Advances in Mathematics: Scientific Journal **11** (2022), no.1, 17–23
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.1.2

ON SOME ASPECTS OF VECTOR MEASURES

Stallone O. Hazoume and Yaogan Mensah[1]

ABSTRACT. This paper addresses some properties of vector measures (Banach space valued measures) as well as topological results on some spaces of vector measures of bounded variation.

1. PRELIMINARIES AND NOTATIONS

Let X be a nonempty set and Σ a $\sigma-$algebra of subsets of X. Let E be a Banach space. A *vector measure* is a set function $m : \Sigma \to E$ such that for all sequences of pairwise disjoint elements $(A_n)_n$ of Σ one has

$$m(\bigcup_n A_n) = \sum_n m(A_n), \tag{1.1}$$

where the convergence of the series $\sum\limits_n m(A_n)$ is considered with respect to the norm topology of E. Taking $E = \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) we recover the notion of scalar measure. Also when we take $E = \mathbb{R}_+$ we obtain positive scalar measures. We denote by $\mathcal{M}(X, \mathbb{R}^+)$ the set of bounded positive measures on X and by $\mathcal{M}(X, E)$ the set of all $E-$valued measures on X that have bounded variation. Let us recall the latter concept. Let m be a vector measure on X with values in E. *The variation*

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 28B05, 46E27, 46G10.
Key words and phrases. scalar measure, vector measure, orthogonal measure.
Submitted: 21.12.2022; *Accepted:* 08.01.2022; *Published:* 10.01.2022.

of m is the set function $|m| : \Sigma \to \mathbb{R}_+$ defined by

$$|m|(A) = \sup \left\{ \sum_{B\in\pi} \|m(B)\| : \pi \subset \Sigma \text{ is a finite partition of } A \right\}. \tag{1.2}$$

When $|m|(X) < \infty$ then m is said to be of *bounded variation*. Then one defines a norm on $\mathcal{M}(X, E)$ by setting

$$\|m\| = |m|(X). \tag{1.3}$$

Two measures μ_1 and μ_2 are called *mutually singular* or *orthogonal* , which we denote by $\mu_1 \perp \mu_2$, if there exists a measurable set A such that μ_1 is supported in A and μ_2 is supported in the complement of A with respect to X.

Let us consider the following subsets of $\mathcal{M}(X, E)$:

- $\mathcal{M}_0(X, E)$ denotes the subset of $\mathcal{M}(X, E)$ of elements of the form $m = F.\mu$ where $\mu \in \mathcal{M}(X, \mathbb{R}^+)$ and F is an E-valued function on X that is Bochner-integrable with respect to μ.
- $\mathcal{M}_S(X, E)$ denotes the subset of $\mathcal{M}(X, E)$ of all $m = \sum\limits_{n\in\mathbb{N}} v_n\mu_n$ such that $(v_n)_{n\in\mathbb{N}}$ is a sequence in E and $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{M}(X, \mathbb{R}^+)$ and $\sum\limits_{n\in\mathbb{N}} ||v_n||\mu_n < \infty$.
- $\mathcal{M}_{S_0}(X, E)$ denotes the subset of $\mathcal{M}_S(X, E)$ containing measures of the form $m = \sum\limits_{n\in\mathbb{N}} v_n\mu_n$ where the measures μ_n are pairwise orthogonal, that is $\mu_i \perp \mu_j$ whenever $i \neq j$.

Throughout the paper, χ_A stands for the characteristic function of the set A. For scalar measures theory one may consult [5,6]. For detailed informations on vector measures we refer to [2, 4] and references therein. For analytic use of vector measures, see for instance [1,3].

2. Main results

We set our main results in this section.

Theorem 2.1. *$\mathcal{M}_0(X, E)$ is a subset of the closure of $\mathcal{M}_S(X, E)$.*

Proof. Let m be in $\mathcal{M}_0(X, E)$. Then $m = F.\mu$ where $\mu \in \mathcal{M}(X, \mathbb{R}_+)$ and F an E-valued function on X that is Bochner-integrable with respect to μ. Since F is

Bochner-integrable, for each $\varepsilon > 0$ there exists π_ε a finite partition of X and a finite sequence $(v_A)_{A\in\pi_\varepsilon}$ of elements of E such that

$$\int_X \|F - \sum_{A\in\pi_\varepsilon} v_A\chi_A\| d\mu < \varepsilon.$$

Therefore $|F\mu - \sum_{A\in\pi_\varepsilon} v_A\chi_A\mu|(X) < \varepsilon$. Now we set $\omega_A = \chi_A\mu$ and $m_\varepsilon = \sum_{A\in\pi_\varepsilon} v_A\omega_A$. Then $m_\varepsilon \in \mathcal{M}_S(X,E)$ and $|m - m_\varepsilon|(X) < \varepsilon$. □

Theorem 2.2. *Let m be in $\mathcal{M}(X,E)$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint members of Σ. Then $\sum_{n\in\mathbb{N}} \|m(A_n)\| < \infty$.*

Proof. Let m be in $\mathcal{M}(X,E)$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint members of Σ. Set $A = \underset{n\in\mathbb{N}}{\cup} A_n$ and $\forall n \in \mathbb{N}$ set $\pi_n = \{A_j, j \le n\} \cup \{\underset{j>n}{\cup} A_j\}$. Then π_n is a finite partition of A. We have

$$\|\sum_{j>n} m(A_j)\| + \sum_{j\le n} \|m(A_j)\| \le |m|(A).$$

So $\forall n \in \mathbb{N}$, $\sum_{j\le n} \|m(A_j)\| \le |m|(A)$. Letting $n \to \infty$ we have the convergence of the series $\sum_{n\in\mathbb{N}} \|m(A_n)\|$. □

Theorem 2.3. *Let $m = \sum_{n\in\mathbb{N}} v_n\mu_n$ be an element of $\mathcal{M}_{S_0}(X,E)$. Then $|m| = \sum_{n\in\mathbb{N}} \|v_n\|\mu_n$.*

Proof. Let $m = \sum_{n\in\mathbb{N}} v_n\mu_n$ be an element of $\mathcal{M}_{S_0}(X,E)$. For $n \in \mathbb{N}$ set $V_n = supp(\mu_n)$ and $V_0 = X \setminus \underset{n\in\mathbb{N}}{\cup} V_n$. Then $(V_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint elements of Σ and it is also a partition of X. Now pick A in Σ and let π to be a finite partition of A. Set $S_n = S \cap V_n$ where $S \in \pi$. Then $(S_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint members of Σ such that $\underset{n\in\mathbb{N}}{\cup} S_n = S$. Therefore

$$\begin{aligned}
\sum_{S\in\pi} \|m(S)\| &= \sum_{S\in\pi} \|m(\underset{n\in\mathbb{N}}{\cup} S_n)\| = \sum_{S\in\pi} \|\sum_{n\in\mathbb{N}} m(S_n)\| \\
&\le \sum_{S\in\pi}\sum_{n\in\mathbb{N}^*} \|m(S_n)\| = \sum_{S\in\pi}\sum_{n\in\mathbb{N}^*} \|v_n\|\mu_n(S) \\
&= \sum_{n\in\mathbb{N}^*}\sum_{S\in\pi} \|v_n\|\mu_n(S) = \sum_{n\in\mathbb{N}^*} \|v_n\| \sum_{S\in\pi} \mu_n(S) \\
&= \sum_{n\in\mathbb{N}^*} \|v_n\|\mu_n(A) < \infty
\end{aligned}$$

So, $|m|(A) \leq \sum\limits_{n\in\mathbb{N}^*} ||v_n||\mu_n(A)$. For the converse inequality let us set $A_n = A \cap V_n$. Then we have $\sum\limits_{n\in\mathbb{N}^*} ||v_n||\mu_n(A) = \sum\limits_{n\in\mathbb{N}^*} ||m(A_n)|| \leq |m|(A)$. □

Theorem 2.4. *Let $m \in \mathcal{M}(X,E)$. Assume that there exists a sequence $(\omega_n, F_n)_{n\in\mathbb{N}}$ such that $\omega_n \in \mathcal{M}(X,\mathbb{R}_+)$, $\omega_i \perp \omega_j$ if $i \neq j$, $F_n : X \longrightarrow E$ is Bochner ω_n- integrable and $m = \sum\limits_{n\in\mathbb{N}} F_n\omega_n$. Then $m \in \mathcal{M}_0(X,E)$.*

Proof. Since the elements in the sequence $(\omega_n)_{n\in\mathbb{N}}$ are pairwise orthogonal, there exists a pairwise disjoint sequence $(V_n)_{n\in\mathbb{N}}$ of elements of Σ such that for each $n \in \mathbb{N}$, ω_n is supported by V_n. Set $\Omega = \dfrac{1}{2}\sum\limits_{n\in\mathbb{N}} \dfrac{1}{2^n}\dfrac{\omega_n}{\omega_n(V_n)}$ and $F = 2\sum\limits_{n\in\mathbb{N}} 2^n\omega_n(V_n)\chi_{V_n}F_n$. Then $\Omega \in \mathcal{M}(X,\mathbb{R}_+)$ since $\Omega(X) = \frac{1}{2}\sum\limits_{n\in\mathbb{N}} \frac{1}{2^n} = 1$. We have

$$F\Omega = [2\sum_{n\in\mathbb{N}} 2^n\omega_n(V_n)\chi_{V_n}F_n][\frac{1}{2}\sum_{n\in\mathbb{N}} \frac{1}{2^n}\frac{\omega_n}{\omega_n(V_n)}] = \sum_{n\in\mathbb{N}} F_n\omega_n = m.$$

So $m \in \mathcal{M}_0(X,E)$. □

Theorem 2.5. *Let $(\mu_i)_{i\in\mathbb{N}}$ be a sequence of pairwise orthogonal elements of $\mathcal{M}(X,\mathbb{R}_+)$. Let $(F_{i,n})_{n\in\mathbb{N},i\leq n}$ be a double sequence of functions from X into E such that $F_{i,n}$ is Bochner-integrable with respect to the measure ω_i for all n. Assume that $m = \lim\limits_{n\longrightarrow+\infty} \sum\limits_{i\leq n} F_{i,n}\mu_i$ exists. Then there exists a sequence of $E-$valued functions $(F_i)_{i\in\mathbb{N}}$ such that each F_i is Bochner-integrable with respect to the measure ω_i and $m = \sum\limits_{i\in\mathbb{N}} F_i\mu_i$.*

Proof. Since the sequence $(\mu_i)_{i\in\mathbb{N}}$ consists of pairwise orthogonal elements, we can find a sequence $(V_i)_{i\in\mathbb{N}}$ a pairwise disjoint elements of Σ such that each $i \in \mathbb{N}$, μ_i is supported by V_i. For $i \in \mathbb{N}$ one has $\chi_{V_i}m = \lim\limits_{n\longrightarrow\infty} \chi_{V_i}F_{i,n}\mu_i$ where χ_{V_i} is the characteristic function of V_i. Thus for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n, p \in \mathbb{N}$ are such that $n, p \geq n_0$ then

$$|\chi_{V_i}F_{i,n}\mu_i - \chi_{V_i}F_{i,p}\mu_i|(X) = \int_X \chi_{V_i}\|F_{i,n} - F_{i,p}\|d\mu_i < \epsilon.$$

This means that $(\chi_{V_i}F_{i,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(X,\mu_i,E)$ which is a Banach space. So it converges to a certain function F_i, that is, $F_i = \lim\limits_{n\longrightarrow+\infty} F_{i,n}$ in $L^1(X,\mu_i,E)$. So $m = \sum\limits_{i\in\mathbb{N}} F_i\mu_i$. □

Theorem 2.6. *If* $m = \sum_{i\in\mathbb{N}} v_i\mu_i$ *is an element of* $\mathcal{M}_S(X,E)$ *then there exists a sequence* $(\omega_n, F_n)_{n\in\mathbb{N}}$ *with* $\omega_n \in \mathcal{M}(X,\mathbb{R}_+)$, $\omega_i \perp \omega_j$ *if* $i \neq j$ *and* $F_n : X \longrightarrow E$ *Bochner* ω_n-*integrable such that* $m = \sum_{n\in\mathbb{N}} F_n\omega_n$.

Proof. Let m be an element of $\mathcal{M}_S(X,E)$. Let us first prove by induction the assertion (R_n) : "$\forall\, n \geq 0, \forall i \leq n, \exists\, F_{i,n} : X \longrightarrow E$ and $\omega_i \in \mathcal{M}(X,\mathbb{R}_+)$, such that $\omega_i \perp \omega_j$ if $i \neq j$ and $\sum_{i\leq n} v_i\mu_i = \sum_{i\leq n} F_{i,n}\omega_i$, with $F_{i,n}$ is Bochner ω_j-integrable $\forall j \leq i$."
Set $\omega_0 = \mu_0$ and $F_{0,0}(x) = v_0, \forall x \in X$, then the initialization is done.

Now let us assume (R_n) and let us prove (R_{n+1}). Consider the vectors v_i, $i = 0,\cdots,n+1$ in E and the measures μ_i, $i = 0,\cdots,n+1$. By the Radon-Nikodym-Lebesgue theorem, we have $\mu_{n+1} = \omega_{n+1} + f_{n+1}\sum_{i\leq n}\omega_i$ where f_{n+1} is a real nonnegative function that is integrable with respect to $\sum_{i\leq n}\omega_i$ and $\omega_{n+1} \perp \sum_{i\leq n}\omega_i$. It follows that $\sum_{i\leq n+1} v_i\mu_i = v_{n+1}\omega_{n+1} + \sum_{i\leq n}[F_{i,n} + v_{n+1}f_{n+1}]\omega_i$.

Now for all $i \leq n$ set $F_{i,n+1} = F_{i,n} + v_{n+1}f_{n+1}$ and $F_{n+1,n+1} = v_{n+1}$. Then $\sum_{i\leq n+1} v_i\mu_i = \sum_{i\leq n+1} F_{i,n+1}\omega_i$. Then our induction is done.

Now, $m = \sum_{i\in\mathbb{N}} v_i\mu_i = \lim_{n\longrightarrow+\infty}\sum_{i\leq n} v_i\mu_i = \lim_{n\longrightarrow+\infty}\sum_{i\leq n} F_{i,n}\omega_i$. By Theorem 2.5 we deduce the existence of a sequence $(F_i)_{i\in\mathbb{N}}$ such that $m = \sum_{n\in\mathbb{N}} F_n\omega_n$. □

Theorem 2.7. $\mathcal{M}_{S_0}(X,E)$ *is dense in* $\mathcal{M}_S(X,E)$.

Proof. Let m be an element of $\mathcal{M}_s(X,E)$. Then by Theorem 2.6 there exists a sequence $(\omega_n, F_n)_{n\in\mathbb{N}}$ where $F_n : X \longrightarrow E$ is a map, $\omega_n \in \mathcal{M}(X,\mathbb{R}_+)$, $\omega_i \perp \omega_j$ if $i \neq j$, F_n is Bochner ω_n- integrable and $m = \sum_{n\in\mathbb{N}} F_n\omega_n$. Now $\forall\varepsilon > 0, \exists\, m_n \in \mathbb{N}^*, \exists\,(A_{j_n})_{j_n\leq m_n} \subset \Sigma, \exists\,(t_{j_n})_{j_n\leq m_n} \subset E$ such that $\int_X ||F_n - \sum_{j_n\leq m_n} t_{j_n}\chi_{A_{j_n}}||d\omega_n < \frac{\varepsilon}{2^n}$.
Let us set

$$\begin{aligned} k_n &= F_n - \sum_{j_n\leq m_n} t_{j_n}\chi_{A_{j_n}}, \\ \nu_n &= k_n\omega_n, \\ \omega_{j_n} &= \chi_{A_{j_n}}\omega_n, \\ \text{and } m_\varepsilon &= \sum_{n\in\mathbb{N}}\sum_{j_n\leq m_n} t_{j_n}\omega_{j_n}. \end{aligned}$$

Then $(A_{j_n})_{j_n \leq m_n, n\in\mathbb{N}}$ is a sequence of pairwise disjoint elements of Σ and $(\omega_{j_n})_{j_n \leq m_n, n\in\mathbb{N}}$ is a sequence of pairwise orthogonal members of $\mathcal{M}(X, \mathbb{R}_+)$. Let us prove that $|m_\varepsilon - m| < 2\varepsilon$. Let π be a finite partion of X. Then

$$\sum_{S\in\pi} \|\nu_n(S)\| = \sum_{S\in\pi} \| \int_S k_n d\omega_n\| \leq \sum_{S\in\pi} \int_S \|k_n\| d\omega_n = \int_X \|k_n\| d\omega_n < \frac{\varepsilon}{2^n}.$$

It follows that $|\nu_n| < \frac{\varepsilon}{2^n}$ and $|m_\varepsilon - m| = |\sum_{n\in\mathbb{N}} \nu_n| \leq \sum_{n\in\mathbb{N}} |\nu_n| < \sum_{n\in\mathbb{N}} \frac{\varepsilon}{2^n} = 2\varepsilon$.

Let us prove that $\sum_{n\in\mathbb{N}} \sum_{j_n\leq m_n} \|t_{j_n}\| \omega_{j_n} < \infty$.

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \sum_{j_n\leq m_n} t_{j_n}\omega_{j_n} &= m - \sum_{n\in\mathbb{N}} \nu_n \\
t_{j_n}\omega_{j_n}(A_{j_n}) &= m(A_{j_n}) - \nu_n(A_{j_n}) \\
\|t_{j_n}\|\omega_{j_n}(A_{j_n}) &= \|m(A_{j_n})\| + \|\nu_n(A_{j_n})\| \\
\sum_{j_n\leq m_n} \|t_{j_n}\|\omega_{j_n}(A_{j_n}) &\leq \sum_{j_n\leq m_n} \|m(A_{j_n})\| + \sum_{j_n\leq m_n} \|\nu_n(A_{j_n})\| \\
&< \sum_{j_n\leq m_n} \|m(A_{j_n})\| + \frac{\varepsilon}{2^n} \\
\sum_{n\in\mathbb{N}} \sum_{j_n\leq m_n} \|t_{j_n}\|\omega_{j_n}(A_{j_n}) &< \sum_{n\in\mathbb{N}} \sum_{j_n\leq m_n} \|m(A_{j_n})\| + 2\varepsilon.
\end{aligned}$$

Since m is of bounded variation and $(A_{j_n})_{j_n\leq m_n, n\in\mathbb{N}}$ is a sequence of pairwise disjoint members of Σ, then by the use of Theorem 2.2, one obtains the convergence of the left hand series. □

REFERENCES

[1] G.P. CURBERA: *Banach space properties of L^1 of a vector measure*, Proc. Amer. Math. Soc., **123** (1995), 3797–3806.

[2] J. DIESTEL, J-J. UHL: *Vector measures*, Amer. Math. Soc., Providence, 1977.

[3] E.J. FERNANDEZ : *Orthogonal systems in L^2 spaces of a vector measure*, Turk. J. Math., **37** (2013), 981–992.

[4] I. KLUVANEK, G. KNOWLES: *Vector measures and Control Systems*, North Holland Publishing Company, 1975.

[5] H.L. ROYDEN, P.M. FITZPATRICK: *Real Analysis*, 4th edition, China Machine Press, 2010.

[6] W. RUDIN: *Real and Complex Analysis*, McGraw-Hill inc., 1970.

Department of Mathematics
University of Lomé
Lomé, Togo.
Email address: hazoume.stallone@outlook.com

Department of Mathematics
University of Lomé
Lomé, Togo
and ICMPA-Unesco Chaire
University of Abomey-Calavi
Cotonou, Benin.
Email address: mensahyaogan2@gmail.com, ymensah@univ-lome.org