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GENERAL DECAY OF THE SOLUTION ENERGY OF AN AXIALLY MOVING VISCOELASTIC BEAM WITH LOGARITHMIC SOURCE TERMS

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ABSTRACT. In this paper, we study the stabilization of an axially moving viscoelastic beam with Logarithmic Source Terms. We obtain an asymptotic stability result of global solution, for certain class of relaxation functions. The proofs is obtained by using the multiplier technique. We extend a recent result in Kelleche and Tatar and Khemmoudj [41].

1. INTRODUCTION

In recent decades, axially moving systems have been extensively researched. Many time-dependent physical events are modelled by such equations or systems with limitations.(se for example [3, 20]). A string, a beam, or a plate model can be used to model axially moving systems. These structures are harmed due to the presence of transverse vibration as a result of certain factors such as: noise, non-uniform material properties, erratic speed, or the environment disturbance.

The aim of this work, is studied the stability of an axially moving viscoelastic structure modeled as an Euler-Bernoulli beam.The problem we are dealing with

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can be formulated as follow

$$(1.1)\qquad \begin{cases} \rho\left(w_{tt}+2vw_{xt}+v^2w_{xx}\right)+EIw_{xxxx} \\ \qquad -EI\displaystyle\int_0^t g(t-s)w_{xxxx}(s)ds=\kappa w\ln|w|, \quad x\in(0,l),\ t\geq 0, \\ w(0,t)=w_x(0,t)=w_{xx}(l,t)=0, \quad t\geq 0, \\ \rho v^2 w_x(l,t)+EIw_{xxx}(l,t)-EI\displaystyle\int_0^t g(t-s)w_{xxx}(l,s)ds \\ \qquad = f(w(l,t)), \quad t\geq 0, \\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x),\ x\ \in(0,l) \end{cases}$$

where $w = w(x,t)$ is the transversal displacement of the beam, v is the axial speed (assumed constant here), EI is the flexural rigidity of the beam and ρ is the mass per unit length of the beam and κ ia a small positive real number. The functions $w_0(x)$, $w_1(x)$ are given and the nonnegative function g represents the kernel of the memory term or the relaxation function. For more details about the physical meaning, see [6, 7, 10] and the nonlinear term f will be specified later. The first term describes the net inertia force where w_{tt} is the local acceleration in the transversal direction of the beam, w_{xt} is the Coriolis' acceleration, and w_{xx} is the centripetal acceleration. The second term represents the bending stiffness (see [22]). The integral term represents the memory term or the viscoelastic damping term. It is derived from the constitutive relationship between the stress and the history of the strain according to Boltzmann principle. For more details, we refer the reader to [3, 4, 7].

Many disciplines of physics use this type of problem with the logarithmic source term. such as inflationary cosmology (see [30]) and optics (see [32]), quantum mechanics, and nuclear physics (see [31,33]). Some Many results in the literature are improved by these writers [34–38]. With all this specific underlying meaning in physics, the global-intime well-posedness of solution to the problem of evolution equation with such logarithmic-type nonlinearity captures lots of attention. We begin our review with Birula and Mycielski's seminal work [39, 40], studied the following problem:

$$(1.2)\qquad \begin{cases} w_{tt}-w_{xx}+w=\kappa w\ln|w| \quad (x,t)\in[a,b]\times(0,T), \\ w(a,t)=w(b,t)=0, \quad (0,T), \\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x),\ x\ \in[a,b] \end{cases}$$

see [1, 3, 21] for further information on how to compute the time derivative correctly. The evaluation of the energy derivative using this rule usually gives rise to new boundary terms whose manipulation is often difficult. If we denote the partial derivatives by $\frac{\partial}{\partial t} = (.)_t$ and $\frac{\partial}{\partial x} = (.)_x$ then, the total derivative operator with respect to time is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}. \tag{1.3}$$

Many authors have addressed the subject of a beam's stability and stabilization in the absence of axial movement. Few papers directly dealing with boundary stabilization of Euler-Bernoulli type beams are cited here,(See, for example, [4, 7, 9, 10, 15, 19]). There are many other references which we cannot insert here.

This paper is divided into two parts : In the first part, is reserved to some preliminaries, assumptions on the relaxation function and some useful notation. We collect the necessary package used in the proof of the main result and give the well-posedness of the problem. In the second part is concerned with the main result where an exponential decay result is proved by using the multiplier method.

2. Preliminaries

We prepare the necessary materials for the proof of our result in this section. First, we'll go over the following helpful notation.:

For every measurable set $\mathcal{A} \subset \mathbb{R}_+$, we define for all $t \geq 0$,

$$\hat{g}(\mathcal{A}) = \frac{1}{k}\int_{\mathcal{A}} g(s)ds \tag{2.1}$$

and

$$\mathcal{A}_t = \mathcal{A} \cap [0, t].$$

The flatness set and the the flatness rate of g are defined by

$$\mathcal{F}_g = \{s \in \mathbb{R}_+ : g(s) > 0 \text{ and } g'(s) = 0\} \tag{2.2}$$

and

$$\mathcal{R}_g = \hat{g}(\mathcal{F}_g),$$

respectively. We also define

$$\tilde{F}_g = \{s \in \mathbb{R}_+ : \ g(t-s) > 0 \text{ and } g'(t-s) = 0\}.$$

We now formulate our assumptions on the relaxation function $g(t)$ and the non-linear term

(A1): $g(t) \geq 0$ for all $t \geq 0$ and $0 < k = \int_0^\infty g(s)ds < 1$.

(A2): $g'(t) \leq 0$ for almost all $t > 0$.

(A3): f satisfies the following hypotheses

$$f(0) = 0, \quad |f(u) - f(v)| \leq m(1 + |u|^\alpha + |v|^\alpha) |u - v| ,$$

for all $u, v \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$,

$$0 \leq F(u) \leq uf(u), \quad \forall u \in \mathbb{R}$$

where $F(z) = \int_0^z f(s)ds$.

Let $t_* > 0$ be a number such that $\int_0^{t_*} g(s)ds = g_* > 0$. For simplicity, we consider kernels continuous everywhere and continuously differentiable *a.e.*

We now state a lemma (containing Poincaré inequality) which will be useful later.

Lemma 2.1. *(see [12]) Let $\Phi(x,t) \in \mathbb{R}$ be a function defined for $x \in [0,l]$ and $t \in \mathbb{R}_+$ that satisfies the boundary condition*

$$\Phi(0,t) = 0, \quad t \geq 0, \tag{2.3}$$

then the following inequalities hold

$$\Phi^2(x,t) \leq l \left\| \Phi_x \right\|^2, \quad x \in [0,l], \ t \geq 0$$

and

$$\left\| \Phi \right\|^2 \leq l^2 \left\| \Phi_x \right\|^2, \quad t \geq 0.$$

If in addition to (2.3) the function satisfies the boundary condition

$$\Phi_x(0,t) = 0, \quad t \geq 0,$$

then the following inequalities also hold

$$\Phi_x^2(x,t) \leq l \left\| \Phi_{xx} \right\|^2, \quad x \in [0,l], \ t \geq 0$$

and

$$\left\| \Phi_x \right\|^2 \leq l^2 \left\| \Phi_{xx} \right\|^2, \quad t \geq 0.$$

We introduce the modified energy associated to (1.1) by

$$\begin{aligned} \mathcal{E}(t) &= \frac{\rho}{2}\|w_t\|^2 - \frac{\rho v^2}{2}\|w_x\|^2 + \frac{EI}{2}\left(1-\int_0^t g(s)ds\right)\|w_{xx}\|^2 \\ &+ \frac{EI}{2}\int_0^l (g\circ w_{xx})(t)dx + F(w(l)) - \frac{\kappa}{2}\int_0^l |w(s)|^2\ln|w(s)|ds + \frac{\kappa}{4}\|w\|^2, \end{aligned} \tag{2.4}$$

$t\geq 0$, where $\|.\|$ is the L^2-norm and

$$(g\circ w)(t) = \int_0^t g(t-s)\,|w(t)-w(s)|^2\,ds, \quad t\geq 0.$$

The following lemma will be used repeatedly in the sequel

Lemma 2.2. *We have*

$$ab \leq \delta a^2 + \frac{b^2}{4\delta}, \quad a,b\in\mathbb{R}, \quad \delta>0.$$

Lemma 2.3. *(See [42]) $\epsilon_0\in(0,1)$then, there exists $d_{\epsilon_0}>0$ such that*

$$s|\ln s| \leq s^2 + d_{\epsilon_0}s^{1-\epsilon_0}, \quad \forall s>0. \tag{2.5}$$

Lemma 2.4. *(See [43, 44])(Logarithmic Sobolev inequality) Let w be any function in $H_0^1(0,l)$ and $a>0$ be any number. Then,*

$$\int_0^l w^2\ln|w(s)|dx \leq \frac{1}{2}\|w\|^2 ln\|w\|^2 + \frac{a^2}{2\pi}\|w_t\|^2 - (1+lna)\|w\|^2. \tag{2.6}$$

Proposition 2.1. *If $v^2 < \frac{EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|w\|^2}{2}+l^4(1+\ln a)\right]}{\rho l^2}$, and then $EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|w\|^2}{2}+l^4(1+\ln a)\right]>0$, we have*

$$\mathcal{E}(t)\geq 0, \quad t\geq 0.$$

Proof. As $w(0,t)=w_x(0,t)=0$, it follows from Lemma 2.1 that

$$\|w_x\|^2 \leq l^2\|w_{xx}\|^2, \quad t\geq 0,$$

$$\|\Phi\|^2 \leq l^2\|\Phi_x\|^2, \quad t\geq 0,$$

and we apply Lemma 2.4, which implies that

$$\begin{aligned}\mathcal{E}(t) &\geq \frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|w\|^2}{2}+l^4(1+\ln a)\right.\\ &+ \left. EI-\rho v^2l^2-EI\int_0^t g(s)ds\right]\|w_{xx}\|^2\\ &+ \frac{EI}{2}\int_0^l (g\circ w_{xx})(t)dx+\frac{\rho}{2}\|w_t\|^2+F(w(l)),\end{aligned}\tag{2.7}$$

$t \geq 0$. Since $v^2 < \frac{EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|w\|^2}{2}+l^4(1+\ln a)\right]}{\rho l^2}$, and

$$EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|w\|^2}{2}+l^4(1+\ln a)\right]>0,$$

the assertion of Proposition 2.1 is satisfied. □

Lemma 2.5. *(Young's inequality). Let* $f\in L^p(\mathbf{R})$ *and* $g\in L^q(\mathbf{R})$ *with* $1\leq p,q\leq\infty$ *and* $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. *Then* $(f*g)\in L^r(\mathbf{R})$ *and*

$$\|f*g\|_{L^r}\leq\|f\|_{L^p}\|g\|_{L^q}.$$

Lemma 2.6. *(See [28]) We have for* $g\in C(0,\infty)$ *and* $w\in C\left((0,\infty);L^2(0,l)\right)$

$$\begin{aligned}\int_0^l w\int_0^t g(t-s)w(s)dsdx &= \frac{1}{2}\left(\int_0^t g(s)ds\right)\|w\|^2\\ &+ \frac{1}{2}\int_0^t g(t-s)\int_0^l w^2(s)dxds-\frac{1}{2}\int_0^l (g\circ w)\,dx,\quad t\geq 0.\end{aligned}$$

The well posedness of the problem (1.1) can be proved using Faedo Galerkin method (see for instance [18]). For this, we need to define the following spaces

$$V=\left\{u\in H^2(0,l),\quad u(0)=u_x(0)=0\right\}$$

and

$$W=\left\{u\in V\cap H^4(0,l),\ u_{xx}(l)=0\right\}.$$

Theorem 2.1. *Let* $(w_0,w_1)\in W\times L^2(0,l)$ *and* $g(t)$ *be a nonnegative summable kernel, under the hypotheses* $v^2 < \frac{EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|w\|^2}{2}+l^4(1+\ln a)\right]}{\rho l^2}$, *the problem (1.1) has a unique (weak) solution such that*

$$w\in L^\infty_{loc}(0,\infty;V),\quad w_t\in L^\infty_{loc}(0,\infty;V),\ w_{tt}\in L^\infty_{loc}\left(0,\infty;L^2(0,l)\right).$$

3. Exponential Stability

In this section we state and prove our decay result.

Lemma 3.1. *The energy $\mathcal{E}(t)$ satisfies, along solutions of (1.1)*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &\le \frac{EI}{2}\int_0^l \left(g'\circ w_{xx}\right)(t)dx - \frac{\rho v}{2}w_t^2(l) - \frac{\rho v^3}{2}w_x^2(l)\\ &- \frac{EIv}{2}\left(1-\int_0^t g(s)ds\right)w_{xx}^2(0) - \frac{EIv}{2}\left(g\circ w_{xx}\right)(0), \quad t\ge 0.\end{aligned} \tag{3.1}$$

Proof. Notice that if $g'\le 0$, the energy $\mathcal{E}(t)$ is nonincreasing and bounded above uniformly by $\mathcal{E}(0)$. The total derivative of $\mathcal{E}(t)$ can be derived as follows (see [22])

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\int_0^l \widetilde{\mathcal{E}}(x,t)dx + \frac{d}{dt}F(w(l)) = \int_0^l \frac{d}{dt}\widetilde{\mathcal{E}}(x,t)dx + \frac{d}{dt}F(w(l))\\ &= \int_0^l \left[\frac{\partial}{\partial t}\widetilde{\mathcal{E}}(x,t) + \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\widetilde{\mathcal{E}}(x,t)\right]dx + \frac{d}{dt}F(w(l))\\ &= \int_0^l \frac{\partial}{\partial t}\widetilde{\mathcal{E}}(x,t)dx + v\,\widetilde{\mathcal{E}}(x,t)\Big|_0^l + \frac{d}{dt}F(w(l)),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\widetilde{\mathcal{E}}(x,t) &= \frac{\rho}{2}w_t^2(x,t) - \frac{\rho v^2}{2}w_x^2(x,t) + \frac{EI}{2}\left(1-\int_0^t g(s)ds\right)w_{xx}^2(x,t)\\ &+ \frac{EI}{2}\left(g\circ w_{xx}\right)(x,t) - \frac{\kappa}{2}w^2\ln|w| + \frac{\kappa}{4}w^2, \quad x\in[0,l], \quad t\ge 0.\end{aligned}$$

Using the equation in (1.1), the relation (3.2) becomes

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= -\int_0^l w_t\left[2\rho v w_{xt} + \rho v^2 w_{xx} + EI\left(w_{xxxx} - \int_0^t g(t-s)w_{xxxx}(s)ds\right)\right]dx\\ &-\rho v^2\int_0^l w_x w_{xt}dx + EI\left(1-\int_0^t g(s)ds\right)\int_0^l w_{xxt}w_{xx}dx - \frac{EI}{2}g(t)\int_0^l w_{xx}^2 dx\\ &+\frac{EI}{2}\int_0^l \frac{\partial}{\partial t}\left(g\circ w_{xx}\right)(t)dx - \frac{\partial}{\partial t}\left(\frac{\kappa}{2}\int_0^l |w(s)|^2\ln|w(s)|\right)ds\\ &+\frac{\partial}{\partial t}\left(\frac{\kappa}{4}\|w\|^2\right) + w_t(l)f(w(l)) + v\,\widetilde{\mathcal{E}}(x,t)\Big|_0^l, \quad t\ge 0.\end{aligned}$$

Integrating by parts and taking into account the boundary conditions in (1.1) we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &= -\rho v w_t^2(l) - w_t(l)\left[\rho v^2 w_x(l) + EI w_{xxx}(l)\right.\\
&- \left. EI\int_0^t g(t-s)w_{xxx}(l,s)ds\right] + EI\int_0^l w_{xxt}\int_0^t g(t-s)w_{xx}(s)dsdx\\
&- EI\left(\int_0^t g(s)ds\right)\int_0^l w_{xxt}w_{xx}dx - \frac{EI}{2}g(t)\int_0^1 w_{xx}^2dx + w_t(l)f(w(l))\\
&+ \frac{EI}{2}\int_0^l \frac{\partial}{\partial t}\left(g\circ w_{xx}\right)(t)dx - \frac{\partial}{\partial t}\left(\frac{\kappa}{2}\int_0^l |w(s)|^2\ln|w(s)|\right)ds\\
&+ \frac{\partial}{\partial t}\left(\frac{\kappa}{4}\|w\|^2\right) + v\,\widetilde{\mathcal{E}}(x,t)\Big|_0^l, \quad t\geq 0.
\end{aligned}$$

Next, in view of the boundary conditions in (1.1) and the definition of $\widetilde{\mathcal{E}}(x,t)$, we have

$$\begin{aligned}
\widetilde{\mathcal{E}}(x,t)\Big|_0^l &= \frac{\rho}{2}w_t^2(l) - \frac{\rho v^2}{2}w_x^2(l) - \frac{EI}{2}\left(1-\int_0^t g(s)ds\right)w_{xx}^2(0)\\
&- \frac{EI}{2}\left(g\circ w_{xx}\right)(0), \quad t\geq 0.
\end{aligned} \tag{3.3}$$

Clearly

$$\begin{aligned}
\frac{\partial}{\partial t}\left(g\circ w_{xx}\right)(x,t) &= \left(g'\circ w_{xx}\right)(x,t) - 2w_{xxt}(x,t)\int_0^t g(t-s)w_{xx}(x,s)ds\\
&+ 2\left(\int_0^t g(s)ds\right)w_{xxt}(x,t)w_{xx}(x,t), \quad t\geq 0.
\end{aligned} \tag{3.4}$$

Therefore, taking into account (3.3), (3.4) and the boundary conditions in (1.1), we obtain (3.1). □

To formulate our results, we define the functionals

$$\begin{aligned}
\Psi_1(t) =& \rho\int_0^l w_t y dx + \frac{\rho v}{2}w^2(l), \quad t\geq 0,\\
\Psi_2(t) =& -\rho\int_0^l w_t\int_0^t g(t-s)(w(t)-w(s))dsdx, \quad t\geq 0,\\
\Psi_3(t) =& \int_0^t\left(\int_t g(\tau-s)d\tau\right)\|w_{xx}(s)\|^2 ds, \quad t\geq 0,
\end{aligned}$$

$$\Psi_4(t) = \int_0^t \left(\int_t^\infty g(\tau - s)d\tau \right) w_{xx}^2(l, s)ds, \quad t \geq 0.$$

and

$$L(t) = M\mathcal{E}(t) + \lambda\Psi_1(t) + \Psi_2(t) + \mu\Psi_3(t) + \gamma\Psi_4(t), \quad t \geq 0$$

where λ, μ and γ are positive constants. The first result is to establish an equivalence between the modified energy $\mathcal{E}(t)$ and the modified energy functional $L(t)$.

Proposition 3.1. *If* $v^2 < \frac{EI(1-k)+\frac{\kappa}{2}\left[\frac{l^4}{2}-\frac{a^2l^2}{2\pi}-\frac{l^4\ln\|w\|^2}{2}+l^4(1+\ln a)\right]}{\rho l^2}$, *then we have*

$$\mathcal{E}(t) \leq L(t) \leq C(\mathcal{E}(t) + \Psi_3(t) + \gamma\Psi_4(t)), \quad t \geq 0$$

for some $C > 0$ *and*

$$L'(t) + \alpha_1\mathcal{E}(t) \leq 0, \quad t \geq 0 \tag{3.5}$$

for some $\alpha_1 > 0$.

Proof. First, Young inequality and Poincaré inequality leads to

$$\begin{aligned} \Psi_1(t) &\leq \frac{\rho}{2}\|w_t\|^2 + \frac{\rho}{2}\|w\|^2 + \frac{\rho v}{2}w^2(l) \\ &\leq \frac{\rho}{2}\|w_t\|^2 + \frac{l\rho}{2}(l+v)\|w_x\|^2 . \end{aligned}$$

Similarly , we get

$$\begin{aligned} \Psi_2(t) &\leq \frac{\rho}{2}\|w_t\|^2 + \frac{\rho}{2}\int_0^l \left(\int_0^t g(t-s)(w(t)-w(s))ds \right)^2 dx \\ &\leq \frac{\rho}{2}\|w_t\|^2 + \frac{\rho k l^4}{2}\int_0^l (g \circ w_{xx})(t)dx \end{aligned}$$

Then, Lemma 2.2 and Lemma 2.1 allow us to write

$$\begin{aligned} L(t) &\leq \frac{\rho}{2}(M+\lambda)\|w_t\|^2 + \frac{\rho}{2}\left[\lambda\rho(l+v) - Mv^2 + \frac{M\kappa l^2}{2}\right]\|w_x\|^2 \\ &+ \frac{MEI}{2}\int_0^l (g \circ w_{xx})(t)dx \\ &+ (\frac{\rho k l^4}{2} + MEI)\int_0^l (g \circ w_{xx})(t)dx - \frac{M\kappa}{2}\int_0^l |w(s)|^2 \ln|w(s)|ds \\ &+ \mu\Psi_3(t) + \gamma\Psi_4(t), \quad t \geq 0. \end{aligned}$$

which implies that there exists a positive constant C such that $\mathcal{E}(t) \leq C(L(t) + \Psi_3(t) + \Psi_4(t))$, $t \geq 0$. On the other side

$$\begin{aligned}
& L(t) - \mathcal{E}(t) \\
= \; & (M-1)\mathcal{E}(t) + \lambda\Psi_1(t) + \Psi_2(t) + \mu\Psi_3(t) + \gamma\Psi_4(t) \\
\geq \; & \frac{\rho}{2}(M+\lambda)\|w_t\|^2 + (M-1)F(w(l) + \left[(M-1)\frac{EI}{2} - \frac{\rho k l^4}{2}\right]\int_0^l (g \circ w_{xx})(t)dx \\
+ \; & \frac{1}{2}\left\{\left[(M-1)\left(\frac{EI}{l^2}(1-K) - \rho v^2 + \frac{M\kappa l^2}{2}\right)\right] - \lambda l\rho(l+v)\right\}\|w_x\|^2 \\
- \; & \frac{\kappa(M-1)}{2}\int_0^l |w(s)|^2 \ln|w(s)|ds + \mu\Psi_3(t) + \gamma\Psi_4(t), \quad t \geq 0.
\end{aligned}$$

Choosing M large enough we find that $L(t) - \mathcal{E}(t) \geq 0$. Thus, the first assertion is proved.

Now, we proceed to show the second assertion.

The total derivative of $\Psi_1(t)$ can be derived as follows

$$\begin{aligned}
\frac{d}{dt}\Psi_1(t) &= \int_0^l \frac{d}{dt}\widetilde{\Psi_1}(x,t)dx = \int_0^l \left(\frac{\partial}{\partial t}\widetilde{\Psi_1}(x,t)\right)dx + v\widetilde{\Psi_1}(x,t)\Big|_0^l \\
&= \rho\|w_t\|^2 + \rho\int_0^l w w_{tt}dx - \rho v w_t(l)w(l) + v\widetilde{\Psi_1}(x,t)\Big|_0^l, \quad t \geq 0, \qquad (3.6)
\end{aligned}$$

where

$$\widetilde{\Psi_1}(x,t) = \rho w(x,t)w_t(x,t), \quad t \geq 0.$$

In virtue of the boundary conditions in (1.1) we see that

$$\widetilde{\Psi_1}(x,t)\Big|_0^l = \rho w_t(l)w(l), \quad t \geq 0. \qquad (3.7)$$

The Eq. in (1.1) and an integration by parts lead to

$$\begin{aligned}
\frac{d}{dt}\Psi_1(t) \leq \; & \rho\|w_t\|^2 + 2\rho v\int_0^l w_x w_t dx + \rho v^2\|w_x\|^2 - EI\|w_{xx}\|^2 - w(l)f(w(l)) \\
+ \; & EI\int_0^l w_{xx}\int_0^t g(t-s)w_{xx}(s)dsdx + \frac{\kappa}{2}\int_0^l w^2(s)\ln|w(s)|ds, \quad t \geq 0. \qquad (3.8)
\end{aligned}$$

We now estimate the second term in the right hand side of (3.8) as follows

$$2\rho v\int_0^l w_x w_t dx \leq \rho\|w_t\|^2 + \rho v^2\|w_x\|^2, \quad t \geq 0. \qquad (3.9)$$

The insertion of (3.9) into (3.8) and the use of Lemma 2.6 we obtain

$$\begin{aligned} \frac{d}{dt}\Psi_1(t) &\leq 2\rho \|w_t\|^2 + 2\rho v^2 \|w_x\|^2 - EI\left(1-\frac{k}{2}\right)\|w_{xx}\|^2 \\ &+ \frac{EI}{2}\int_0^t g(t-s)\|w_{xx}(s)\|^2 ds - \frac{EI}{2}\int_0^l (g\circ w_{xx})(t)dx \\ &- w(l)f(w(l)) + \frac{\kappa}{2}\int_0^l w^2(s)\ln|w(s)|. \end{aligned} \tag{3.10}$$

For $\Psi_2(t)$, we have

$$\begin{aligned} \frac{d}{dt}\Psi_2(t) &= \int_0^l \frac{d}{dt}\widetilde{\Psi_2}(x,t)dx = \int_0^l \left(\frac{\partial}{\partial t}\widetilde{\Psi_2}(x,t)\right)dx + v\widetilde{\Psi_2}(x,t)\Big|_0^l \\ &= -\rho\int_0^l w_{tt}\int_0^t g(t-s)(w(t)-w(s))dsdx - \rho\left(\int_0^t g(s)ds\right)\|w_t\|^2 \\ &- \rho\int_0^l w_t\int_0^t g'(t-s)(w(t)-w(s))dsdx + v\widetilde{\Psi_2}(x,t)\Big|_0^l, \quad t\geq 0, \end{aligned} \tag{3.11}$$

where

$$\widetilde{\Psi_2}(x,t) = -\rho w_t(x,t)\int_0^t g(t-s)(w(x,t)-w(x,s))ds, \quad t\geq 0. \tag{3.12}$$

On the other hand, the boundary conditions in (1.1) yield

$$\widetilde{\Psi_2}(x,t)\Big|_0^l = -\rho w_t(l)\int_0^t g(t-s)(w(l,t)-w(l,s))ds, \quad t\geq 0. \tag{3.13}$$

Using integration by parts and taking into account (3.13), the total derivative in (3.11) is equal to

$$\begin{aligned}
\frac{d}{dt}\Psi_2(t) &= EI\left(1-\int_0^t g(s)ds\right)\int_0^l w_{xx}\int_0^t g(t-s)(w_{xx}(t)-w_{xx}(s))ds \\
&+ EI\int_0^l \left|\int_0^t g(t-s)(w_{xx}(t)-w_{xx}(s))ds\right|^2 dx \\
&- \rho v^2\int_0^l w_x\int_0^t g(t-s)(w_x(t)-w_x(s))dsdx \\
&- 2\rho v\int_0^l w_t\int_0^t g(t-s)(w_x(t)-w_x(s))dsdx \\
&- \rho\int_0^l w_t\int_0^t g'(t-s)(w(t)-w(s))dsdx - \rho\left(\int_0^t g(s)ds\right)\|w_t\|^2 \\
&+ f(w(l))\int_0^t g(t-s)(w(l,t)-w(l,s))ds \\
&+ \rho v w_t(l)\int_0^t g(t-s)(w(l,t)-w(l,s))ds \\
&- \kappa\int_0^l w\ln|w|\int_0^t g(t-s)(w(t)-w(s))dsdx, \quad t\geq 0. \qquad (3.14)
\end{aligned}$$

We now estimate the terms in the right hand side of (3.14). For all measurable sets $\mathcal{A}$ and $\mathcal{F}$ such that $\mathcal{A}=\mathbb{R}_+\backslash\mathcal{F}$, we have

$$\begin{aligned}
&\int_0^l w_{xx}\int_0^t g(t-s)(w_{xx}(t)-w_{xx}(s))dsdx \\
= &\int_0^l w_{xx}\left(\int_{\mathcal{A}_t} g(t-s)(w_{xx}(t)-w_{xx}(s))ds \right. \qquad (3.15)\\
+ &\left.\int_{\mathcal{F}_t} g(t-s)(w_{xx}(t)-w_{xx}(s))ds\right)dx \\
\leq &\int_0^l w_{xx}\int_{\mathcal{A}_t} g(t-s)(w_{xx}(t)-w_{xx}(s))dsdx \\
+ &\left(\int_{\mathcal{F}_t} g(t-s)ds\right)\|w_{xx}\|^2 - \int_0^l w_{xx}\int_{\mathcal{F}_t} g(t-s)w_{xx}(s)dsdx, \quad t\geq 0. \qquad (3.16)
\end{aligned}$$

Using Lemma 2.2 it is easy to see for $t\geq 0$

$$\int_0^l w_{xx}\int_{\mathcal{A}_t} g(t-s)(w_{xx}(t)-w_{xx}(s))dsdx$$

$$\leq \eta_1 \|w_{xx}\|^2 + \frac{k}{4\eta_1}\int_0^l \int_{\mathcal{A}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx, \quad \eta_1 > 0.$$

and

$$-\int_0^l w_{xx}\int_{\mathcal{F}_t} g(t-s)w_{xx}(s)dsdx \leq \frac{1}{2}\|w_{xx}\|^2 + \frac{k}{2}\int_0^t g(t-s)\|w_{xx}(s)\|^2 ds.$$

Therefore, from (3.16) we can deduce that

$$\begin{aligned} & \int_0^l w_{xx}\int_0^t g(t-s)(w_{xx}(t)-w_{xx}(s))dsdx \\ \leq\ & \left((\frac{1}{2}+\eta_1)+k\hat{g}(\mathcal{F})\right)\|w_{xx}\|^2 \qquad (3.17) \\ +\ & \frac{k}{4\eta_1}\int_0^l \int_{\mathcal{A}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx \\ +\ & \frac{k}{2}\int_0^t g(t-s)\|w_{xx}(s)\|^2 ds, \quad t \geq 0, \qquad (3.18) \end{aligned}$$

where $\hat{g}(\mathcal{F})$ is defined in (2.1). Similarly

$$\begin{aligned} & \int_0^l \left|\int_0^t g(t-s)(w_{xx}(t)-w_{xx}(s))ds\right|^2 dx \\ \leq\ & \left(1+\frac{1}{\eta_2}\right)k\int_0^l \int_{\mathcal{A}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx \\ +\ & (1+\eta_2)\,k\hat{g}(\mathcal{F})\int_0^l \int_{\mathcal{F}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx, \quad \eta_2 > 0. \qquad (3.19) \end{aligned}$$

For the third term we can write

$$\begin{aligned} & -\int_0^l w_x \int_0^t g(t-s)(w_x(t)-w_x(s))dsdx \\ =\ & -\left(\int_0^t g(s)ds\right)\|w_x\|^2 + \int_0^l w_x \int_0^t g(t-s)w_x(s)dsdx \\ \leq\ & \left(\frac{1}{2}-g_*\right)\|w_x\|^2 + \frac{l^2k}{2}\int_0^t g(t-s)\|w_{xx}(s)\|^2 ds, \quad t \geq t_*, \quad \eta_3 > 0. \qquad (3.20) \end{aligned}$$

As for the first term, we estimate the fourth term as follows

$$
\begin{aligned}
& -2\rho v \int_0^l w_t \int_0^t g(t-s)(w_x(t)-w_x(s))dsdx \\
(3.21) \quad = & -2\rho v \int_0^l w_t \left(\int_{\mathcal{A}_t} g(t-s)(w_x(t)-w_x(s))ds \right. \\
& \left. + \int_{\mathcal{F}_t} g(t-s)(w_x(t)-w_x(s))ds \right) dx \\
= & -2\rho v \int_0^l w_t \int_{\mathcal{A}_t} g(t-s)(w_x(t)-w_x(s))dsdx \\
(3.22) \quad & -2\rho v \left(\int_{\mathcal{F}_t} g(t-s)ds \right) \int_0^l w_t w_x dx \\
(3.23) \quad & +2\rho v \int_0^l w_t \int_{\mathcal{F}_t} g(t-s) w_x(s) dsdx
\end{aligned}
$$

or

$$
\begin{aligned}
& -2\rho v \int_0^l w_t \int_{\mathcal{A}_t} g(t-s)(w_x(t)-w_x(s))dsdx \\
(3.24) \quad \leq \; & \eta_3 \rho \|w_t\|^2 + \frac{\rho v^2 l^2 k}{\eta_3} \int_0^l \int_{\mathcal{A}_t} g(t-s) \left| w_{xx}(t) - w_{xx}(s) \right|^2 dsdx, \quad t \geq 0
\end{aligned}
$$

and

$$
(3.25) \quad -2\rho v \left(\int_{\mathcal{F}_t} g(t-s)ds \right) \int_0^l w_t w_x dx \leq \rho k \hat{g}(\mathcal{F}) \|w_t\|^2 + \rho v^2 k \hat{g}(\mathcal{F}) \|w_x\|^2
$$

$$
(3.26) \quad 2\rho v \int_0^l w_t \int_{\mathcal{F}_t} g(t-s) w_x(s) dsdx \leq \rho \|w_t\|^2 + \rho v^2 l^2 k \int_0^t g(t-s) \|w_{xx}(s)\|^2 ds
$$

for all $t \geq 0$. Therefore (3.23) becomes

$$
\begin{aligned}
& -2\rho v \int_0^l w_t \int_0^t g(t-s)(w_x(t)-w_x(s))dsdx \\
(3.27) \quad \leq \; & \rho \left(1 + \eta_3 + k\hat{g}(\mathcal{F})\right) \|w_t\|^2 + \rho v^2 k \hat{g}(\mathcal{F}) \|w_x\|^2 \\
& + \rho v^2 l^2 k \int_0^t g(t-s) \|w_{xx}(s)\|^2 ds \\
(3.28) \quad & + \frac{\rho v^2 l^2 k}{\eta_3} \int_0^l \int_{\mathcal{A}_t} g(t-s) \left| w_{xx}(t) - w_{xx}(s) \right|^2 dsdx.
\end{aligned}
$$

The fifth term may be evaluated by

$$-\rho\int_0^l w_t\int_0^t g'(t-s)(w(t)-w(s))dsdx$$
$$\leq \eta_4\rho\left\|w_t\right\|^2-\frac{\rho g(0)l^4}{4\eta_4}\int_0^l\left(g'\circ w_{xx}\right)(t)dx,\quad \eta_4>0,\quad t\geq 0. \tag{3.29}$$

Now, we pass to the nonlinear term, we have

$$\begin{aligned}
&f(w(l))\int_0^t g(t-s)(w(l,t)-w(l,s))ds\\
&=f(w(l))\left(\int_{\mathcal{A}_t} g(t-s)(w(l,t)-w(l,s))ds\right.\\
&\left.+\int_{\mathcal{F}_t} g(t-s)(w(l,t)-w(l,s))ds\right)\\
&=f(w(l))\int_{\mathcal{A}_t} g(t-s)(w(l,t)-w(l,s))ds\\
&-f(w(l))\int_{\mathcal{F}_t} g(t-s)w(l,s)ds\\
&+\left(\int_{\mathcal{F}_t} g(t-s)ds\right)w(l)f(w(l))
\end{aligned} \tag{3.30}$$

or

$$f(w(l))\int_{\mathcal{A}_t} g(t-s)(w(l,t)-w(l,s))ds$$
$$\leq \eta_5\left|f(w(l))\right|^2+\frac{l^3k}{4\eta_5}\int_0^l\int_{\mathcal{A}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx,\quad t\geq 0$$

and

$$-f(w(l))\int_{\mathcal{F}_t} g(t-s)w(l,s)ds$$
$$\leq \frac{\left|f(w(l))\right|^2}{2}+\frac{l^3k}{2}\int_0^t g(t-s)\left\|w_{xx}(s)\right\|^2 ds,\quad \eta_4>0,\quad t\geq 0.$$

From the hypothesis **(A4)** and with the help of (2.7) and Lemma 2.1 we entail

$$\left|f(w(l))\right|^2\leq 2m^2(\left|w(l)\right|^2+\left|w(l)\right|^{2(\alpha+1)})$$
$$\leq 2m^2l^3\left[\left\|w_{xx}\right\|^2+\left(\frac{2}{EIk-\rho v^2l^2}E(0)\right)^{\alpha}\left\|w_{xx}\right\|^2\right]=\beta\left\|w_{xx}\right\|^2.$$

Therefore, the relation (3.30) becomes

$$
\begin{aligned}
& f(w(l))\int_0^t g(t-s)(w(l,t)-w(l,s))ds \\
\leq\; & (\eta_5+\frac{\beta}{2})\left\|w_{xx}\right\|^2+\frac{l^3k}{2}\int_0^t g(t-s)\left\|w_{xx}(s)\right\|^2 ds \\
& +k\hat{g}(\mathcal{F})w(l)f(w(l))+\frac{l^3k}{4\eta_5}\int_0^l\int_{\mathcal{A}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx. \qquad (3.31)
\end{aligned}
$$

For the Seventh term, we can write

$$
\begin{aligned}
& \rho v w_t(l)\int_0^t g(t-s)(w(l,t)-w(l,s))ds \\
\leq\; & \frac{\rho v}{4\eta_7}w_t^2(l)+\eta_7 l^3 k\int_0^l\left(g\circ w_{xx}\right)dx, \quad \eta_7>0, \quad t\geq 0. \qquad (3.32)
\end{aligned}
$$

To estimate the last term, we apply Lemma 2.3 with $\epsilon=(1/2)$ and use repeatedly Young's, Cauchy-Schwartz's, and Lemma 2.1 and the embedding inequalities, as follows:

$$
\begin{aligned}
& -\kappa\int_0^l w\ln|w|\int_0^t g(t-s)(w(t)-w(s))dsdx \\
\leq\; & \kappa\int_0^l\left(w^2+d_{\epsilon_0}\sqrt{|w|}\right)\int_0^t g(t-s)(w(t)-w(s))dsdx \\
\leq\; & \kappa\left(\eta_8\int_0^l(w^2+d_{\epsilon_0}\sqrt{|w|})^2dx\right. \\
& \left.+\frac{1}{4\eta_8}\int_0^l|\int_0^t g(t-s)(w(t)-w(s))ds|^2dx\right) \\
\leq\; & \kappa\eta_8(l^8+l^4)\|w_{xx}\|^2+\frac{\kappa k l^4}{4\eta_8}\int_0^l\left(g\circ w_{xx}\right)(t)dx. \qquad (3.33)
\end{aligned}
$$

The insertion of (3.18)-(3.20), (3.28), (3.29), (3.31),(3.33), (3.32) into (3.14) we obtain

$$
\begin{aligned}
& \frac{d}{dt}\Psi_2(t) \qquad (3.34) \\
\leq\; & -\frac{g(0)l^4}{4\eta_4}\int_0^l\left(g'\circ w_{xx}\right)(t)dx+\rho\left(-g_*+\eta_4+\frac{1}{2}+\eta_3+k\hat{g}(\mathcal{F})\right)\left\|w_t\right\|^2 \\
+\; & \rho v^2\left(-g_*+\frac{1}{2}+k\hat{g}(\mathcal{F})\right)\left\|w_x\right\|^2+\{(\frac{1}{2}+\eta_1)\left(1-g_*\right)EI+(\frac{\beta}{2}+\eta_5\frac{\beta}{2})
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \kappa\eta_8(l^8+l^4) + EI\left(1-g_*\right)k\hat{g}(\mathcal{F})\} \left\|w_{xx}\right\|^2 + (\frac{\kappa k l^4}{4\eta_8} + \eta_7 l^3 k)\int_0^l \left(g\circ w_{xx}\right)(t)dx \\
&+ \quad \frac{1}{2}\left(\left(1-g_*\right)kEI + 5\rho v^2 l^2 k + l^3 k\right)\int_0^t g(t-s)\left\|w_{xx}(s)\right\|^2 ds \\
&+ \quad \left(1+\eta_2\right)EIk\hat{g}(\mathcal{F})\int_0^l\int_{\mathcal{F}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx + k\left\{\left(1-g_*\right)\frac{EI}{4\eta_1}\right. \\
&+ \quad \left.\left(1+\frac{1}{\eta_2}\right)EI + \frac{\rho v^2 l^2}{\eta_3} + \frac{l^3}{4\eta_5}\right\}\int_0^l\int_{\mathcal{A}_t} g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^2 dsdx \\
&+ \quad k\hat{g}(\mathcal{F})w(l)f(w(l)) + \frac{\rho v}{4\eta_7}w_t^2(l), \quad t\geq t_*>0.
\end{aligned}
$$

where $\beta = 2m^2l^3\left[1+\left(\frac{2E(0)}{kEI-l^2\rho v^2}\right)^{\alpha}\right]$ for some positive constants η_i, $i=1,\ldots,7$.

On the other hand, a differentiation of $\Psi_3(t)$ gives

$$
\frac{d}{dt}\Psi_3(t) = k\left\|w_{xx}\right\|^2 - \int_0^t g(t-s)\left\|w_{xx}(s)\right\|^2 ds,\ t\geq 0. \tag{3.35}
$$

The total derivative of $\Psi_3(t)$ is given by

$$
\frac{d}{dt}\Psi_3(t) = \int_0^l\left(\frac{\partial}{\partial t}\widetilde{\Psi_3}(x,t)\right)dx + v\widetilde{\Psi_3}(x,t)\Big|_0^l,\ t\geq 0, \tag{3.36}
$$

where

$$
\tilde{\Psi}_3(x,t) = \int_0^t\left(\int_t^\infty g(\tau-s)d\tau\right)w_{xx}^2(x,s)ds,\ t\geq 0.
$$

Clearly

$$
\frac{d}{dt}\Psi_3(t)\leq k\left\|w_{xx}\right\|^2 - \int_0^t g(t-s)\int_0^l w_{xx}^2(s)dxds + v\int_0^t g(t-s)w_{xx}^2(l,s)ds,\ t\geq 0.
$$

On the other hand, a differentiation of $\psi_4(t)$ gives

$$
\frac{d}{dt}\Psi_4(t) = kw_{xx}^2(l,s) - \int_0^t g(t-s)w_{xx}^2(l,s)ds,\ t\geq 0. \tag{3.37}
$$

From the relations (3.1), (3.10), (3.35) and (3.37) we have, for $t\geq t_*$

$$
\begin{aligned}
\frac{d}{dt}L(t) &\leq \left(\frac{MEI}{2} - \frac{g(0)l^4}{4\eta_4}\right)\int_0^l\left(g'\circ w_{xx}\right)(t)dx \\
&+\rho\left[2\lambda + \left(-g_* + \eta_4 + (1+\eta_3) + k\hat{g}(\mathcal{F})\right)\right]\left\|w_t\right\|^2
\end{aligned} \tag{3.38}
$$

$$
\begin{aligned}
&+\rho v^{2}\left[2\lambda+\left(-g_{*}+\frac{1}{2}+k\hat{g}(\mathcal{F})\right)\right]\|w_{x}\|^{2}\\
&+\left\{[(\frac{1}{2}+\eta_{1})\left(1-g_{*}\right)EI+(\eta_{5}\frac{\beta}{2}+\frac{\beta}{2})+\kappa\eta_{8}(l^{8}+l^{4})+EI\left(1-g_{*}\right)k\hat{g}(\mathcal{F})]\right.\\
&\left.-\lambda EI\left(1-\frac{k}{2}\right)+(\mu+\gamma)k\right\}\|w_{xx}\|^{2}\\
&+\left(\frac{\kappa k l^{4}}{4\eta_{8}}-\lambda\frac{EI}{2}+\eta_{7}l^{3}k\right)\times\int_{0}^{l}\left(g\circ w_{xx}\right)(t)dx+(\mu v-\gamma)\int_{0}^{t}g(t-s)w_{xx}^{2}(l,s)ds\\
&+\left\{\lambda\frac{EI}{2}+\frac{k}{2}\left[\left(1-g_{*}\right)EI+5\rho v^{2}l^{2}+l^{3}\right]-\mu\right\}\int_{0}^{t}g(t-s)\|w_{xx}(s)\|^{2}ds\\
&+\left(1+\eta_{2}\right)EIk\hat{g}(\mathcal{F})\times\int_{0}^{l}\int_{\mathcal{F}_{t}}g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^{2}dsdx\\
&+k\left[\left(1-g_{*}\right)\frac{EI}{4\eta_{1}}+\left(1+\frac{1}{\eta_{2}}\right)EI\right.\\
&\left.+\left(1+\frac{1}{\eta_{2}}\right)EI+\frac{\rho v^{2}l^{2}}{\eta_{3}}+\frac{l^{3}}{4\eta_{5}}\right]\int_{0}^{l}\int_{\mathcal{A}_{t}}g(t-s)\left|w_{xx}(t)-w_{xx}(s)\right|^{2}dsdx\\
&+\left\{-\lambda+k\hat{g}(\mathcal{F})\right\}w(l)f(w(l))+\frac{\rho v}{2}\left(\frac{1}{2\eta_{7}}-1\right)w_{t}^{2}(l).
\end{aligned}
$$

We select M large enough so that

$$
\frac{MEI}{2}-\frac{g(0)l^{4}}{4\eta_{4}}\geq\frac{MEI}{4}. \tag{3.39}
$$

As in [21], we introduce for $n\in\mathbb{N}$ the sets

$$
\mathcal{A}_{n}=\{s\in\mathbb{R}_{+}:nh'(s)+g(s)\leq 0\},\ \mathcal{F}_{n}=\mathbb{R}_{+}\backslash\mathcal{A}_{n}
$$

and also

$$
\tilde{A}_{nt}=\{s\in\mathbb{R}_{+}:0\leq s\leq t,\quad nh'(t-s)+g(t-s)\leq 0\},\quad n\in\mathbb{N}.
$$

Observe that

$$
\bigcup_{n}\mathcal{A}_{n}=\mathbb{R}_{+}\backslash\{\mathcal{F}_{g}\cup\mathcal{N}_{g}\},
$$

where $\mathcal{N}_g$ is the set where g' is not defined and $\mathcal{F}_g$ is defined. Since $\mathcal{F}_{n+1}\subset\mathcal{F}_n$ for all n and $\bigcap_{n}\mathcal{F}_n = \mathcal{F}_g\cup\mathcal{N}_g$, then $\lim_{n\to\infty}\hat{g}(\mathcal{F}_n)=\hat{g}(\mathcal{F}_g)$. Taking $\mathcal{A}_t := \tilde{A}_{nt}$,

$\mathcal{F}_t := \tilde{F}_{nt}$, it follows that

$$\begin{aligned}
&\frac{d}{dt}L(t) \leq \rho\left[2\lambda + (-g_* + \eta_4 + (1+\eta_3) + k\hat{g}(\mathcal{F}))\right]\|w_t\|^2 \\
&+\rho v^2\left[2\lambda + \left(-g_* + \frac{1}{2} + k\hat{g}(\mathcal{F})\right)\right]\|w_x\|^2 \\
&-\left\{\left[\lambda EI\left(1-\frac{k}{2}\right) - (\frac{1}{2}+\eta_1)(1-g_*)EI - (\frac{\beta}{2}\eta_5 + \frac{\beta}{2}) - \kappa\eta_8(l^8+l^4)\right.\right. \\
&\left.\left. -\ EI(1-g_*)k\hat{g}(\mathcal{F})\right]\right\} - (\mu+\gamma)k\}\|w_{xx}\|^2 \\
&+\left(\frac{\kappa k l^4}{4\eta_8} - \lambda\frac{EI}{2} + \eta_7 l^3 k\right) \times \int_0^l (g\circ w_{xx})(t)dx \\
&+(\mu v - \gamma)\int_0^t g(t-s)w_{xx}^2(l,s)ds \\
&+\left\{\lambda\frac{EI}{2} + \frac{k}{2}\left[(1-g_*)EI + 5\rho v^2 l^2 + l^3\right] - \mu\right\}\int_0^t g(t-s)\|w_{xx}(s)\|^2 ds \qquad (3.40) \\
&+(1+\eta_2)EIk\hat{g}(\mathcal{F}) \times \int_0^l \int_{\mathcal{F}_t} g(t-s)|w_{xx}(t) - w_{xx}(s)|^2 dsdx \\
&+k\left[(1-g_*)\frac{EI}{4\eta_1} + \left(1+\frac{1}{\eta_2}\right)EI + \left(1+\frac{1}{\eta_2}\right)EI + \frac{\rho v^2 l^2}{\eta_3} + \frac{l^3}{4\eta_5} - \frac{MEI}{4n}\right] \\
&\cdot\int_0^l \int_{\mathcal{A}_t} g(t-s)|w_{xx}(t) - w_{xx}(s)|^2 dsdx + \{-\lambda + k\hat{g}(\mathcal{F})\}w(l)f(w(l)) \\
&+\frac{\rho v}{2}\left(\frac{1}{2\eta_7} - 1\right)w_t^2(l).
\end{aligned}$$

In (3.41), for $g_* > k\hat{g}(\mathcal{F}) + \frac{1}{2}$ sufficiently large we take $\lambda = \frac{1}{2}\left(g_* - k\hat{g}(\mathcal{F}) - \frac{1}{2} - \varepsilon\right)$. We infer that

$$\frac{\kappa k l^4}{4\eta_8} - \frac{\lambda EI}{2} + (1+\eta_2)EIk\hat{g}(\mathcal{F}_n) + \eta_7 l^3 k \leq 0$$

and $\frac{1}{2\eta_7} - 1 \leq 0$.

II. In order to ensure the negativity of the coefficient of $\|w_{xx}\|^2$, we will need (we neglect η_5 and $\eta_1$as will be chosen small enough)

$$\frac{1}{2}(1-g_*)EI + \frac{\beta}{2} + \kappa\eta_8(l^8+l^4) + EI(1-g_*)k\hat{g}(\mathcal{F}) - \lambda EI\left(1-\frac{k}{2}\right) + (\mu+\gamma)k < 0.$$

Adding and substraction the term $\frac{\delta}{2}\left(g_* - k\hat{g}(\mathcal{F}) - \frac{1}{2} - \varepsilon\right)\left(1 - \frac{k}{2}\right) EI$, the previous term becomes

$$\left[\left(1-\frac{\delta}{2}\right)\left(g_* - k\hat{g}(\mathcal{F}) - \frac{1}{2} - \varepsilon\right) EI \left(1-\frac{k}{2}\right) - \mu k\right]$$
$$+ \left[\frac{\delta}{2}\left(g_* - k\hat{g}(\mathcal{F}) - \frac{1}{2} - \varepsilon\right)\left(1-\frac{k}{2}\right) EI\right.$$
$$\left. -\frac{1}{2}\left(1-g_*\right) EI - \frac{\beta}{2} - \left(1-g_*\right) k\hat{g}(\mathcal{F}_n) - \gamma k\right] > 0.$$

This term is divided into parts.

For the second part, we need

$$\frac{1}{2}\left(1-g_*\right) EI + \frac{\beta}{2} + EI\left(1-g_*\right) k\hat{g}(\mathcal{F}_n) + \gamma k < \frac{\delta}{2}\left(g_* - k\hat{g}(\mathcal{F}) - \frac{1}{2} - \varepsilon\right)\left(1-\frac{k}{2}\right) EI,$$

where

$$\delta = \frac{2\left(1-g_*\right) EI(\frac{2k}{3}+1) + 2\beta + 4\gamma k}{\left(g_* - \frac{k}{3} - \frac{1}{2}\right)\left(2-k\right)} EI,$$

and this relation holds when $\hat{g}(\mathcal{F}_g) < 1/3$. For the first part, we need

$$\mu < \left(1-\frac{\delta}{2}\right) \frac{\left(g_* - k\hat{g}(\mathcal{F}_n) - \frac{1}{2} - \varepsilon\right)}{k} \left(1-\frac{k}{2}\right) EI$$

and taking into account that the coefficient of term $\int_0^t g(t-s)\left\|w_{xx}(s)\right\|^2 ds$ must be negative. We choose μ such that

$$\frac{1}{2}\left(g_* - k\hat{g}(\mathcal{F}) - \frac{1}{2} - \varepsilon\right)\frac{EI}{2} + \frac{k}{2}\left(1-g_*\right) EI + 5\rho v^2 l^2 + l^3 < \mu$$

$$\mu < \left(1+\frac{\delta}{2}\right) \frac{\left(g_* - k\hat{g}(\mathcal{F}_n) - \frac{1}{2} - \varepsilon\right)}{k} \left(1-\frac{k}{2}\right) EI. \tag{3.41}$$

Note that (ε is ignored)

$$\left(1-\frac{\delta}{2}\right) \frac{\left(g_* k\hat{g}(\mathcal{F}_n) - \frac{1}{2} - \varepsilon\right)}{k} \left(1-\frac{k}{2}\right) EI =$$
$$\frac{\left(g_* - \frac{k}{3} - \frac{1}{2}\right)\left(2-k\right) - \left(1-g_*\right)EI(\frac{-2k}{3}+1) - 2\gamma k - \beta)}{2k}.$$

After simplification by replacing δ and neglecting ε, the choice of μ in the relation (3.41) is possible if

$$1 < \frac{(g_* - \frac{k}{3} - \frac{1}{2})\,(2-k) - (1-g_*)EI(\frac{-2k}{3}+1) - 2\gamma k - \beta)}{2k}, \tag{3.42}$$

that is when

$$g_* > \frac{2k\gamma + 2\beta + (\frac{k}{3}+\frac{3}{2})(2-k) + EI(2\frac{k}{3}+1)}{(2-k) + (2\frac{K}{3}+1)EI}. \tag{3.43}$$

Now, is the choice (3.43) possible?

We know that $g_* < k$ and to make (3.43) possible, we must have

$$\frac{2k\gamma + 2\beta + (\frac{k}{3}+\frac{3}{2})(2-k) + EI(2\frac{k}{3}+1)}{(2-k) + (2\frac{K}{3}+1)EI} < g_* < k. \tag{3.44}$$

The previous relation holds provided that $\mathcal{E}\,(0)$ is so small that

$$\beta < \frac{(2-k)(\frac{1}{2}-\frac{k}{3}) - \gamma k}{2}.$$

Finally, we choose $\gamma > \mu v$. These choices together with (3.41) imply that there exists a positive constant α_1 such that

$$L'(t) + \alpha_1 \mathcal{E}(t) \leq 0, \quad t \geq t_*.$$

This completes the proof of the second assertion. □

Theorem 3.1. *Under assumptions **(A1)-(A3)** and if*

$$v^2 < \frac{EI(1-k) + \frac{\kappa}{2}\left[\frac{l^4}{2} - \frac{a^2 l^2}{2\pi} - \frac{l^4 \ln \|w\|^2}{2} + l^4(1+\ln a)\right]}{\rho l^2},$$

is sufficiently small, then there exist two positive constants A and ϱ such that

$$E(t) \leq A e^{-\varrho t}, \quad t \geq 0.$$

Proof. In view of the equivalence result (3.5), we see that

$$\frac{d}{dt}L(t) \leq -\frac{\alpha_1}{C}L(t), \quad t \geq \bar{t}. \tag{3.45}$$

An integration of (3.45) over $(\bar{t}, t)$ conducts to

$$L(t) \leq L(\bar{t})e^{-\frac{\alpha_1}{C}(t-\bar{t})}, \quad t \geq \bar{t}.$$

Therefore the main result follows again by virtue of (3.5). □

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