(t))^{-\alpha-1}J_i(t)$$
$$+\gamma'(U_i(t))\frac{dU_i(t)}{dt}A'(U_i(t))\frac{dU_i(t)}{dt},$$

$0 \leq i \leq I-1$,

$$\frac{dJ_I(t)}{dt} - \gamma(U_I(t))\delta^2 J_I(t) \geq \alpha\gamma(U_I(t))(1-U_I(t))^{-\alpha-1}J_I(t)$$
$$+\gamma'(U_I(t))\frac{dU_I(t)}{dt}A'(U_I(t))\frac{dU_I(t)}{dt} + \frac{2}{h}\gamma(U_I(t))$$
$$\cdot\left(\alpha\tau(1-U_I(t))^{-\alpha-1}B(U_I(t)) - B'(U_I(t))\frac{dU_I(t)}{dt}\right).$$

From (4.1)–(4.2), we observe that $J_i(0) \geq 0$ for $0 \leq i \leq I$. We deduce from Lemma 3.1 that $J_i(t) \geq 0$, $0 \leq i \leq I$. Which implies that

$$dU_i(t) \geq \tau(1-U_i(t))^{-\alpha}dt, \quad 0 \leq i \leq I, \quad t \in [0, T_q^h).$$

These inequalities can be rewritten as follows

$$(1-U_i(t))^{\alpha}dU_i(t) \geq \tau dt, \quad 0 \leq i \leq I, \quad t \in [0, T_q^h).$$

Integrating the above inequalities over the interval (t, T_q^h), we get

$$T_q^h - t \leq \frac{(1-U_i(t))^{\alpha+1}}{\tau(\alpha+1)}, \quad 0 \leq i \leq I, \quad t \in [0, T_q^h). \tag{4.4}$$

Taking $t = 0$ and $i = 0$, we obtain:

$$T_q^h \leq \frac{(1-\varphi_0)^{\alpha+1}}{\tau(\alpha+1)}.$$

Using the fact that $\|\varphi_h\|_\infty = \varphi_0$ thanks to the Lemma 3.3, we get:

$$T_q^h \leq \frac{(1-\|\varphi_h\|_\infty)^{\alpha+1}}{\tau(\alpha+1)}.$$

We have the desired result. □

Remark 4.1. *By replacing t by t_0 and i by 0 in* (4.4)*, we obtain*

$$T_q^h - t_0 \leq \frac{(1 - \|U_h(t_0)\|_\infty)^{\alpha+1}}{\tau(\alpha+1)}, \quad t_0 \in [0, T_q^h),$$

and

$$\|U_h(t_0)\|_\infty \leq 1 - C_1(T_q^h - t_0)^{\frac{1}{\alpha+1}},$$

where $C_1 = (\tau(\alpha+1))^{\frac{1}{\alpha+1}}$.

The Remark 4.1 is crucial to prove the convergence of the semidiscrete quenching time.

5. Convergence of Semidiscrete Quenching Time

Theorem 5.1. *Assume that the problem* (1.4)-(1.6) *has a solution $u \in C^{4,1}([0,1] \times [0,T])$ such that $\sup_{t\in[0,T]} \|u(.,t)\|_\infty = \zeta < 1$. Suppose that the initial data at* (2.4) *verifies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \longrightarrow 0. \tag{5.1}$$

Then, for h small enough, the semidiscrete problem (2.1)–(2.4) *has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that*

$$\max_{t\in[0,T]} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h) \quad as \quad h \longrightarrow 0,$$

where $T < \min\{T_q; T_q^h\}$.

Proof. Since $u \in C^{4,1}([0,1] \times [0,T])$, there exists a positive constant ξ such that

$$\frac{\|u_{xxx}\|_\infty}{3} \leq \xi \quad \text{and} \quad \frac{\|u_{xxxx}\|_\infty}{12} \leq \xi. \tag{5.2}$$

The problem (2.1)–(2.4) has for each h, a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$. Let $t(h) \leq T$ the greatest value of $t > 0$ such that there exists a positive constant β (with $\zeta < \beta < 1$) such that

$$\|U_h(t) - u_h(t)\|_\infty < \frac{\beta - \zeta}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{5.3}$$

The relation (5.1) implies that $t(h) > 0$ for h small enough. By the triangular inequality, we obtain

$$\|U_h(t)\|_\infty \leq \|u(.,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad for \quad t \in (0, t(h)),$$

which implies that

$$\|U_h(t)\|_\infty \leq \zeta + \frac{\beta - \zeta}{2} = \frac{\beta + \zeta}{2} < 1, \quad for \quad t \in (0, t(h)). \tag{5.4}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0, t(h))$ and $0 \leq i \leq I-1$,

$$\begin{aligned}
&\frac{de_0(t)}{dt} - \gamma(u(x_0,t))\delta^2 e_0(t)\\
&= [\alpha\gamma(u(x_0,t))(1-\beta_0(t))^{-\alpha-1} + \gamma'(\eta_0(t))(1-U_0(t))^{-\alpha}\\
&+ \gamma'(\eta_0(t))\delta^2 U_0(t)]e_0(t) + \gamma(u(x_0,t))h\left(\frac{h}{12}u_{xxxx}(\tilde{x}_0,t) + \frac{2}{3}u_{xxx}(x_0,t)\right)\\
&\frac{de_i(t)}{dt} - \gamma(u(x_i,t))\delta^2 e_i(t)\\
&= [\alpha\gamma(u(x_i,t))(1-\beta_i(t))^{-\alpha-1} + \gamma'(\eta_i(t))(1-U_i(t))^{-\alpha}\\
&+ \gamma'(\eta_i(t))\delta^2 U_i(t)]e_i(t) + \gamma(u(x_i,t))\frac{h^2}{12}u_{xxxx}(\tilde{x}_i,t),\\
&\frac{de_I(t)}{dt} - \gamma(u(x_I,t))\delta^2 e_I(t)\\
&= [\alpha\gamma(u(x_I,t))(1-\lambda_I(t))^{-\alpha-1} + \gamma'(\theta_I(t))(1-U_I(t))^{-\alpha}\\
&+ \gamma'(\theta_I(t))\delta^2 U_I(t) - \frac{2}{h}\gamma'(\theta_I(t))B(U_I(t)) - \frac{2}{h}\gamma(u(x_I,t))B'(\sigma_I(t))]e_I(t)\\
&+ \gamma(u(x_I,t))h\left(\frac{h}{12}u_{xxxx}(\tilde{x}_I,t) - \frac{2}{3}u_{xxx}(x_I,t)\right).
\end{aligned}$$

Using (5.2) and (5.4), there exist M and K nonnegative constants such that

$$\frac{de_0(t)}{dt} - \delta^2 e_0(t) \leq M|e_0(t)| + Kh,$$

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq M|e_i(t)| + Kh^2, \quad 1 \leq i \leq I-1,$$

$$\frac{de_I(t)}{dt} - \delta^2 e_I(t) \leq \frac{M}{h}|e_I(t)| + Kh.$$

Let $Z_h(t)$ the vector defined by

$$Z_i(t) = e^{(M+1)t}(||\varphi_h - u_h(0)||_\infty + Kh), 0 \le i \le I.$$

A simple calculation give

$$\frac{dZ_0(t)}{dt} - \delta^2 Z_0(t) > M|Z_0(t)| + Kh,$$

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) > M|Z_i(t)| + Kh^2, \quad 1 \le i \le I-1,$$

$$\frac{dZ_I(t)}{dt} - \delta^2 Z_I(t) > \frac{M}{h}|Z_I(t)| + Kh,$$

$$Z_i(0) > e_i(0), \quad 0 \le i \le I.$$

From Lemma 3.2, we obtain

$$Z_i(t) > e_i(t), \quad t \in (0, t(h)), \quad 0 \le i \le I.$$

By analogy, we also prove that

$$Z_i(t) > -e_i(t), \quad t \in (0, t(h)), \quad 0 \le i \le I.$$

Hence we have

$$Z_i(t) > |e_i(t)|, \quad t \in (0, t(h)), \quad 0 \le i \le I.$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + Kh)e^{(M+1)t}, \quad t \in (0, t(h)).$$

Next we prove that $t(h) = T$. Suppose that $t(h) < T$. From (5.3), we obtain

$$\frac{\beta - \zeta}{2} \le \|U_h(t(h)) - u_h(t(h))\|_\infty \le (\|\varphi_h - u_h(0)\|_\infty + Kh)e^{(M+1)T}. \tag{5.5}$$

Since $(\|\varphi_h - u_h(0)\|_\infty + Kh)e^{(M+1)T} \longrightarrow 0$ as $h \longrightarrow 0$, we deduce that $\frac{\beta - \zeta}{2} \le 0$, which is impossible. Hence we have $t(h) = T$, and the proof is complete. □

Theorem 5.2. *Suppose that the solution u of problem* (1.4)–(1.6) *quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0, T_q))$ and the initial data at* (2.4) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \longrightarrow 0. \tag{5.6}$$

Under the assumptions of Theorem 4.1, the solution U_h of (2.1)–(2.4) *quenches in finite time T_q^h and we have*

$$\lim_{h\to 0} T_q^h = T_q.$$

Proof. Set $0 < \varepsilon < \frac{T_q}{2}$. There exists $\eta = \beta - \zeta$ (with $0 < \zeta < \beta < 1$) such that

$$\frac{(1-\varrho)^{\alpha+1}}{\tau(\alpha+1)} \leq \frac{\varepsilon}{2}, \quad \varrho \in [1-\eta, 1). \tag{5.7}$$

Since $\lim_{t\to T_q^-} \|u(.,t)\|_\infty = 1$, there exists a time $T_1 < T_q$ and $|T_q - T_1| < \dfrac{\varepsilon}{2}$ such that $1 - \frac{\eta}{2} \leq \|u(.,t)\|_\infty < 1$ for $t \in [T_1, T_q)$. From Theorem 5.1, the problem (2.1)–(2.4) has for each h, a unique solution U_h such that $\|U_h(t) - u_h(t)\|_\infty < \dfrac{\eta}{2}$ for $t \in [0, T_2]$ where $T_2 = \frac{T_1+T_q}{2}$. Using the triangle inequality, we get

$$\|U_h(t)\|_\infty \geq \|u(.,t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty \geq 1 - \frac{\eta}{2} - \frac{\eta}{2} \ \text{ for } \ t \in [T_1, T_2].$$

which implies that

$$\|U_h(t)\|_\infty \geq 1 - \eta \ \text{ for } \ t \in [T_1, T_2].$$

From Theorem 4.1, U_h quenches in a finite time T_q^h. We deduce from Remark 4.1 and (5.7) that

$$|T_q^h - T_1| \leq \frac{(1 - \|U_h(T_1)\|_\infty)^{\alpha+1}}{\tau(\alpha+1)} \leq \frac{\varepsilon}{2},$$

which implies

$$|T_q^h - T_q| \leq |T_q^h - T_1| + |T_1 - T_q| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,$$

and the proof is complete. □

6. Numerical experiments

In this section, we present some numerical approximations of the quenching time of the problem (1.4)–(1.6) in the case where $u_0(x) = 0.7 - \dfrac{1}{2}x^4$, $\gamma(U_i^{(n)}) = \dfrac{(U_i^{(n)})^{(1-p)}}{p}$, $B(U_i^{(n)}) = (U_i^{(n)})^{-q}, 0 \leq i \leq I$, with $0 < p \leq 1$, $q > 0$ and $\alpha = 4$. Firstly, we consider the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} = (U_i^{(n)})^{(1-p)} \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{ph^2} + \frac{(U_i^{(n)})^{(1-p)}}{p}(1 - U_i^{(n)})^{-\alpha},$$

$1 \leq i \leq I-1,$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n^e} = (U_0^{(n)})^{(1-p)} \frac{2U_1^{(n)} - 2U_0^{(n)}}{ph^2} + \frac{(U_0^{(n)})^{(1-p)}}{p}(1 - U_0^{(n)})^{-\alpha},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n^e} = (U_I^{(n)})^{(1-p)} \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{ph^2} + \frac{(U_I^{(n)})^{(1-p)}}{p}(1 - U_I^{(n)})^{-\alpha} - \frac{2(U_I^{(n)})^{(1-p)}}{ph}(U_I^{(n)})^{-q},$$

$$U_i^{(0)} = \varphi_i, 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n^e = \min\left\{\frac{h^2}{2}, h^2(1 - \|U_h^{(n)}\|_\infty)^{\alpha+1}\right\}$. We also consider the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = (U_i^{(n)})^{(1-p)} \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{ph^2} + \frac{(U_i^{(n)})^{(1-p)}}{p}(1 - U_i^{(n)})^{-\alpha},$$

$1 \leq i \leq I-1,$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = (U_0^{(n)})^{(1-p)} \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{ph^2} + \frac{(U_0^{(n)})^{(1-p)}}{p}(1 - U_0^{(n)})^{-\alpha},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = (U_I^{(n)})^{(1-p)} \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{ph^2} + \frac{(U_I^{(n)})^{(1-p)}}{p}(1 - U_I^{(n)})^{-\alpha} - \frac{2(U_I^{(n)})^{(1-p)}}{ph}(U_I^{(n)})^{-q},$$

$$U_i^{(0)} = \varphi_i, 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{\alpha+1}$. In the following tables, in rows, we present the numerical quenching times, the numbers of iterations and the orders of the approximations corresponding to meshes $16, 32, 64, 128, 256, 512$. The numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

The order s of the method is computed from

$$s = \frac{log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{log(2)}.$$

TABLE 1. Numerical quenching times obtained with the explicit Euler method $p = 1$, $q = 0.5$ and $\alpha = 4$

I	T^n	n	s
16	0.00048983809	1292	-
32	0.00048696537	4891	-
64	0.00048624977	18434	2.00
128	0.00048607104	69198	2.00
256	0.00048602636	258629	2.00
512	0.00048601519	961840	2.00

TABLE 2. Numerical quenching times obtained with the implicit Euler method $p = 1$; $q = 0.5$ and $\alpha = 4$

I	T^n	n	s
16	0.00049012477	1292	-
32	0.00048703076	4891	-
64	0.00048626995	18434	2.02
128	0.00048607886	69199	1.99
256	0.00048602982	258631	1.96
512	0.00048601682	961844	1.91

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 64$, $p = 1$, $q = 0.5$ and $\alpha = 4$. Figures 1–4 show that the semidiscrete solution quenches at the first node, which is well known in a theoretical point of view. For figures 5–6, we see that the semidiscrete solution quenches at finite time close to 4.9×10^{-4}.

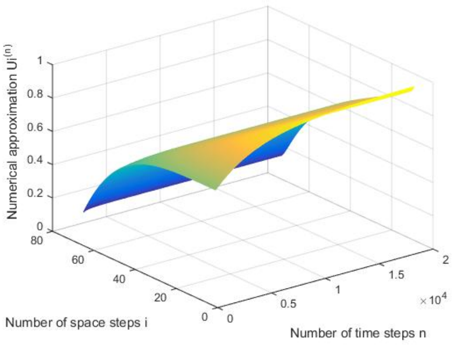

FIGURE 1. Evolution of the numerical solution (explicit scheme).

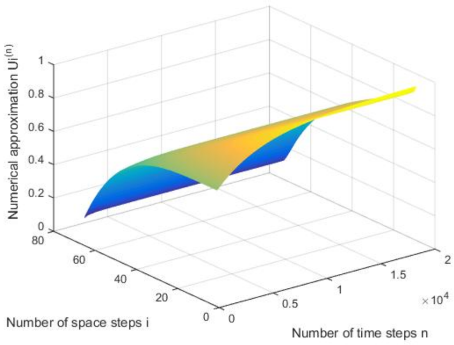

FIGURE 2. Evolution of the numerical solution (implicit scheme).

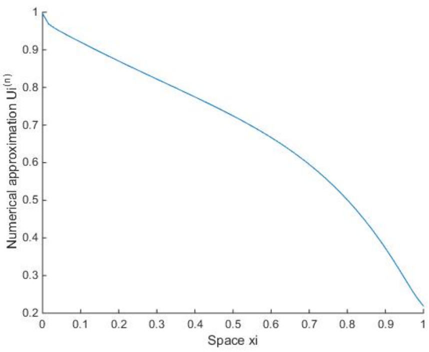

FIGURE 3. The profil of the approximation of u(x,T) where, T is the quenching time (explicit scheme).

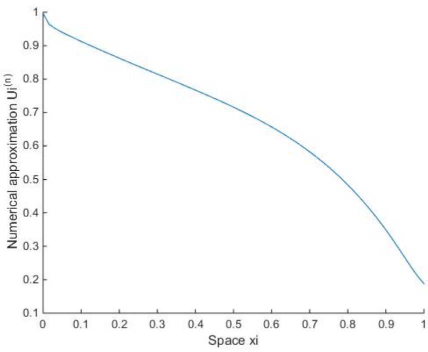

FIGURE 4. The profil of the approximation of u(x,T) where, T is the quenching time (implicit scheme).

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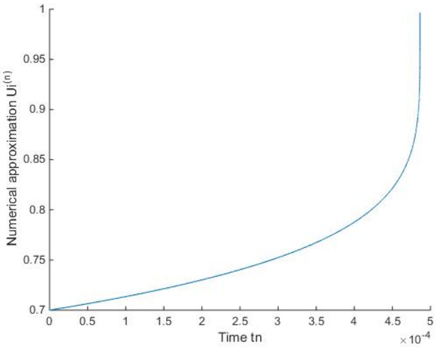

FIGURE 5. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (explicit scheme).

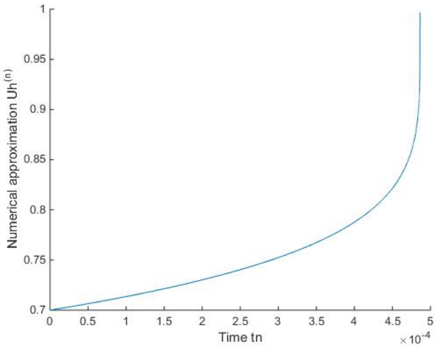

FIGURE 6. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (implicit scheme).

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