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ON MODIFIED MOMENT-TYPE OPERATORS

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ABSTRACT. We propose a modification for moment-type operators in order to preserve the exponential function e^{2cx} with $c > 0$ on real axis. First, we present moment identities. Then, we prove two weighted convergence theorems. Finally, we present a Voronovskaya-type theorem for the new operators.

1. INTRODUCTION

The theorem known as Bohman-Korovkin theorem (see [7, 16]) in the literature has become the key theorem of approximation theory over the years. The details about this theorem and its important aspects can be found in [2]. In papers [12, 13], Gadjiev proved weighted analogues of Korovkin's theorem in the subspace of continuous functions, which become bounded if they are scaled by a specially defined weight function. In the same paper, positive and negative results are valid with respect to different circumstances. Following [12, 13], using two different weight functions, Coskun [9, 10] gave analogous results using a different approach. Also, in [11], Coskun took her approach to the next level and constructed a new sequence of linear positive operators from the original operators satisfying her hypotheses. Some recent works which are related to Coskun's

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approach can be given as [3, 20]. In the year 1932, Voronovskaya [18] obtained an asymptotic formula for Bernstein polynomials. After this study, such theorems became widespread.

Large number of operator modification processes that preserve some polynomials or exponential functions have been studied since King's [15] work. In this context, very interesting studies have been done on singular integrals. These studies are of great significance as their applications in the field of engineering are intense. The interested reader may find necessary information about singular integrals in [8]. In 2017, Agratini et al. [1] derived two new integral operators from classical Picard and Gauss-Weierstrass integral operators. The new operators preserve constant functions and e^{2ax} with $a > 0$ on $\mathbb{R}$. In the same paper, the authors examined some properties of modified operators. In [3], the author obtained a new modification of Picard integral operators which fix e^{ax} and e^{2ax}, $a > 0$ on $\mathbb{R}$ and proved some theorems. Bardaro et al. [6] constructed general type linear positive operators which fix e^{ax} and e^{2ax}, $a > 0$ on $\mathbb{R}$ using a unified approach and proved some general theorems. Many modified operators including Picard-, moment- and Gauss-Weierstrass-type can be obtained from the constructed sequence in this paper by changing its kernel. In this kind of works, it is expected that the performance of the modified operators will be better than the old versions.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers $(-\infty, +\infty)$, respectively. The moment-type operators defined by

$$(\mathcal{M}_n g)(x) = \frac{n}{2} \int\limits_{-\infty}^{\infty} g(x+u)\, \kappa_{\left[-\frac{1}{n}, \frac{1}{n}\right]}(u) du, \tag{1.1}$$

where $x, u \in \mathbb{R}$, $n \in \mathbb{N}$ and $\kappa_{\left[-\frac{1}{n}, \frac{1}{n}\right]}(u)$ with $u \in \mathbb{R}$ stands for the characteristic function of the set $\left[-\frac{1}{n}, \frac{1}{n}\right]$, that is,

$$\kappa_{\left[-\frac{1}{n}, \frac{1}{n}\right]}(u) := \begin{cases} 1, & if \quad u \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 0, & if \quad u \in \mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n}\right], \end{cases}$$

and their different settings were previously considered by many authors, such as Swiderski and Wachnicki [17], Karsli [14], Wachnicki and Krech [19]. Further information about moment-type operators can be found in [4, 5]. The operators are well-defined on suitable function spaces. The operators defined in equation (1.1) are well-defined on suitable function spaces.

In this manuscript, we consider the following modification of the operators defined in equation (1.1):

$$(\mathcal{M}_n^* g)(x) = \frac{n}{2} \int_{-\infty}^{\infty} g(x - \lambda_n^c + u)\, \kappa_{\left[-\frac{1}{n},\frac{1}{n}\right]}(u) du, \tag{1.2}$$

where $x, u \in \mathbb{R}$ and $\kappa_{\left[-\frac{1}{n},\frac{1}{n}\right]}(u)$ with $u \in \mathbb{R}$ stands for the characteristic function of the set $\left[-\frac{1}{n}, \frac{1}{n}\right]$ and $\lambda_n^c := \frac{1}{2c} \ln\left(\frac{n}{2c} \sinh\left(\frac{2c}{n}\right)\right) > 0$ for any fixed $c > 0$ for all $n \in \mathbb{N}$ satisfying $n \geq n_0$, $x \in \mathbb{R}$, and $n_0 \in \mathbb{N}$ is fixed. Here, $\lim\limits_{c \to 0^+} \lambda_n^c = 0$. Using this observation, we deduce that the operators defined in equation (1.2) may approximate to the operators defined in equation (1.1) as $c \to 0^+$ for suitable function spaces under appropriate circumstances. The modified moment operators expressed in equation (1.2) fix the constant functions and e^{2cx} with $c > 0$ which are defined on $\mathbb{R}$.

2. Weighted Approximation

First, we give the following lemma whose proof is based on standart calculations.

Lemma 2.1. *Let $E_i(u) := u^j$ for $j = 0, 1, 2$ with $u \in \mathbb{R}$ be test functions. For the sequence of operators $(\mathcal{M}_n^*)_{n \geq n_0}$ defined in equation (1.2), one has*

$$\begin{aligned} (\mathcal{M}_n^* E_0)(x) &= 1 \\ (\mathcal{M}_n^* E_1)(x) &= x - \lambda_n^c \\ (\mathcal{M}_n^* E_2)(x) &= (x - \lambda_n^c)^2 + \frac{1}{3n^2}, \end{aligned}$$

where $\lambda_n^c := \frac{1}{2c} \ln\left(\frac{n}{2c} \sinh\left(\frac{2c}{n}\right)\right)$ with $c > 0$, $x \in \mathbb{R}$ and $n \geq n_0$.

To prove some Korovkin-type theorems in weighted spaces, Gadjiev [13] defined and considered the following weighted spaces:

$$\begin{aligned} B_v(\mathbb{R}) &: = \{g : |g(x)| \leq M_g v(x) \text{ for all } x \in \mathbb{R}\}, \\ C_v(\mathbb{R}) &: = \{g : g \in B_v(\mathbb{R}) \text{ and } g \text{ is continuous on } \mathbb{R}\} \end{aligned}$$

and

$$C_v^*(\mathbb{R}) := \left\{ g : g \in C_v(\mathbb{R}) \text{ and } \lim_{x\to\pm\infty} \frac{g(x)}{v(x)} \text{ finitely exists} \right\},$$

where $v(x) = 1 + \sigma^2(x)$, σ is a continuous and strictly increasing function on $\mathbb{R}$ such that $\lim\limits_{x\to\pm\infty} v(x) = +\infty$ and M_g is a constant depending on the function g. The norm given by

$$\|g\|_v = \sup_{x\in\mathbb{R}} \frac{|g(x)|}{v(x)}$$

is associated to the space $B_v(\mathbb{R})$. In view of the above definitions, the same norm is also valid for the spaces $C_v(\mathbb{R})$ and $C_v^*(\mathbb{R})$.

Let $(\mathcal{L}_n)_{n\in\mathbb{N}}$ be a sequence of linear positive operators defined on $C_v(\mathbb{R})$. Gadjiev [13] stated that $\mathcal{L}_n$ is a mapping from $C_v(\mathbb{R})$ into $B_v(\mathbb{R})$ if and only if $(\mathcal{L}_n v)(x) \leq M_1 v(x)$ for all $x \in \mathbb{R}$, where $M_1 > 0$ is a real number.

Based on these characterizations, Gadjiev [13] proved that for a sequence of linear positive operators $\mathcal{L}_n : C_v(\mathbb{R}) \to B_v(\mathbb{R})$ satisfying conditions given as

$$\lim_{n\to+\infty} \left\| \mathcal{L}_n \sigma^j - \sigma^j \right\|_v = 0, \; j = 0, 1, 2, \tag{2.1}$$

one has $\lim\limits_{n\to+\infty} \|\mathcal{L}_n g - g\|_v = 0$ for every function $g \in C_v^*(\mathbb{R})$ (see Theorem 2 in [13]).

Now, we are ready to prove the following theorem.

Theorem 2.1. *Let $c > 0$ be fixed and $v(x) = 1 + \sigma^2(x)$ with $\sigma(x) = x$ for all $x \in \mathbb{R}$. For the sequence of operators $(\mathcal{M}_n^*)_{n\geq n_0}$, defined in equation (1.2), there holds*

$$\lim_{n\to+\infty} \|\mathcal{M}_n^* g - g\|_v = 0$$

for every function $g \in C_v^*(\mathbb{R})$.

Proof. Since there can be found numbers $M_1 > 0$ and $n_0 \in \mathbb{N}$ justifying $(\mathcal{M}_n^* v)(x) \leq M_1 v(x)$, $\mathcal{M}_n^*$ is a mapping from $C_v(\mathbb{R})$ into $B_v(\mathbb{R})$ for all $n \geq n_0$. Now, we show that the conditions stated in (2.1) hold for $\mathcal{M}_n^*$.

For $j = 0$, we directly have $\lim\limits_{n\to+\infty} \|\mathcal{M}_n^* E_0 - E_0\|_v = 0$. For $j = 1$, there holds

$$\lim_{n\to+\infty} \|\mathcal{M}_n^* E_1 - E_1\|_v = \lim_{n\to+\infty} \left(\sup_{x\in\mathbb{R}} \frac{|x - \lambda_n^c - x|}{1 + x^2} \right) = \lim_{n\to+\infty} \left(\sup_{x\in\mathbb{R}} \frac{|\lambda_n^c|}{1 + x^2} \right) = 0.$$

For $j=2$, there holds

$$\begin{aligned}\|\mathcal{M}_n^* E_2 - E_2\|_v &= \sup_{x\in\mathbb{R}} \frac{\left|(x-\lambda_n^c)^2 + \frac{1}{3n^2} - x^2\right|}{1+x^2} \\ &\leq \sup_{x\in\mathbb{R}} \frac{1}{3n^2(1+x^2)} + \sup_{x\in\mathbb{R}} \frac{2|x||\lambda_n^c|}{1+x^2} + \sup_{x\in\mathbb{R}} \frac{(\lambda_n^c)^2}{1+x^2}.\end{aligned}$$

This means $\lim_{n\to+\infty} \|\mathcal{M}_n^* E_2 - E_2\|_v = 0$. The proof is completed. □

On the basis of above-mentioned space characterizations of Gadjiev [13], Coskun [9] gave the following three features of the linear positive operators from $C_{v_1}(\mathbb{R})$ into $B_{v_2}(\mathbb{R})$ $(v_1 \neq v_2)$ with the structure consisting of the following statements:

(i) A linear positive operator L defined on $C_{v_1}(\mathbb{R})$ is a mapping from $C_{v_1}(\mathbb{R})$ into $B_{v_2}(\mathbb{R})$ if and only if $\|Lv_1\|_{v_2}$ is bounded above by a positive real number.

(ii) Assume that $L : C_{v_1}(\mathbb{R}) \to B_{v_2}(\mathbb{R})$ is a linear positive operator. Then, there holds that $\|L\|_{C_{v_1}(\mathbb{R})\to B_{v_2}(\mathbb{R})} = \|Lv_1\|_{v_2}$ and thus, for every $g \in C_{v_1}(\mathbb{R})$, one has $\|Lg\|_{v_2} \leq \|Lv_1\|_{v_2} \|g\|_{v_1}$.

(iii) Assume that $T_n : C_{v_1}(\mathbb{R}) \to B_{v_2}(\mathbb{R})$ is a sequence of linear positive operators and there exists a positive real number A such that the inequality $v_1(x) \leq Av_2(x)$ holds for all $x \in \mathbb{R}$. Then, $\left(\|T_n\|_{C_{v_1}(\mathbb{R})\to B_{v_2}(\mathbb{R})}\right)_{n\in\mathbb{N}}$ is uniformly bounded provided that the relation

$$\lim_{n\to+\infty} \|T_n v_1 - v_1\|_{v_2} = 0$$

holds.

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of linear positive operators defined on $C_{v_1}(\mathbb{R})$. Taking the weight functions as $v_j(x) = 1 + \sigma_j^2(x)$ with $j = 1, 2$, where σ_j are two continuous and strictly increasing functions on $\mathbb{R}$ such that $\lim_{x\to\pm\infty} \sigma_j(x) = \pm\infty$, Coskun [10] proved that $\lim_{n\to+\infty} \|\mathcal{T}_n g - g\|_{v_2} = 0$ for every function $g \in C_{v_1}(\mathbb{R})$ provided that a sequence of linear positive operators $(\mathcal{T}_n)_{n\in\mathbb{N}}$, $\mathcal{T}_n : C_{v_1}(\mathbb{R}) \to B_{v_2}(\mathbb{R})$, satisfies the conditions given as

$$\lim_{n\to+\infty} \left\|\mathcal{T}_n \sigma_1^k - \sigma_1^k\right\|_{v_2} = 0, \; k = 0, 1, 2, \tag{2.2}$$

and for the weight functions, the condition given as

$$\lim_{x\to+\infty}\frac{v_1(x)}{v_2(x)}=0$$

holds (see Theorem 2 in [10]).

First, we verify above three features $(i)-(iii)$ for the sequence of operators $(\mathcal{M}_n^*)_{n\geq n_0}$ defined in equation (1.2) using particular weight functions satisfying the above-mentioned properties.

Let $\sigma_1(x)=x$ and $\sigma_2(x)=x^3$ with $x\in\mathbb{R}$. Clearly, both functions are continuous and strictly increasing on $\mathbb{R}$. Therefore, we take the weight functions as $v_1(x)=1+x^2$ and $v_2(x)=1+x^6$ which are defined on $\mathbb{R}$. Observe that $\lim\limits_{x\to\pm\infty}v_1(x)=+\infty$ and $\lim\limits_{x\to\pm\infty}v_2(x)=+\infty$.

First, we know that the sequence of operators $(\mathcal{M}_n^*)_{n\geq n_0}$ defined on $C_{v_1}(\mathbb{R})$ are linear positive operators. For the case (i), we prove the converse part. For each fixed $n\geq n_0$, one has

$$\begin{aligned}\left\|\mathcal{M}_n^*v_1\right\|_{v_2} &= \sup_{x\in\mathbb{R}}\frac{\left|\left(\mathcal{M}_n^*v_1\right)(x)\right|}{1+x^6}\\ &= \sup_{x\in\mathbb{R}}\frac{\left|1+(x-\lambda_n^c)^2+\frac{1}{3n^2}\right|}{1+x^6}.\end{aligned}$$

Obviously, the last term in preceding equality is bounded above by a positive real number. This shows that the operators $\mathcal{M}_n^*$ with $n\geq n_0$ is a mapping from $C_{v_1}(\mathbb{R})$ into $B_{v_2}(\mathbb{R})$. The case (ii) is clear. For the last case, by virtue of the fact that there exists a real number $A>0$ such that $v_1(x)\leq Av_2(x)$ holds for all $x\in\mathbb{R}$, it is sufficient to show that $\lim\limits_{n\to+\infty}\left\|\mathcal{M}_n^*v_1-v_1\right\|_{v_2}=0$. Since

$$\begin{aligned}\left\|\mathcal{M}_n^*v_1-v_1\right\|_{v_2} &= \sup_{x\in\mathbb{R}}\frac{\left|1+(x-\lambda_n^c)^2+\frac{1}{3n^2}-(1+x^2)\right|}{1+x^6}\\ &= \sup_{x\in\mathbb{R}}\frac{\left|-2\lambda_n^cx+(\lambda_n^c)^2+\frac{1}{3n^2}\right|}{1+x^6}\\ &\leq \sup_{x\in\mathbb{R}}\frac{2\left|x\right|\left|\lambda_n^c\right|}{1+x^6}+\sup_{x\in\mathbb{R}}\frac{(\lambda_n^c)^2+\frac{1}{3n^2}}{1+x^6},\end{aligned}$$

we have $\lim\limits_{n\to+\infty}\left\|\mathcal{M}_n^*v_1-v_1\right\|_{v_2}=0$. This means $\left(\left\|\mathcal{M}_n^*\right\|_{C_{v_1}(\mathbb{R})\to B_{v_2}(\mathbb{R})}\right)_{n\geq n_0}$ is uniformly bounded.

Now, we are ready to prove the following theorem.

Theorem 2.2. *Let $c>0$ be fixed and $v_1(x)=1+\sigma_1^2(x)$ and $v_2(x)=1+\sigma_2^2(x)$ with $\sigma_1(x)=x$ and $\sigma_2(x)=x^3$ for all $x\in\mathbb{R}$. For the sequence of operators $(\mathcal{M}_n^*)_{n\geq n_0}$ defined in equation (1.2), there holds*

$$\lim_{n\to+\infty}\|\mathcal{M}_n^*g-g\|_{v_2}=0$$

for every function $g\in C_{v_1}(\mathbb{R})$.

Proof. Since $\mathcal{M}_n^*$ is a mapping from $C_{v_1}(\mathbb{R})$ into $B_{v_2}(\mathbb{R})$ for all $n\geq n_0$, we show that the conditions stated in (2.2) hold for $\mathcal{M}_n^*$.

For $k=0$, we directly have $\lim\limits_{n\to+\infty}\|\mathcal{M}_n^*E_0-E_0\|_{v_2}=0$. For $k=1$, we may write

$$\lim_{n\to+\infty}\|\mathcal{M}_n^*E_1-E_1\|_{v_2} = \lim_{n\to+\infty}\left(\sup_{x\in\mathbb{R}}\frac{|\lambda_n^c|}{1+x^6}\right)=0.$$

For $k=2$, there holds

$$\|\mathcal{M}_n^*E_2-E_2\|_{v_2}\leq\sup_{x\in\mathbb{R}}\frac{1}{3n^2\left(1+x^6\right)}+\sup_{x\in\mathbb{R}}\frac{2\,|x|\,|\lambda_n^c|}{1+x^6}+\sup_{x\in\mathbb{R}}\frac{(\lambda_n^c)^2}{1+x^6}.$$

Therefore, $\lim\limits_{n\to+\infty}\|\mathcal{M}_n^*E_2-E_2\|_{v_2}=0$. The proof is completed. □

3. Voronovskaya-type Theorem

Now, we prove the following Voronovskaya-type theorem.

Theorem 3.1. *Let $c>0$ be fixed. If $g\in C_v^*(\mathbb{R})$ and g is a two times differentiable function with $g',g''\in C_v^*(\mathbb{R})$, then for the sequence of operators $(\mathcal{M}_n^*)_{n\geq n_0}$ defined in equation (1.2), we have*

$$\lim_{n\to+\infty}n^2\left[\left(\mathcal{M}_n^*g\right)(x)-g(x)\right]=-\frac{c}{3}g'(x)+\frac{1}{6}g''(x)$$

for every fixed $x\in\mathbb{R}$.

Proof. Let $x\in\mathbb{R}$ and $c>0$ be fixed. For all $u\in\mathbb{R}$, using local Taylor expansion, we have

$$g(x-\lambda_n^c+u)=g(x)+g'(x)\left(u-\lambda_n^c\right)+\frac{1}{2}g''(x)\left(u-\lambda_n^c\right)^2+h_x(t)\left(u-\lambda_n^c\right)^2,$$

where $h_x(t) = h_x(t\,(u))$, $t\,(u) = u - \lambda_n^c$ and $h_x \in C_v^*\,(\mathbb{R})$ with $\lim_{t\to 0} h_x(t) = 0$. We can write

$$\begin{aligned}(\mathcal{M}_n^* g)\,(x) - g\,(x) &= g'\,(x)\,\frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} (u-\lambda_n^c)\,du + g''\,(x)\,\frac{n}{4}\int_{-\frac{1}{n}}^{\frac{1}{n}} (u-\lambda_n^c)^2\,du \\ &+ \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} h_x(t)\,(u-\lambda_n^c)^2\,du \\ &=: I_1(n,c) + I_2(n,c) + I_3\,(n,c)\,.\end{aligned}$$

For $I_1(n,c)$ and $I_2(n,c)$, we obtain $\lim_{n\to+\infty} n^2 I_1(n,c) = -\frac{c}{3} g'\,(x)$ and $\lim_{n\to+\infty} n^2 I_2(n,c) = \frac{1}{6} g''\,(x)\,.$

Using Cauchy-Schwarz inequality in $I_3\,(n,c)\,,$ we get

$$\left|n^2 I_3\,(n,c)\right| \le \left(\left(\mathcal{M}_n^* h_x^2\right)(x)\right)^{1/2} \left(n^4 \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} (u-\lambda_n^c)^4\,du\right)^{1/2}.$$

Since $\lim_{n\to+\infty} n^4 \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} (u-\lambda_n^c)^4\,du = \frac{1}{5}$ and $\lim_{n\to+\infty} (\mathcal{M}_n^* h_x^2)\,(x) = 0$, we see that $\lim_{n\to+\infty} n^2 I_3(n,c) = 0$. The proof is completed. □

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