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ON GRADED *S*-PRIMARY SUBMODULES OF GRADED MODULES OVER GRADED COMMUTATIVE RINGS

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ABSTRACT. Let G be a group, R be a G-graded commutative ring with identity, M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. In this paper, a new concept of graded S-primary submodules of M is introduced as a generalization of graded Primary submodules as well as a generalization of graded S-prime submodules of M. Also, some properties of this class are investigated.

1. INTRODUCTION AND PRELIMINARIES

Through out this article, we assume that R is a commutative G-graded ring with identity and M is a unitary graded R-Module.

Refai and Al-Zoubi in [19] introduced the concept of graded primary ideal. The concept of graded primary submodules has attracted the attentions of many mathematicians, see for example [1, 7, 8, 10, 11, 18]. Then many generalizations of graded primary submodules were studied such as graded 2-absorbing primary (see [12]), graded weakly 2-absorbing primary (see [2]), graded classical primary (see [4]), graded weakly classical primary (see [6]) and graded quasi-primary submodules (see [5]).

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Recently, Al-Zoubi and M. Al-Azaizeh in [3] introduced the concept of graded S-prime submodule over a graded commutative rings. Here, we introduce the concept of graded S-primary submodule as a new generalization of a graded primary submodule on the one hand and a generalization of a graded S-prime submodule on other hand. Let us recall some basic properties of graded rings and graded modules, which will be used in the sequel. For more information about the properties of the graded rings and graded modules, the readers are referred to consults [14–16] among others.

Let G be a multiplicative group with identity element e. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_h of R indexed by the elements $h \in G$ such that $R = \oplus_{h\in G} R_h$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The non-zero elements of R_h are said to be homogeneous of degree h and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{h\in G} R_h$. If $a \in R$, then a can be written uniquely as $\sum_{h\in G} a_h$, where a_g is called a homogeneous component of a in R_h (see [17]). Let $R = \oplus_{h\in G} R_h$ be a G-graded ring. An ideal J of R is said to be a graded ideal if $J = \sum_{h\in G}(J \cap R_h) := \sum_{h\in G} J_h$ (see [17]).

Let $R = \oplus_{h\in G} R_h$ be a G-graded ring. A left R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_h\}_{h\in G}$ of M such that $M = \oplus_{h\in G} M_h$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{h\in G} M_h = h(M)$, then it is called a homogeneous. Note that M_h is an R_e-module for every $h \in G$. Let $R = \oplus_{h\in G} R_h$ be a G-graded ring. A submodule N of M is said to be a graded submodule of M if $N = \oplus_{h\in G}(N \cap M_h) := \oplus_{h\in G} N_h$. In this case, N_h is called the h-component of N (see [17]).

Let I a graded ideal of R. The *graded radical* of I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [19]). Let K be a graded submodule of M and I a graded ideal of R. Then $(K :_R M)$ is defined as $(K :_R M) = \{a \in R : aM \subseteq K\}$. It is shown in [9] that if K is a graded submodule of M, then $(K :_R M)$ is a graded ideal of R. The graded submodule $\{m \in M : Im \subseteq K\}$ will be denoted by $(K :_M I)$. Particularly, we use $(N :_M s)$ instead of $(N :_M Rs)$.

2. GRADED S-PRIMARY SUBMODULES

In this section, we introduce our new concept and study their basic properties. Let us start with the following:

Definition 2.1. *Let R be a G-graded ring and M a graded R-module, N a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R with $(N :_R M) \cap S = \emptyset$. Then N is said to be a graded S-primary submodule of M if there exists a fixed $s_g \in S$ and whenever $a_h m_i \in N$, where $a_h \in h(R)$ and $m_i \in h(M)$, then $s_g a_h \in Gr(N :_R M)$ or $s_g m_i \in N$. In particular, we say that a graded ideal K of R is a graded S-primary ideal if K is a graded S-primary submodule of R-module M.*

Theorem 2.1. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodule of M with $(N :_R M) \cap S = \emptyset$. Then the following are equivalent*

(i) *N is a graded S-primary submodule of M;*
(ii) *There exists $s_g \in S$ such that whenever $a_h K \subseteq N$ and K is a graded submodule of M and $a_h \in h(R)$, then $s_g a_h \in Gr(N :_R M)$ or $s_g K \subseteq N$;*
(iii) *There exists $s_g \in S$ such that whenever $IK \subseteq N$, where K is a graded submodule of M and I is a graded ideal of R, then $s_g I \subseteq Gr(N :_R M)$ or $s_g K \subseteq N$.*

Proof.
$(i) \Rightarrow (ii)$. It is clear.

$(ii) \Rightarrow (iii)$. Let $IK \subseteq N$ for some graded ideal I of R and K is a graded submodule of M. We want show that there exists $s_g \in S$ such that $s_g K \subseteq N$ or $s_g I \subseteq Gr(N :_R M)$. Since $a_h K \subseteq N$ for every $a_h \in I$, there exists $s_g \in S$ such that $s_g K \subseteq N$ or $s_g a_h \in Gr(N :_R M)$ for every $a_h \in I$.

$(iii) \Rightarrow (i)$. Let $a_h \in h(R)$ and $m_i \in h(M)$ with $a_h m_i \in N$. Now, set $I = Ra_h$ and $K = Rm_i$, then I is a graded ideal of R and K is a graded submodule of M. Then we can conclude that $IK = Ra_h m_i \subseteq N$. Then there exists $s_g \in S$ so that $s_g I = Ra_h s_g \subseteq Gr(N :_R M)$ or $s_g K = Rs_g m_i \subseteq N$ and hence either $s_g a_h \in Gr(N :_R M)$ or $s_g m_i \in N$. Therefore, N is graded S-primary submodule of M. $\square$

Recall from [18] that a proper graded submodule P of M is said to be *a graded primary submodule of* M if whenever $r_h \in h(R)$ and $m_g \in h(M)$ with $r_h m_g \in P$, then either $m_g \in P$ or $r_h \in Gr((P :_R M))$.

The following result can be easily obtain. (We omit the proof).

Theorem 2.2. *Let $S_1 \subseteq S_2 \subseteq h(R)$ be two multiplicatively closed subsets of R. If N is a graded S_1-primary submodule of M such that $(N :_R M) \cap S_2 = \emptyset$, then N is also graded S_2-primary submodule of M. In particular, every graded primary submodule N with $(N :_R M) \cap S = \emptyset$ is graded S-primary.*

The converse of Theorem 2.2 is not true in general, the following example illustrates that.

Example 1. *Let $G = (\mathbb{Z}, +)$ and $R = (\mathbb{Z}, +, .)$. Define*

$$R_g = \left\{ \begin{array}{cc} \mathbb{Z} & \textit{if } g = 0 \\ 0 & \textit{otherwise} \end{array} \right\}.$$

Then R is a G-graded ring. Let $M = \mathbb{Q} \times \mathbb{Z}_3$. Then M is G-graded R-module with

$$M_g = \left\{ \begin{array}{cc} \mathbb{Q} \times \{0\} & \textit{if } g = 0 \\ \{0\} \times \mathbb{Z}_3 & \textit{if } g = 1 \\ \{0\} \times \{0\} & \textit{otherwise} \end{array} \right\}.$$

Take the graded submodule $P = (0)$ of M and the multiplicatively closed subset $S = \{1, 3^n : n \in \mathbb{N} \cup \{0\}\} \subseteq h(R)$. First note that $(P :_R M) = \{0\}$ and $3 \cdot (0, \overline{1}) \in P$. Since $2 \notin Gr((P :_R M))$ and $(0, \overline{1}) \notin P$, P is not graded primary submodule of M. At the same time, P is a graded S-primary. This can easily be shown by taking $s = 3$ and whenever $a(m_1, \overline{m_2}) \in P$, then $sa \in Gr((P :_R M))$ or $s(m_1, \overline{m_2}) \in P$ for any $a \in h(R)$ and $(m_1, \overline{m_2}) \in h(M)$.

Theorem 2.3. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodule of M such that $(N :_R M) \cap S = \emptyset$. If $S \subseteq u(R) \cap h(R)$, then N is graded S-primary submodule of M if and only if N is graded primary submodule of M, where $u(R)$ is the set of all units in R.*

Proof. Suppose that N is a graded S-primary submodule of M and $s_g \in S$ be the element which satisfies the graded S-primary submodule condition. Let $r_h m_j \in N$, where $r_h \in h(R)$, $m_j \in h(M)$. So we have either $s_g m_j \in N$ or $s_g r_h \in Gr(N :_R M)$. Now s_g is unit, so it has an inverse $s_g^{-1} \in h(R)$. Thus either $m_j \in N$ or $r_h \in Gr(N :_R M)$. Thus, N is graded primary submodule of M. For the converse, see Theorem 2.2. $\square$

Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N a graded submodule of M with $(N :_R M) \cap S = \emptyset$. Then N is said to be graded S-prime submodule of M if there exists a fixed $s_g \in S$ and whenever $a_h m_i \in N$, where $a_h \in h(R)$ and $m_i \in h(M)$, then either $s_g a_h \in (N :_R M)$ or $s_g m_i \in N$ (see [3]).

Clearly, every graded S-prime submodule is graded S-primary. However, Theorem 2.5 shows that graded S-primary submodule need not to be graded S-prime. This can be seen by taking $S = u(R) \cap h(R)$ and recalling that graded primary submodule does not imply graded prime.

Let R be a G-graded ring, M a graded R-module, and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Recall that the saturation S^* of S is defined by $S^* = \{s_g \in h(R) : \frac{s_g}{1}$ is a unit in $S^{-1}R\}$. Then S^* is a multiplicatively closed subset of R containing S. Also a multiplicatively closed subset $S \subseteq h(R)$ is a saturated set if $S = S^*$ (see [3]).

Theorem 2.4. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Assume that S^* is the saturation of S. Then a graded submodule N of M is graded S-primary submodule of M if and only if it is graded S^*-primary submodule of M.*

Proof. Assume that N is a graded S-primary submodule of M. Now, we want to show that $(N :_R M) \cap S^* = \emptyset$. Suppose there exists $x_g \in (N :_R M) \cap S^*$. As $x_g \in S^*$, then $\frac{x_g}{1}$ is a unit of $S^{-1}R$, it follows that there exist $a_h \in h(R)$ and $s_i \in S$ such that $\frac{x_g}{1}\frac{a_h}{s_i} = 1$. This yields that $n_j s_i = n_j x_g a_h$ for some $n_j \in S$. This gives that $n_j s_i = n_j x_g a_h \in (N :_R M) \cap S$, which is a contradiction. Thus $(N :_R M) \cap S^* = \emptyset$. Since $S \subseteq S^*$, then by Theorem 2.2, N is a graded S^*-primary submodule of M. For the converse, assume that N is a graded S^*-primary submodule of M. Since $S \subseteq S^*$, then $(N :_R M) \cap S = \emptyset$. Now, let $a_h m_i \in N$, where $a_h \in h(R)$ and $m_i \in h(M)$. Then there exists $x_j \in S^*$ such that $x_j a_h \in Gr(N :_R M)$ or $x_j m_i \in N$. As $\frac{x_j}{1}$ is a unit of $S^{-1}R$, there exists $n_k, s_g \in S$ and $b_t \in h(R)$ such that $n_k s_g = n_k x_j b_t$. Take $s_l = n_k s_g \in S$. It follows that either $s_l a_h = (n_k x_j b_t) a_h \in Gr(N :_R M)$ or $s_l m_i \in N$. Therefore, N is a graded S-primary submodule of M. □

Theorem 2.5. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded S-primary submodule of M, then $S^{-1}N$ is a graded primary submodule of $S^{-1}M$.*

Proof. Assume that N is a graded S-primary submodule of M. Let $(\frac{a_h}{s_g})(\frac{m_i}{t_j}) \in S^{-1}N$, where $\frac{a_h}{s_g} \in h(S^{-1}R)$ and $\frac{m_i}{t_j} \in h(S^{-1}M)$. Then there exists $n_k \in S$ such that $n_k a_h m_i \in N$. Since N is graded S-primary submodule of M, there is an $s'_l \in S$ so that $s'_l n_k a_h \in Gr(N :_R M)$ or $s'_l m_i \in N$. This implies that $\frac{a_h}{s_g} = \frac{s'_l n_k a_h}{s'_l n_k s_g} \in S^{-1}Gr(N :_R M) \subseteq Gr(S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $\frac{m_i}{t_j} = \frac{s'_l m_i}{s'_l t_j} \in S^{-1}N$. Therefore, $S^{-1}N$ is a graded primary submodule of $S^{-1}M$. $\square$

Let M and T be two G-graded R-modules. Recall that an R-module homomorphism $f : M \to T$ is called a G-graded R-module homomorphism if $f(M_g) \subseteq T_g$ for all $g \in G$, see [17].

Theorem 2.6. *Let R be a G-graded ring, M, M' be two graded R-module and $f : M \longrightarrow M'$ be a graded epimorphism. Let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then we have the following:*

(i) *If N' is a graded S-primary submodule of M', then $f^{-1}(N)$ is a graded S-primary submodule of M.*

(ii) *If N is a graded S-primary submodule of M with $Ker(f) \subseteq N$, then $f(N)$ is a graded S-primary submodule of M'.*

Proof.

(i) Assume that N' is a graded S-primary submodule of M'. First, we want to show that $(f^{-1}(N') :_R M) \bigcap S = \emptyset$. Suppose on the contrary that there exists $s_h \in (f^{-1}(N') :_R M) \bigcap S$. So $s_h M \subseteq f^{-1}(N')$ and hence $s_h f(M) = s_h M' \subseteq N'$, which is a contradiction since $(N' :_R M') \bigcap S = \emptyset$. Now, let $a_h m_i \in f^{-1}(N')$ for some $a_h \in h(R)$ and $m_i \in h(M)$. Hence $f(a_h m_i) = a_h f(m_i) \in N'$. Then there is exists $s_g \in S$ such that $s_g a_h \in Gr(N' :_R M')$ or $s_g f(m_i) \in N'$ as N' is graded S-primary submodule of M. Now we will show that $(N' :_R M') \subseteq (f^{-1}(N') :_R M)$. Let $x_j \in (N' :_R M') \cap h(M')$. Then we have $x_j M' \subseteq N'$. So $f(x_j M) = x_j f(M) = x_j M' \subseteq N'$. This implies that $x_j M \subseteq f^{-1}(f(M)) \subseteq f^{-1}(N')$ and so $x_j \in (f^{-1}(N') :_R M)$. As $Gr(N' :_R M') \subseteq Gr(f^{-1}(N') :_R M)$, we get either $s_g a_h \in Gr(f^{-1}(N') :_R M)$ or $s_g m_i \in f^{-1}(N')$. Therefore, $f^{-1}(N')$ is graded S-primary submodule of M.

(ii) First, we want to show that $(f(N)) :_R M') \cap S = \emptyset$. Suppose on the contrary that there exists $s_g \in (f(N)) :_R M') \cap S$. Hence $s_g M' \subseteq f(N)$, it follows that $f(s_g M) = s_g f(M) = s_g M' \subseteq f(N)$. Which implise, $s_g M = s_g M + Ker(f) \subseteq f^{-1}(f(N)) = N + Ker(f) = N$. Hence $s_g \in (N :_R M)$, which is a contradictions since $(N :_R M) \cap S = \emptyset$. Now, let $a_h m'_i \in f(N)$ for some $a_h \in h(R)$ and $m'_i \in h(M')$.

As f is a graded epimorphism, there is an $m_j \in h(M)$ such that $m'_j = f(m_j)$. Then $a_h m'_j = a_h f(m_j) = f(a_h m_j) \in f(N)$. Hence there exists $n_k \in N \cap h(M)$ such that $f(a_h m_j) = f(n_k)$. Hence $a_h m_j - n_k \in \ker(f) \subseteq N$, and so $a_h m_j \in N$. Since N is a graded S-primary submodule of M, there is an $s_g \in S$ such that $s_g a_h \in Gr(N :_R M)$ or $s_g m_j \in N$. Since $Gr(N :_R M) \subseteq Gr(f(N) :_R M')$ we have $s_g a_h \in Gr(f(N) :_R M')$ or $f(s_g m_j) = s_g f(m_j) = s_g m'_i \in f(N)$. Hence $f(N)$ is graded S-primary submodule of M'. □

Recall that a graded R-module M over G-graded ring R is said to be *a graded multiplication module (gr-multiplication module)* if for every graded submodule C of M there exists a graded ideal I of R such that $C = IM$. It is clear that M is gr-multiplication R- module if and only if $C = (C :_R M)M$ for every graded submodule C of M (see [13]).

Theorem 2.7. *Let N be a graded submodule of M and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The following statements hold:*

(i) *If N is a graded S-primary submodule of M, then $(N :_R M)$ is a graded S-primary ideal of R.*

(ii) *If M is a graded multiplication module and $(N :_R M)$ is a graded S-primary ideal of R, then N is a graded S-primary submodule of M.*

Proof.

(i) Assume that N is a graded S-primary submodule of M. Let $a_h r_j \in (N :_R M)$ for some $a_h, r_j \in h(R)$. Then $a_h r_j m_i \in N$ for all $m_i \in h(M)$. As N is a graded S-primary submodule, there exists $s_g \in S$ such that either $s_g a_h \in Gr(N :_R M)$ or $s_g r_j m_i \in N$, for all $m_i \in h(M)$. If $s_g a_h \in Gr(N :_R M)$, then we are done. Suppose that $s_g a_h \notin Gr(N :_R M)$. Then $s_g r_j m_i \in N$, for all $m_i \in h(M)$, it follows that $s_g r_j \in (N :_R M)$. Therefore, $(N :_R M)$ is graded S-primary ideal of R.

(ii) Assume that M is a graded multiplication module and $(N :_R M)$ is a graded S-primary ideal of R. Let I be a graded ideal of R and K be a graded submodule of M with $IK \subseteq N$. Then $I(K :_R M) \subseteq (IK :_R M) \subseteq (N :_R M)$. As $(N :_R M)$ is graded S-primary ideal of R, there is $s_g \in S$ so that $s_g(K :_R M) \subseteq (N :_R M)$ or $s_g I \subseteq Gr(N :_R M)$. Thus we can conclude that $s_g K = s_g(K :_R M)M \subseteq (N :_R M)M = N$ or $s_g I \subseteq Gr(N :_R M)$. Therefore, by Theorem 2.1(iii), N is a graded S-primary submodule of M. □

The *graded radical* of a graded submodule N of M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$ (see [10]).

Theorem 2.8. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded S-primary submodule of a graded multiplication R-module M. Suppose that $K \cap L \subseteq N$ for some graded submodules K and L of M. Then either $s_g K \subseteq N$ or $s_g L \subseteq Gr_M(N)$ for some $s_g \in S$.*

Proof. Suppose N is a graded S-primary submodule, so there exists $s_g \in S$ such that for every $a_h \in h(R)$ and $m_i \in h(M)$, if $a_h m_i \in N$, then $s_g a_h \in Gr(N :_R M)$ or $s_g m_i \in N$. Assume that $K \cap L \subseteq N$ and $s_g K \nsubseteq N$ for some graded submodules K and L of M. Then $\exists m'_j \in h(K)$ such that $s_g m'_j \notin N$. Let $x = \sum_{t \in G} x_t \in (L :_R M)$. As $(L :_R M)$ is a graded ideal of R, we have $x_t \in (L :_R M)$ for all $t \in G$. This yields that $x_t m'_j \in (L :_R M)K \subseteq L \cap K \subseteq N$ for all $t \in G$. As N is a graded S-primary submodule of M and $s_g m'_j \notin N$, so we get $s_g x_t \in Gr(N :_R M)$ for all $t \in G$, and hence $s_g x \in Gr(N :_R M)$. This yields that $s_g(L :_R M) \subseteq Gr(N :_R M)$. As M is a graded multipliction module by [18, Theorem 9], we get $s_g L = s_g(L :_R M)M \subseteq Gr(N :_R M)M = Gr_M(N)$. □

Lemma 2.1. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded S-primary submodule of M. Then the following statements hold for some $s_g \in S$.*

(i) $(N :_M s'_g) \subseteq (N :_M s_g)$ *for all* $s'_g \in S$.

(ii) $((N :_R M) :_R s'_g) \subseteq ((N :_R M) :_R s_g)$ *for all* $s'_g \in S$.

Proof. Assume that N is a graded S-primary submodule of M. Then there exists $s_g \in S$ so that $r_h m_i \in N$ implies $s_g r_h \in Gr((N :_R M))$ or $s_g m_i \in N$ for each $r_h \in h(N)$ and $m_i \in h(M)$.

(i). Let $m' = \sum_{j \in G} m'_j \in (N :_M s'_g)$. Then $m'_j \in (N :_M s'_g)$ for all $j \in G$ as $(N :_M s'_g)$ is a graded submodule of M. Then $s'_g m'_j \in N$. Since N is graded S-primary submodule of M, so there exists $s_g \in S$ such that $s_g s'_g \in Gr(N :_R M)$ or $s_g m'_j \in N$. As $(N :_R M) \cap S = \emptyset$, we get $s_g m'_j \in N$ for all $j \in G$, hence $m' \in (N :_M s_g)$. Therefore $(N :_M s'_g) \subseteq (N :_M s_g)$.

(ii). Follows from part (i). □

Theorem 2.9. *Let M be a finitely generated graded R-module, $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodule of M provided that $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:*

(i) *N is a graded S-primary submodule of M.*

(ii) *$S^{-1}N$ is a graded primary submodule of $S^{-1}M$ and there is an $s_g \in S$ satisfying $(N :_M s'_g) \subseteq (N :_M s_g)$ for all $s'_g \in S$.*

Proof.

$(i) \Rightarrow (ii)$: By Theorem 2.5 and Lemma 2.1.

$(ii) \Rightarrow (i)$: Let $x_h \in h(R)$ and $y_i \in h(M)$ with $x_h y_i \in N$. Then $\frac{x_h}{1}\frac{y_i}{1} \in S^{-1}N$. Since $S^{-1}N$ is a graded primary submodule of $S^{-1}M$ and M is a graded finitely generated, we get $\frac{x_h}{1} \in Gr(S^{-1}N :_{S^{-1}R} S^{-1}M) = Gr(S^{-1}(N :_R M))$ or $\frac{y_i}{1} \in S^{-1}N$. Then $r_j x_h \in Gr(N :_R M)$ or $b_t y_i \in N$ for some $r_j, b_t \in S$. By assumption, there is an $s_g \in S$ such that $(N :_M s'_g) \subseteq (N :_M s_g)$ for all $s'_g \in S$. If $r_j x_h \in Gr(N :_R M)$, then $(x_h)^n M \subseteq (N :_M (r_j)^n) \subseteq (N :_M s_g)$ for some $n \in \mathbb{N}$ and so $s_g x_h \in G((N :_R M))$. If $b_t y_i \in N$, so $y_i \in (N :_M b_t) \subseteq (N :_M s_g)$ and so $s_g y_i \in N$. Therefore, N is a graded S-primary submodule of M. □

Theorem 2.10. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodule of M such that $(N :_R M) \cap S = \emptyset$. Then N is a graded S-primary submodule of M if and only if $(N :_M s_g)$ is a graded primary submodule of M for some $s_g \in S$.*

Proof. Assume that N is a graded S-primary submodule of M. Then there exists $s_g \in S$ such that whenever $x_h y_i \in N$ where $x_h \in h(R)$ and $y_i \in h(M)$, then $s_g x_h \in Gr(N :_R M)$ or $s_g y_i \in N$. Now to prove that $(N :_M s_g)$ is a graded primary submodule of M. Let $u_j y_i \in (N :_M s_g)$ where $u_j \in h(R)$ and $y_i \in h(M)$. Then $s_g u_j y_i \in N$. As N is a graded S-primary submodule of M, we get $s_g^2 u_j \in Gr(N :_R M)$ or $s_g y_i \in N$. If $s_g y_i \in N$, then there is nothing to show. Assume that $s_g y_i \notin N$. Then $s_g^2 u_j \in Gr(N :_R M)$ and so $s_g u_j \in Gr(N :_R M)$ that is $s_g^n u_j^n \in (N :_R M)$. Then by Lemma, 2.1 $u_j^n \in ((N :_R M) :_R s_g^n) \subseteq ((N :_R M) :_R s_g)$ for some $n \in \mathbb{N}$. Thus we can conclude that $u_j^n \in ((N :_M s_g) :_R M)$ so $u_j \in Gr((N :_M s_g) :_R M)$. Therefore, $(N :_M s_g)$ is a graded primary submodule of M. Conversely, assume that $(N :_M s_g)$ is a graded primary submodule of M for some $s_g \in S$. Let $x_h y_i \in N$ where $x_h \in h(R)$ and $y_i \in h(M)$. Then $x_h y_i \in (N :_M s_g)$. Since $(N :_M s_g)$ is a graded primary submodule, we get $x_h \in Gr((N :_M s_g) :_R M)$ or $y_i \in (N :_M s_g)$. Which implies that either $x_h s_g \in Gr(N :_R M)$ or $s_g y_i \in N$. Hence N is a graded S-primary submodule of M. $\square$

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