

PROPERTIES OF FIBONACCI AND LUCAS SEQUENCES VIA FIBONACCI LIKE MATRICES

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ABSTRACT. Some interesting identities of Fibonacci and Lucas Sequences by using Fibonacci like Matrices are discussed.

1. INTRODUCTION

The Fibonacci Matrix Q , given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

satisfies the relation,

$$(0.1) \quad Q^2 = Q + I_2,$$

where I_2 is 2x2 unit matrix. The matrix Q also occurs on page 362 of [2]. This relation further implies that

$$(0.2) \quad Q^{n+2} = Q^{n+1} + Q^n, n \geq 0,$$

where Q^0 we mean the unit matrix I_2 of order 2 and n is a non-negative integer. This property helps us to introduce the following definitions.

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Definition 1.1. A matrix M of order n is said to be a Fibonacci like matrix (FLM), if the relation

$$(0.3) \quad M^2 = M + I_n$$

holds, where I_n the unit matrix of order n .

Definition 1.2. A sequence A_n of $n \times n$ matrices is said to be a Fibonacci like matrix sequence (FLMS) if the following relation holds:

$$(0.4) \quad A_{n+2} = A_{n+1} + A_n, \quad n \geq 0,$$

where A_0 and A_1 are known matrices.

For example the sequence Q_n defined by $Q_n = Q^n$, $n \geq 0$ is an FLMS where, Q is as in (0.1) here Q_0 is and is Q itself. Before giving a method of constructing such matrices and sequences and discussing their properties we require more definitions.

Definition 1.3. We shall call a sequence G_n of real or complex numbers, Fibonacci like sequence (FLS) if the relation

$$G_{n+2} = G_{n+1} + G_n, \quad n \geq 0, \quad G_0 = x_0, G_1 = x_1$$

holds where x_0 and x_1 , are known numbers. For example the famous Fibonacci sequence F_n and Lucas sequence L_n defined by the relation

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0, \quad F_0 = 0, F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad n \geq 0, \quad L_0 = 2, L_1 = 1$$

are both FLS's. Similarly, the sequences α^n and β^n are both FLS's where the numbers α and β are the positive and negative roots of the equation

$$x^2 - x - 1 = 0.$$

Remark 1.1. The sequences F_n and L_n can be extended to the left of F_0 and L_0 respectively and observed that

$$F_{-n} = (-1)^{n+1} \cdot F_n \quad \text{and} \quad L_{-n} = (-1)^n L_n.$$

For example, $F_{-1} = 1$, $F_{-2} = -1$ and $L_{-1} = -1$, $L_{-2} = 3$, etc.

Similarly, for a matrix sequence A_n by $A - n$ we mean the inverse of an A_n and it is also equal to $A^n - 1$. There are more than 200 identities related to the sequence F_n and L_n proved by different authors and a list of which up to the year 2001 is given in chapter 5 of [2].

In the next section, we shall discuss a few of them and show that the method of deriving these with the help of FLM and FLMS is also worth considerable and comparatively easier to understand.

2. FIBONACCI LIKE MATRICES AND SEQUENCES OF ORDER 2

In order to construct a Fibonacci like matrix [FLM] of order 2, we begin with the general 2×2 non-zero matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c and d are any four numbers real or complex. We shall determine the numbers a, b, c and d such that the following relations holds

$$(1.1) \quad M^2 = M + I_2.$$

After determining M^2 and $M + I_2$ this relation lead to

$$(1.2) \quad M = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} = M + I_2 = \begin{bmatrix} a + 1 & b \\ c & d + 1 \end{bmatrix}$$

Using the equality of two matrices, we then get the following relation

$$(1.3) \quad a^2 + bc = a + 1, \quad b(a + d) = b,$$

$$(1.4) \quad c = c(a + d), \quad bc + d^2 = d + 1.$$

Let $c \neq b \neq 0$ then using this and (1.3) and (1.4), we have

$$d = 1 - a, \quad bc = 1 + a - a^2,$$

i.e.

$$c = \frac{(1 + a - a^2)}{b}.$$

Using this relation we observe that the matrix M depends only on two parameters a and b , hence we denote it by Symbol $M(a,b)$. It takes the form

$$(1.5) \quad M(a, b) = \begin{bmatrix} a, & b \\ \frac{(1+a-a^2)}{b}, & 1-a \end{bmatrix}, \quad b \neq 0.$$

When, $a = b = 1$, $M(a, b)$ coincides with the matrix Q mentioned in the beginning. Further, it is easy to verify that $\text{Det}(M(a, b)) = -1$, clearly this matrix satisfies the relation (1.1) with M replaced by $M(a, b)$. Hence, we have

$$(1.6) \quad \left[M^{n+2}(a, b) = M^{n+1}(a, b) + (a, b), n \geq 0 \right],$$

where by $M^0(a, b)$ are mean the Unit Matrix I_2 . Using (1.5), (1.6) and the definition of F_n we observe that,

$$M^1(a, b) = \begin{bmatrix} aF_1 + F_0, & bF_1 \\ \frac{(1+a-a^2)}{b}, & F_2 - aF_1 \end{bmatrix}$$

and

$$\begin{aligned} M^2(a, b) &= \begin{bmatrix} a, & b \\ \frac{(1+a-a^2)}{b}, & 1-a \end{bmatrix} + \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+a, & b \\ (1+a-a^2)b, & 2-a \end{bmatrix} = \begin{bmatrix} aF_2 + F_1, & bF_2 \\ F_2 \frac{(1+a-a^2)}{b}, & F_3 - aF_2 \end{bmatrix}. \end{aligned}$$

From these observation and the use of the principle of mathematical induction (PMI) it is easy to prove the result

$$(1.7) \quad M^n(a, b) = \begin{bmatrix} aF_n + F_{n-1}, & bF_n \\ F_n \frac{(1+a-a^2)}{b}, & F_{n+1} - aF_n \end{bmatrix}, \quad n \geq 0.$$

Further using PMI on m and n it is easy to prove the identity

$$(1.8) \quad M^{m+n}(a, b) = M^m(a, b) \times M^n(a, b), \quad m, n \geq 0.$$

Remark 2.1. Let the notation M_n stand for $M^n(a, b)$, $n \geq 0$. Then the identities (1.6) and (1.8) can be expressed in the form

$$(1.9) \quad M_{n+2} = M_{n+1} + M_n \text{ and } M_{m+n} = M_m \cdot M_n, \quad m, n \geq 0,$$

respectively, where M_m, M_n stands for the matrix product $(M_m \times M_n)$.

We now prove the theorem related to the matrix sequence M_n .

Theorem 2.1. Let $M_n = M^n(a, b)$, $n \geq 0$ where a, b are any two numbers then the following properties hold:

- (P_1) the sequence M_n of matrices is a FLMS.
- (P_2) $\sum_{k=0}^n M_k = M_{n+2} - M_1$
- (P_3) $\sum_{k=1}^n M_{2k-1} = M_{2n} - I_2$
- (P_4) $\sum_{k=1}^n M_{2k} = \sum_{k=1}^n M_k^2 = M_{2n+1} - M_1$
- (P_5) Cassini's Type Formula $M_{n+1}, M_{n-1} - M^n = 0$ (zero matrix), $n \geq 0$.

Proof. The properties (P_1) and (P_5) are direct consequences of Remark 1.1 and the definition of FLMS. The property P_2 can be proved directly or by using PMI. The property (P_3) and (P_4) follow by the use of PMI. The proof of the theorem is complete. $\square$

We now obtain some properties of the sequence F_n in the form of a theorem

Theorem 2.2. Let F_n be the sequence defined by the relation (0.6) then the following properties hold:

- (i) Cassini's formula $F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n$, $n \geq 1$.
- (ii)

$$(1.10) \quad F_{m+n} = F_n \cdot F_{m-1} + F_m \cdot F_{n+1}$$

- (iii) Convolution property

$$(2.1) \quad F_n = F_n - m \cdot F_{m-1} + F_m \cdot F_{n-m+1}$$

- (iv) $F_{r+s+t} = F_{r+1} \cdot F_{s+1} \cdot F_{t+1} + F_r \cdot F_s \cdot F_t - F_{r-1} \cdot F_{s-1} \cdot F_{t-1}$, $r, s, t \geq 0$

- (v) $\sum_{k=1}^n F_k = F_{n+2} - F_1$, $n \geq 1$

- (vi)

$$(1.11) \quad (a) \sum_{k=1}^n F_{2k-1} = F_{2n} \quad (b) \sum_{k=1}^n F_{2k} = F_{2n+1} - 1, \quad n \geq 1.$$

- (vii)

$$(1.12) \quad (a) F_{n+1}^2 + F_n^2 = F_{2n+1} \quad (b) F_{n+1}^2 - F_{n-1}^2 = F_{2n}, \quad n \geq 1.$$

Proof. The proof of the (i) follows by taking the determinants of both side (1.7). And using the fact that the determinants of M_1 is -1 and that $M_n = M^n$. For the

proof of (ii) we take $a=\alpha$ and $b=1$ in (1.7) and use the identity

$$(1.13) \quad M_{m+n} = M_m \times M_n$$

we then get

$$(1.14) \quad M_{m+n} = \begin{bmatrix} aF_{m+n} + F_{m+n-1}, & F_{m+n} \\ 0, & F_{m+n+1} - \alpha F_{m+n} \end{bmatrix} = \begin{bmatrix} aF_m + F_{m-1}, & F_m \\ 0, & F_{m+1} - \alpha F_m \end{bmatrix}.$$

Equating the 1, 2 of both sides of this identity, we get

$$F_{m+n} = (\alpha F_m + F_{m-1})F_n + F_m(F_{n+1} - \alpha F_n).$$

Cancelling two terms in this result we get the desired result (ii).

Similarly, the property (iii) can be proved by using the identity

$$M_n = M_m \times M_{n-m}$$

and taking $a = \alpha$ and $b=1$ in (1.7). Using $m= r+s$ and $n=t$ in (ii) we get the property (iv) after a few calculation. it can also be proved by using the identity $M_{r+s+t} = M_t \times M_{r+s}$ as in the case of property (ii) but the proof is lengthy. The proof of the property (v) follows by the 1,2 entries in the property (P_2) of theorem 1.1 and equating the coefficients of b . Taking $a = b = 1$ in (1.7) and using in the properties (P_3) and (P_4), we get the property (vi). For the proof of (vii) we use (1.7) in the identity

$$M_n^2 = M_n \times M_n$$

and get

$$(1.15) \quad \begin{bmatrix} aF_m + F_{n-1}, & bF_n \\ F_n \frac{(1+a-a^2)}{b}, & F_{n+1} - \alpha F_n \end{bmatrix} \times \begin{bmatrix} aF_n + F_{n-1}, & bF_n \\ F_n \frac{(1+a-a^2)}{b}, & F_{n+1} - \alpha F_n \end{bmatrix} \\ = \begin{bmatrix} aF_{2n} + F_{2n-1}, & bF_{2n} \\ \frac{H\alpha-a^2}{b}F_{2n}, & F_{2n+1} - \alpha F_{2n} \end{bmatrix}, \quad n \geq 1.$$

Multiplying the two matrices on the left side of this identity and equating 2, 2 entries on both sides, we get

$$(1 + a - a^2)F_n^2 t + F_{n+1}^2 - 2a.F_n.F_{n+1} - F_n^2 = F_{2n+1} - aF_{2n}.$$

Simplifying this further we get

$$F_{n+1}^2 + F_n^2 - a(2F_n \cdot F_{n+1} - F_n^2) = F_{2n+1} - aF_{2n}.$$

This is a linear identity in a and is true for all values of a . Hence, equating the constant terms of both sides of this equation, we get (a) part of (vii). For (b) part, we have to equate the coefficients of a on both sides of the above linear identities and also the relation

$$F_n = F_{n+1} - F_{n-1}.$$

□

Remark 2.2.

- (i) *The identities mentioned in the above theorem occur in the chapter 5 and 32 of [2]. For example, the properties 1,2 and 5 occur on page 363, 364 and 365 (chapter 32) of [2] respectively.*
- (ii) *Since the parameters a and b are arbitrary, we conclude that there are infinitely many FLM's and corresponding FLMS's. A particular case of such FLMS is the Q_n mentioned in the first section and is obtained by taking $a = b = 1$ in the definition of M_n .*

3. THE FLMS M_α AND M_β

In the matrix $M(a,b)$ discussed in the previous section $a = \alpha, b = 1$ we get a matrix $M_\alpha, 1$. Let us denote it by M_α . Similarly we denote the matrix $M(\beta, -1)$ by M_β where α, β are the roots of $x^2 - 1 = 0$. It is observed that

$$(2.1) \quad M(\alpha) = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix} \quad \text{and} \quad M(\beta) = \begin{bmatrix} \beta & -1 \\ 0 & \alpha \end{bmatrix}.$$

With this change the result (1.7) takes the following forms

$$(2.2) \quad M(\alpha) = \begin{bmatrix} \alpha F_n + F_{n-1} & F_n \\ 0 & F_{n+1} - \alpha F_n \end{bmatrix},$$

$$M(\beta) = \begin{bmatrix} \beta F_n + F_{n-1} & -F_n \\ 0 & F_{n+1} - \beta F_n \end{bmatrix}.$$

Further from (2.1) and using PMI it is also easy to show that

$$(2.3) \quad M(\alpha) = \begin{bmatrix} \alpha^n & F_n \\ 0 & \beta^n \end{bmatrix} \quad \text{and} \quad M^n(\beta) = \begin{bmatrix} \beta^n & F_n \\ 0 & \alpha^n \end{bmatrix}.$$

Using (2.2) and (2.3) and the equality of two matrices, we conclude that

$$(2.4) \quad \alpha^n = \alpha F_n + F_{n-1} = F_{n+1} - \beta F_n \quad \text{and} \quad \beta^n = \beta F_n + F_{n-1} - \alpha F_n$$

These identities are given on page 78 of [2] in the form of Lemma 5.1 and corollary (5.3), respectively. Further using (2.3) and $M^{m+n}(\alpha) = M^m(\alpha) \times M^n(\alpha)$, we get

$$\begin{bmatrix} \alpha^{m+n} & F_{m+n} \\ 0 & \beta^{m+n} \end{bmatrix} = \begin{bmatrix} \alpha^m & F_m \\ 0 & \beta^m \end{bmatrix} \times \begin{bmatrix} \alpha^n & F_n \\ 0 & \beta^n \end{bmatrix}.$$

Equating 1,2 terms on both sides of this identity, we get

$$F_{m+n} = \alpha^m \cdot F_n + \beta^n \cdot F_m.$$

Interchanging the role m and n in this identity, we get

$$F_{m+n} = \alpha^n F_m + \alpha^m F_n.$$

Adding these identities and using

$$L_n = \alpha^n + \beta^n$$

we get $F_{m+n} = \frac{L_m F_n + L_n F_m}{2}$ and taking $m=n$ in this identity we get, $F_{2n} = L_n F_n$.

Further adding the proper parts in the identities (2.4) and using the relation

$$L_n = \alpha^n + \beta^n$$

we get

$$L_n = F_{n+1} + F_{n-1}$$

Again multiplying the two results in (2.4) and using $\alpha\beta = -1$, we get the following results:

$$(2.5) \quad \begin{aligned} (-1)^n &= \alpha^n \times \beta^n = [\alpha F_n + F_{n-1}] \cdot [F_{n+1} - \alpha F_n] \\ (-1)^n &= \alpha F_n \cdot F_{n+1} \cdot F_{n-1} - \alpha F_{n-1} \cdot F_n - \alpha^2 F_n^2 \\ (-1)^n &= \alpha F_n (F_{n+1} - F_{n-1}) + F_{n+1} \cdot F_{n-1} - \alpha^2 \cdot F_n^2 \\ (-1)^n &= \alpha F_n (F_{n+1} - F_{n-1}) + F_{n+1} \cdot F_{n-1} - \alpha^2 \cdot F_n^2 \end{aligned}$$

$$\begin{aligned}(-1)^n &= \alpha F_n^2 + F_{n+1} \cdot f_{n-1} - (\alpha + 1) \cdot F_n^2 \\(-1)^n &= F_{n+1} \cdot F_{n-1} - F_n^2.\end{aligned}$$

This result is known as cassini's formula

Remark 3.1. Among number sequences can be observed that the sequences α^n and β^n are the only two types of FLMS. It is simple to show that characteristic equation $M(a, b)$ is $x^2 - x - 1$, this is independent of the parameters a and b , hence this equation also coincides with the characteristics equation of the matrix Q given by (0.1) and also for the matrices $M(\alpha)$ and $M(\beta)$

4. CONCLUSION

With the help of the definition of Fibonacci Like Matrix (FLM) of n th order and corresponding Fibonacci Like Matrix Sequence (FLMS), we construct different FLM and FLMS of order two and apply them in deriving several known properties of Fibonacci and Lucas Sequences.

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