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VARIATIONAL ANALYSIS OF AN ELASTIC-THERMO-VISCOPLASTIC CONTACT PROBLEM WITH NORMAL DAMPED RESPONSE

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ABSTRACT. We consider a quasistatic frictional contact problem between an elastic-viscoplastic body and an obstacle. The contact is modelled with normal damped response and a local friction law. The material is elastic-viscoplastic with two internal variables which may describe a temperature parameter and the damage of the contacting surface. We provide a variational formulation of the problem and prove the existence of a unique weak solution to the model. The proof is based on arguments of evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed point.

1. INTRODUCTION

In this work, we deal with a model for the frictional contact between an elastic viscoplastic body and a reactive obstacle, the so-called foundation. Phenomena of frictional contact between deformable bodies abound in the industry and everyday life. A major problem related to the modelling of the contact phenomena which, currently, is still under investigation, is the choice of the contact boundary conditions. One of the most popular boundary conditions, used both in the engineering

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and the mathematical literature is the normal compliance condition. It was used in a large number of papers, see [8] and the references therein. Note that these contact conditions, used in most of the related works on the subject, are formulated in terms of the displacement field. However, when the contact surfaces are lubricated, normal compliance conditions expressed in terms of the normal velocity seem to be more appropriate. Such kind of conditions, called normal damped response conditions, have been used in various papers (see, e.g., [4, 9, 13, 20] and the references therein).

In the current paper, we assume that the foundation is deformable and that there exists a thin lubricant layer located on the contact boundary between the two bodies. Then, a normal damped response contact condition is considered and the associated frictional law is also included.

The constitutive laws which utilize internal variables to characterize the changing state of a material during a deformation process have been proposed by several investigators in recent years. The temperature is one of these internal state variables, considered by many authors, we can see [3, 9, 15].

The damage is another internal state variable, it is an extremely important topic in design engineering, since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. General models of mechanical damage, derived from thermodynamical considerations and the virtual power principle, were introduced in [14]. Contact problems, involving viscoelastic and viscoplastic materials and including the effect due to the damage, were studied in [2, 8, 15].

Various works in the study of elastic-viscoplastic materials can be found in [2,13, 19] and the references therein. Both quasistatic and dynamic problems involving elastic-viscoplastic materials with thermal effect have studied in recent papers. For such kinds of materials, the wear of the contacting surface was included in [10], the contact was modeled with normal compliance in [6], the adhesion field was considered in [1], the mathematical problem modeled with Tresca's friction law was studied in [21]. Then, in this paper we continue in this line of research, where we extend a part of these results to more contact conditions for elastic-thermo-viscoplastic materials.

The aim of this paper is to study a quasistatic frictional contact problem for general elastic-thermo-viscoplastic materials. For this, we consider a rate-type constitutive equation with two internal variables of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(t))\big) + \mathcal{B}\big(\varepsilon(\mathbf{u}(t))\big) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}(s)), \theta(s), \beta(s)\Big) ds,$$

in which $\mathbf{u}$, $\boldsymbol{\sigma}$ represent, respectively, the displacement field and the stress field, θ represents the temperature, β is the damage field, $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the viscoplastic behavior of the material.

In this paper the differential inclusion for the evolution of the damage field is

$$\dot{\beta} - k_1 \Delta\beta + \partial\varphi_K(\beta) \ni \phi\big(\sigma, \varepsilon(\mathbf{u}), \theta, \beta\big),$$

where *K* denotes the set of admissible damage functions defined by

$$K = \{\xi \in V : 0 \leq \xi(x) \leq 1 \text{ a.e. } x \in \Omega\},$$

k_1 represents the damage diffusion constant, assumed positive, φ_K is the indicator function of the set *K* and $\partial\varphi_K$ represents its subdifferential. ϕ is a given constitutive function which describes the sources of the damage in the system which results from tension or compression.

The evolution of the temperature field is governed by the heat equation, obtained from the conservation of energy and defined by a differential equation for the temperature of the form $\dot{\theta} - k_0 \Delta\theta = \psi\big(\boldsymbol{\sigma}, \varepsilon(\dot{\mathbf{u}}), \theta, \beta\big) + q$, where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source.

The rest of the paper is structured as follows. In Section 2 we present the notation we shall we use as well as some preliminaries. In Section 3 we present the mechanical problem, list the assumptions on the data, give the variational formulation of the problem and state our main existence and uniqueness result, Theorem 3.1. In section 4 we give the proof of Theorem 3.1 based on arguments of evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point.

2. Notations and Preliminaries

In this section we present the notation and some preliminary material which will be used in the next sections. For further details, we refer reader to [11, 17].

We denote by $\mathbb{S}_d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$ (d = 2, 3), while " $\cdot$ " and $|\cdot|$ represent the inner product and Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}_d$, respectively.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Here and throughout this paper, the indices i and j run from 1 to d, the summation convention over repeated indices is adopted and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. Next, we use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) \,|\, u_i \in L^2(\Omega)\right\}, \qquad H_1 = \{\mathbf{u} \in H \,|\, \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$
$$\mathcal{H} = \left\{\boldsymbol{\tau} = (\tau_{ij}) \,|\, \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\}, \qquad \mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} \,|\, \tau_{ij,j} \in H\}.$$

These are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
(\mathbf{u}, \mathbf{v})_H &= \int_\Omega u_i v_i \, dx \quad \forall \mathbf{u}, \mathbf{v} \in H, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} \, dx \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}, \\
(\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \in H_1, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathrm{Div}\, \boldsymbol{\tau})_H, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.
\end{aligned}$$

Here $\varepsilon : H_1 \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The associated norms on the spaces $H, \mathcal{H}, H_1 \, and \, \mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively. For every element $\mathbf{u} \in H_1$ we denote by $\mathbf{u}$ its trace on Γ and by $\mathbf{u}_\nu$ and $\mathbf{u}_\tau$ its normal and the tangential components on Γ given by

$$\mathbf{u}_\nu = \mathbf{u} \cdot \boldsymbol{\nu}, \qquad \mathbf{u}_\tau = \mathbf{u} - \mathbf{u}_\nu \boldsymbol{\nu}. \tag{2.1}$$

Similarly, for a tensor field $\boldsymbol{\sigma} : \Omega \to S_d$ we denote by $\boldsymbol{\sigma}_\nu$ and $\boldsymbol{\sigma}_\tau$ its normal and tangential components on $\mathcal{H}_1$, and when $\boldsymbol{\sigma}$ is a regular function (say C^1) then

(2.2) $$\boldsymbol{\sigma}_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{\sigma}_\nu \boldsymbol{\nu}.$$

We recall that the following Green's formula holds

(2.3) $$(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \mathbf{u})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{u} \, da \quad \forall \mathbf{u} \in H_1.$$

We recall the following standard result for parabolic variational inequalities which may be found in [5, p.124] and which will be used in Section 4 of this paper.

Theorem 2.1. *Let $V \subset H \subset V'$ be a Gelfand triple, Let K be a nonempty closed, and convex set of V, and let $a : (.,.) : V \times V \to \mathbb{R}$ be a continuous and symmetric bilinear form which satisfies*

$$a(\mathbf{v}, \mathbf{v}) + c_0 \, |\mathbf{v}|_H^2 \geq \kappa \, |\mathbf{v}|_V^2 \quad \forall \mathbf{v} \in V$$

for some constants κ and c_0. Then, for every $\mathbf{u}_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $\mathbf{u} \in L^2(0,T;V) \cap H^1(0,T;H)$ which satisfies

$$\begin{aligned}
&\mathbf{u}(t) \in K \quad \forall t \in (0,T),\\
&(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{V' \times V} + a\big(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)\big) \geq \big(f(t), \mathbf{v} - \mathbf{u}(t)\big)_H\\
&\forall \mathbf{v} \in K, \ a.e. \ t \in (0,T),\\
&\mathbf{u}(0) = \mathbf{u}_0.
\end{aligned}$$

We conclude this section with a fixed point result which is a consequence of the Bannch contraction principle and which will be used in Section 4 of this paper, its proof can be found in [16].

Lemma 2.1. *Given a Bannach space X with a norm $|\,.\,|_X$ and $T > 0$, let $\Lambda : L^2(0,T;X) \to L^2(0,T;X)$ be an operator such that*

$$|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)|_X^2 \leq C \int_0^t |\eta_1(s) - \eta_2(s)|_X^2 \, ds,$$

for every $\eta_1, \eta_2 \in L^2(0,T;X)$, a.e. $t \in (0,T)$ with a constant $C > 0$. Then Λ has a unique fixed point in $L^2(0,T;X)$, in other words, there exists a unique $\eta^* \in L^2(0,T;X)$ such that $\Lambda\eta^* = \eta^*$.

3. Problem Statement and Variational Formulation

In this section we describe the model for the process, present its variational formulation. The physical setting is as follows.

An elastic-thermo-viscoplastic body occupies a domain Ω with a regular surface $\Gamma,$ and let Γ_1, Γ_2, Γ_3 be a partition of Γ into three disjoint measurable parts such that $meas\,(\Gamma_1) > 0.$ Let $T > 0$ and let $[0, T]$ denote the time interval considered. We assume that the body is fixed on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. Surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$ and a volume force of density f_0 acts on $\Omega \times (0, T)$. We admit a possible external heat source applied in $\Omega \times (0, T)$, given by the function q. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach of the process. Finally, the body is in contact with a reactive foundation over the potential contact surface Γ_3.

We assume that the normal stress $\boldsymbol{\sigma}_\nu$ satisfies a general normal damped response condition

$$-\boldsymbol{\sigma}_\nu = p_\nu(\dot{\mathbf{u}}_\nu), \tag{3.1}$$

where $\dot{\mathbf{u}}_\nu$ represents the normal velocity and p_ν is a prescribed function.

Equality (3.1) states a general dependence of the normal stress on the normal velocity. In the case when

$$p_\nu(r) = kr, \tag{3.2}$$

with k $\geq$ 0, the resistance of the foundation to penetration is proportional to the normal velocity. This type of behavior was considered in [20] when modeling the motion of a deformable body on sand or a granular material.

The contact is frictional and the associated friction law is chosen as follows

$$-\boldsymbol{\sigma}_\tau = p_\tau(\dot{\mathbf{u}}_\tau), \tag{3.3}$$

where $\boldsymbol{\sigma}_\tau$ represents the tangential force on the contact boundary, $\dot{\mathbf{u}}_\tau$ denotes the tangential velocity and p_τ is a prescribed vector-valued function.

We use a thermo-elastic-viscoplastic constitutive law with damage to model the material's behavior, and a differential inclusion of parabolic type to describe the evolution of the damage.

With the assumptions above, the mechanical problem of the quasistatic contact with normal damped response may be formulated classically as follows.

Problem P Find the displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$, the damage field $\beta : \Omega \times [0,T] \to \mathbb{R}$ and the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned}(3.4)\qquad \boldsymbol{\sigma}(t) &= \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(t))\big) + \mathcal{B}\big(\varepsilon(\mathbf{u}(t))\big)\\ &\quad + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}(s)), \theta(s), \beta(s)\Big) ds, \quad \text{in } \Omega \times (0,T),\end{aligned}$$

$$(3.5)\qquad \operatorname{Div} \boldsymbol{\sigma} + f_0 = 0, \quad \text{in } \Omega \times (0,T),$$

$$(3.6)\qquad \dot{\theta} - k_0 \Delta\theta = \psi\big(\boldsymbol{\sigma}, \varepsilon(\dot{\mathbf{u}}), \theta, \beta\big) + q, \quad \text{in } \Omega \times (0,T),$$

$$(3.7)\qquad \dot{\beta} - k_1 \Delta\beta + \partial\varphi_K(\beta) \ni \phi\big(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \theta, \beta\big), \quad \text{in } \Omega \times (0,T),$$

$$(3.8)\qquad \mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T),$$

$$(3.9)\qquad \boldsymbol{\sigma}\nu = f_2, \quad \text{on } \Gamma_2 \times (0,T),$$

$$(3.10)\qquad -\boldsymbol{\sigma}_\nu = p_\nu(\dot{\mathbf{u}}_\nu), \quad -\boldsymbol{\sigma}_\tau = p_\tau(\dot{\mathbf{u}}_\tau), \quad \text{on } \Gamma_3 \times (0,T),$$

$$(3.11)\qquad k_0 \frac{\partial\theta}{\partial\nu} + B\theta = 0, \quad \text{on } \Gamma \times (0,T),$$

$$(3.12)\qquad \frac{\partial\beta}{\partial\nu} = 0, \quad \text{on } \Gamma \times (0,T),$$

$$(3.13)\qquad \mathbf{u}(0) = \mathbf{u}_0, \qquad \beta(0) = \beta_0, \qquad \theta(0) = \theta_0, \quad \text{in } \Omega.$$

This problem represents the quasistatic evolution of damage in thermo-elastic-viscoplastic materials. Equation (3.4) represents the thermo-elastic-viscoplastic constitutive law, already introduced in the first section. The relation (3.5) is the equilibrium equation, we use it here since we assume that process is quasistatic. Equation (3.6) represents the energy conservation where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source. Inclusion (3.7) describes the evolution of damage field, governed by the source damage function ϕ, where $\partial\varphi_K$ is the subdifferential of indicator function of the set K of admissible damage functions.

Conditions (3.8) and (3.9) are the displacement and traction boundary conditions, respectively. (3.10) represents the normal damped response condition and its associated friction law, described above on the potential contact surface Γ_3. Equation (3.11) represents a Fourier boundary condition for the temperature on Γ. Equation (3.12) represents an homogeneous Newmann boundary condition for the damage field on Γ, where $\frac{\partial\beta}{\partial\nu}$ is the normal derivative of β. Finally the functions $\mathbf{u}_0$, β_0 and θ_0 in (3.13) are the initial data.

To obtain the variational formulation of the problem (3.4)–(3.13), we need additional notation.

Thus, let V denote the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 \,|\, \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) \geq 0$ and Γ is Lipschitz, Korn's inequality holds that there exists a posetive constant C_k which depends only on Ω and Γ_1 such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}} \geq C_k \, |\mathbf{v}|_{H_1}, \quad \forall \, \mathbf{v} \in V. \tag{3.14}$$

The proof of this inequality may be found in [18, p. 79].

We consider on V the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \, \mathbf{u}, \mathbf{v} \in V, \tag{3.15}$$

$$|\, \mathbf{v} \,|_V = |\varepsilon(\mathbf{v})|_{\mathcal{H}}, \quad \forall \, \mathbf{v} \in V. \tag{3.16}$$

It follows from (3.14) and (3.15) that $|\,.\,|_{H_1}$ and $|\,.\,|_V$ are equivalent norms on V and therefore $(V, \, |\,.\,|_V)$ is a real Hilbert space. We note that the assumption that Γ is Lipschitz continuous is sufficient for our purposes. First, it ensures that the outer normal $\boldsymbol{\nu}$ is defined a.e. on Γ, and then the normal and tangential components of various functions make sense. Second, it is sufficient for Korn's inequality (3.14) to hold true. Moreover, by the Sobolev trace theorem and (3.14), we have a posetive constant $C_0 > 0$ depending only on Ω, Γ_1, and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \leq C_0 \, |\mathbf{v}|_V, \quad \forall \mathbf{v} \in V. \tag{3.17}$$

In the study of the mechanical problem (3.4)–(3.13), we consider the following assumptions

The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

$$
(3.18) \qquad \begin{cases}
(a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)| \leq L_{\mathcal{A}} \, |\varepsilon_1 - \varepsilon_2| \, , \\
\qquad \forall \, \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega. \\
(b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \, |\varepsilon_1 - \varepsilon_2|^2 \, , \\
\qquad \forall \, \varepsilon_1, \varepsilon_2 \in S_d, \, a.e. \; x \in \Omega. \\
(c) \text{ The mapping } x \mapsto \mathcal{A}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \forall \, \varepsilon \in S_d. \\
(d) \text{ The mapping } x \mapsto \mathcal{A}(x, 0) \text{ belongs to } \mathcal{H}.
\end{cases}
$$

The elasticity operator $\mathcal{B} : \Omega \times S_d \to S_d$ satisfies

$$
(3.19) \qquad \begin{cases}
(a) \text{ There exists a constant } L_{\mathcal{B}} > 0 \text{ such that} \\
\qquad |\mathcal{B}(x, \varepsilon_1) - \mathcal{B}(x, \varepsilon_2)| \leq L_{\mathcal{B}}(|\varepsilon_1 - \varepsilon_2| \, , \\
\qquad \forall \, \varepsilon_1, \varepsilon_2 \in S_d, \, a.e. \; x \in \Omega. \\
(b) \text{ The mapping } x \mapsto \mathcal{B}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \forall \, \varepsilon \in S_d. \\
(c) \text{ The mapping } x \mapsto \mathcal{B}(x, 0,) \text{ belongs to } \mathcal{H}.
\end{cases}
$$

The plasticity operator $\mathcal{G} : \Omega \times S_d \times S_d \times \mathbb{R} \times \mathbb{R} \to S_d$ satisfies

$$
(3.20) \quad \begin{cases}
(a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\
\qquad |\mathcal{G}(x, \boldsymbol{\sigma}_1, \varepsilon_1, \theta_1, \beta_1) - \mathcal{G}(x, \boldsymbol{\sigma}_2, \varepsilon_2, \theta_2, \beta_2)| \leq L_{\mathcal{G}}(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| \\
\qquad + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\beta_1 - \beta_2|), \\
\qquad \forall \, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in S_d, \, \forall \, \varepsilon_1, \varepsilon_2 \in S_d, \, \forall \, \theta_1, \theta_2 \in \mathbb{R}, \, \forall \, \beta_1, \beta_2 \in \mathbb{R}, \; a.e. \; x \in \Omega. \\
(b) \text{ The mapping } x \mapsto \mathcal{G}(x, \sigma, \varepsilon, \theta, \beta) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \forall \, \boldsymbol{\sigma}, \varepsilon \in S_d, \, \forall \, \theta, \beta \in \mathbb{R}. \\
(c) \text{ The mapping } x \mapsto \mathcal{G}(x, 0, 0, 0, 0) \text{ belongs to } \mathcal{H}.
\end{cases}
$$

The nonlinear constitutive function $\psi : \Omega \times S_d \times S_d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\text{(3.21)} \quad \begin{cases} (a) \text{ There exists a constant } L_\psi > 0 \text{ such that} \\ \quad |\psi(x, \boldsymbol{\sigma}_1, \varepsilon_1, \theta_1, \beta_1) - \psi(x, \boldsymbol{\sigma}_2, \varepsilon_2, \theta_2, \beta_2)| \le L_\psi(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| \\ \quad + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\beta_1 - \beta_2|), \\ \quad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in S_d,\ \forall\, \varepsilon_1, \varepsilon_2 \in S_d,\ \forall\, \theta_1, \theta_2 \in \mathbb{R},\ \forall\, \beta_1, \beta_2 \in \mathbb{R},\ a.e.\ x \in \Omega. \\ (b) \text{ The mapping } x \mapsto \psi(x, \boldsymbol{\sigma}, \varepsilon, \theta, \beta) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall\, \boldsymbol{\sigma}, \varepsilon \in S_d,\ \forall\, \theta, \beta \in \mathbb{R}. \\ (c) \text{ The mapping } x \mapsto \psi(x, 0, 0, 0, 0) \text{ belongs to } \mathcal{H}. \end{cases}$$

The damage source function $\phi : \Omega \times S_d \times S_d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\text{(3.22)} \quad \begin{cases} (a) \text{ There exists a constant } L_\phi > 0 \text{ such that} \\ \quad |\phi(x, \boldsymbol{\sigma}_1, \varepsilon_1, \theta_1, \beta_1) - \phi(x, \boldsymbol{\sigma}_2, \varepsilon_2, \theta_2, \beta_2)| \le L_\phi(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| \\ \quad + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\beta_1 - \beta_2|), \\ \quad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in S_d,\ \forall\, \varepsilon_1, \varepsilon_2 \in S_d,\ \forall\, \theta_1, \theta_2 \in \mathbb{R},\ \forall\, \beta_1, \beta_2 \in \mathbb{R},\ a.e.\ x \in \Omega. \\ (b) \text{ The mapping } x \mapsto \phi(x, \boldsymbol{\sigma}, \varepsilon, \theta, \beta) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall\, \boldsymbol{\sigma}, \varepsilon \in S_d,\ \forall\, \theta, \beta \in \mathbb{R}. \\ (c) \text{ The mapping } x \mapsto \phi(x, 0, 0, 0, 0) \text{ belongs to } \mathcal{H}. \end{cases}$$

The normal contact function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\text{(3.23)} \quad \begin{cases} (a) \text{ There exists a constant } L_\nu > 0 \text{ such that} \\ \quad |p_\nu(x, r_1) - p_\nu(x, r_2)| \le L_\nu \,|r_1 - r_2|\,, \quad \forall\, r_1, r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3. \\ (b)\ (p_\nu(x, r_1) - p_\nu(x, r_2))(r_1 - r_2) \ge 0 \quad \forall\, r_1, r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3. \\ (c) \text{ The mapping } x \mapsto p_\nu(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \quad \forall\, r \in \mathbb{R}. \\ (d) \text{ The mapping } r \mapsto p_\nu(x, r) \text{ is continuous on } \mathbb{R},\ a.e.\ x \in \Gamma_3. \end{cases}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$(3.24) \quad \begin{cases} (a) \text{ There exists a constant } L_\tau > 0 \text{ such that} \\ \qquad |p_\tau(x,r_1) - p_\tau(x,r_2)| \le L_\tau\, |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}^d,\ a.e.\ x \in \Gamma_3. \\ (b)\ (p_\tau(x,r_1) - p_\tau(x,r_2)) \cdot (r_1 - r_2) \ge 0 \quad \forall\, r_1, r_2 \in \mathbb{R}^d,\ a.e.\ x \in \Gamma_3. \\ (c) \text{ The mapping } x \mapsto p_\tau(x,r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \qquad \forall\, r \in \mathbb{R}^d. \\ (d) \text{ The mapping } r \mapsto p_\tau(x,r) \text{ is continuous on } \mathbb{R}^d,\ a.e.\ x \in \Gamma_3. \\ (e)\ p_\tau(x,r).\nu(x) = 0\ \forall r \in \mathbb{R}^d \text{ such that } r.\nu(x) = 0,\ a.e.\ x \in \Gamma_3. \end{cases}$$

We suppose that the body forces and surface tractions satisfy

$$(3.25) \qquad f_0 \in L^2(0,T;H), \quad f_2 \in L^2(0,T;L^2(\Gamma_2)^d).$$

The volume heat source satisfies

$$(3.26) \qquad q \in L^2(0,T;L^2(\Omega)).$$

We also suppose that

$$(3.27) \qquad B > 0, \qquad k_i > 0 \quad (i = 0,1).$$

Finally we assume that the initial data satisfy the following conditions

$$(3.28) \qquad \mathbf{u}_0 \in V, \quad \theta_0 \in V, \quad \beta_0 \in K..$$

Next, we denote by $f(t)$ the element of V' given by

$$(3.29) \qquad (f(t), \mathbf{v})_{V' \times V} = \int_\Omega f_0(t).\mathbf{v}\, dx + \int_{\Gamma_2} f_2(t).\mathbf{v}\, da \quad \forall\, \mathbf{v} \in V.$$

We note that the conditions (3.25) imply

$$(3.30) \qquad f \in L^2(0,T;V').$$

We define the following bilinear forms

$$(3.31) \qquad a_0 : V \times V \to \mathbb{R}, \quad a_0(\zeta, \xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi\, dx + B \int_\Gamma \zeta\xi d\gamma,$$

$$(3.32) \qquad a_1 : V \times V \to \mathbb{R}, \quad a_1(\zeta, \xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi\, dx.$$

We consider the functional $j : V \times V \to \mathbb{R}$ defined by

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} \big(p_\nu(\mathbf{u}_\nu)\mathbf{v}_\nu + p_\tau(\mathbf{u}_\tau)\cdot\mathbf{v}_\tau\big)da, \quad \forall \mathbf{u},\mathbf{v}\in V. \tag{3.33}$$

Keeping in mind (3.23) and (3.24) we observe that the integral in (3.33) is well defined. Next, we assume that $\{\mathbf{u},\boldsymbol{\sigma}\}$ are regular functions satisfying (3.5)-(3.10) and let $\mathbf{v}\in V$, $t\in[0,T]$. Using Green's formula (2.3) and (3.5) we have

$$(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} = \int_\Omega f_0(t)\cdot\mathbf{v}\,dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\mathbf{v}\,da, \quad \forall \mathbf{v}\in V. \tag{3.34}$$

Applying the boundary conditions (3.8) and (3.9), we have

$$\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\mathbf{v}\,da = \int_{\Gamma_2} f_2(t)\cdot\mathbf{v}\,da + \int_{\Gamma_3}\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\mathbf{v}\,da. \tag{3.35}$$

It now follows from (3.34), (3.35) and (3.29) that

$$(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} = (f(t),\mathbf{v})_{V'\times V} + \int_{\Gamma_3}\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\mathbf{v}\,da, \quad \forall \mathbf{v}\in V. \tag{3.36}$$

On the other hand, from (2.1), (2.2) and (3.10) we obtain

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\mathbf{v} = -p_\nu(\dot{\mathbf{u}}_\nu(t))\mathbf{v}_\nu - p_\tau(\dot{\mathbf{u}}_\tau(t))\cdot\mathbf{v}_\tau, \quad \text{ on } \Gamma_3. \tag{3.37}$$

Finally from (3.33), (3.36) and (3.37) we find

$$(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t),\mathbf{v}) = (f(t),\mathbf{v})_{V'\times V}.$$

Thus, we can obtain the variational formulation of the quasistatic problem with normal damped response, friction and damage as follows

Problem PV Find the displacement field $\mathbf{u} : [0,T]\to V$, the stress field $\boldsymbol{\sigma} : [0,T]\to \mathcal{H}$, the temperature $\theta : [0,T]\to V$, the damage field $\beta : [0,T]\to H^1(\Omega)$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(t))\big) + \mathcal{B}(\varepsilon(\mathbf{u}(t)))\\ &\quad + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}(s)), \theta(s), \beta(s)\Big)ds, \ \text{ a.e. } t\in(0,T),\end{aligned} \tag{3.38}$$

$$(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t),\mathbf{v}) = (f(t),\mathbf{v})_{V'\times V}, \quad \forall\mathbf{v}\in V,\ \forall t\in[0,T], \tag{3.39}$$

$$\begin{aligned}(\dot\theta(t),\mathbf{v})_{V'\times V} + a_0(\theta(t),\mathbf{v}) &= \big(\psi(\boldsymbol{\sigma}(t)),\varepsilon(\dot{\mathbf{u}}(t)),\theta(t),\beta(t),\mathbf{v}\big)_{V'\times V}\\ &\quad + (q(t),\mathbf{v})_{V'\times V} \quad \forall\mathbf{v}\in V, \ \text{ a.e. } t\in(0,T),\end{aligned} \tag{3.40}$$

$$\beta(t) \in K, \quad (\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + a_1(\beta(t), \xi - \beta(t)) \geq \tag{3.41}$$
$$(\phi\big(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \theta(t), \beta(t), \xi - \beta(t)\big)_{L^2(\Omega)},$$
$$\forall \xi \in K, \quad \text{a.e. } t \in (0, T)\,,$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0, \qquad \beta(0) = \beta_0. \tag{3.42}$$

We notice that the variational problem PV is formulated in terms of displacement field, stress field, temperature and damage field. Our main result that we state here and prove in the next section is the following.

Theorem 3.1. *Assume that (3.18)–(3.24) hold. Then there exists a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \theta, \beta\}$ *to problem* PV*. Moreover, the solution has the regularity*

$$\mathbf{u} \in C^1(0, T; V), \tag{3.43}$$

$$\boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1), \tag{3.44}$$

$$\theta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; V), \tag{3.45}$$

$$\beta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \tag{3.46}$$

The quadruple $\{\mathbf{u}, \boldsymbol{\sigma}, \theta, \beta\}$ which satisfies (3.38)–(3.42) is called a weak solution to the frictional contact problem P. We conclude that under the stated assumptions, problem (3.4)–(3.13) has a unique weak solution satisfying (3.43)–(3.46).

4. Proof of Theorem 3.1

The proof of Theorem 3.1 will be carried out in several steps and is based on arguments of evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point.

To this end, we assume in the following that (3.18)–(3.24) hold and, everywhere in this section C will represent a strictly positive constant which may depend on the problem's data but it is independent on time, and whose value may change from place to place.

Moreover, for the sake of simplicity, we suppress, in what follows, the explicit dependence of various functions on $\mathrm{x} \in \Omega \cup \Gamma$.

Let $\eta \in L^2(0, T; V')$ be given. In the first step we consider the following variational problem.

Problem PV_η. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that

$$\big(\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(t))), \varepsilon(\mathbf{v})\big)_{\mathcal{H}} + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) + (\eta(t), \mathbf{v})_{V'\times V} = (f(t), \mathbf{v})_{V'\times V}, \quad \forall \mathbf{v} \in V \quad a.e.\ t \in (0,T), \tag{4.1}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{4.2}$$

In the study of Problem PV_η we have the following result.

Lemma 4.1. *There exists a unique solution to problem PV_η and it has the regularity expressed in (3.43). Moreover, if $\mathbf{u}_i$ represents the solution of problem PV_{η_i} for $\eta_i \in L^2(0,T;V')$, $i = 1,2$ then there exists $C > 0$ such that*

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V \leq C \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}\,. \tag{4.3}$$

Proof. Using Riesz's representation theorem we define the operator $T : V \to V'$ and the element $f_\eta(t) \in V'$ by

$$(T\mathbf{u}, \mathbf{v})_{V'\times V} = (\mathcal{A}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\mathbf{u}, \mathbf{v}), \tag{4.4}$$

$$(f_\eta(t), \mathbf{v})_{V'\times V} = (f(t), \mathbf{v})_{V'\times V} - (\eta(t), \mathbf{v})_{V'\times V}, \tag{4.5}$$

for all $\mathbf{u}, \mathbf{v} \in V$, $t \in [0,T]$. Let $\mathbf{u}_1, \mathbf{u}_2 \in V$. Using (4.4) and (3.33) we find

$$\begin{aligned}(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{V'\times V} &= \big(\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\big)_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} \big(p_\nu(\mathbf{u}_{1\nu}) - p_\nu(\mathbf{u}_{2\nu})\big)(\mathbf{u}_{1\nu} - \mathbf{u}_{2\nu})da \\ &\quad + \int_{\Gamma_3} \big(p_\tau(\mathbf{u}_{1\tau}) - p_\tau(\mathbf{u}_{2\tau})\big).(\mathbf{u}_{1\tau} - \mathbf{u}_{2\tau})da,\end{aligned}$$

and, keeping in mind (3.18)(b), (3.23)(b) and (3.24)(b), we obtain

$$(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{V'\times V} \geq m_{\mathcal{A}}\, |\mathbf{u}_1 - \mathbf{u}_2|_V^2\,. \tag{4.6}$$

Using again (4.4) and (3.33) it follows that

$$\begin{aligned}(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{v})_{V'\times V} &= \big(\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{v})\big)_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} \big(p_\nu(\mathbf{u}_{1\nu}) - p_\nu(\mathbf{u}_{2\nu})\big)(\mathbf{v}_\nu)da + \int_{\Gamma_3} \big(p_\tau(\mathbf{u}_{1\tau}) - p_\tau(\mathbf{u}_{2\tau})\big).(\mathbf{v}_\tau)da,\end{aligned}$$

for all $\mathbf{v} \in V$ and, by (3.18)(a), we deduce that

$$|T\mathbf{u}_1 - T\mathbf{u}_2|_{V'} \leq L_{\mathcal{A}}\, |\mathbf{u}_1 - \mathbf{u}_2|_V \quad \forall\, \mathbf{u}_1, \mathbf{u}_2 \in V. \tag{4.7}$$

Inequality (4.6) shows that $T : V \to V'$ is a strongly monotone operator. Moreover, inequality (4.7) implies that T is Lipschitz continuous. Therefore, using a standard result for nonlinear equations (see, e.g., [7]), there exists a unique element $\mathbf{w}_\eta$ which satisfies

$$T\mathbf{w}_\eta(t) = f_\eta(t) \quad a.e.\ t \in (0,t), \tag{4.8}$$

$$\mathbf{w}_\eta \in C(0,T;V), \tag{4.9}$$

Consider now the function $\mathbf{u}_\eta : [0,T] \to V$ defined by

$$\mathbf{u}_\eta = \int_0^t \mathbf{w}_\eta(s)ds + \mathbf{u}_0. \tag{4.10}$$

It follows from (4.4), (4.8)–(4.10) that $\mathbf{u}_\eta$ is a solution of the equation (4.1) and it satisfies (3.43).

It remains to show estimate (4.3). Let $\eta_1, \eta_2 \in L^2(0,T,V')$ and use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$ for $i = 1,2.$

Moreover, using (4.1) and subtracting the two obtained equations, by choosing $\mathbf{v} = \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2$ as test function, lead to

$$\begin{aligned}
&\Big(\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_1(t))) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_2(t))), \varepsilon(\dot{\mathbf{u}}_1(t)) - \varepsilon(\dot{\mathbf{u}}_2(t))\Big)_{\mathcal{H}} \\
&+ j\big(\dot{\mathbf{u}}_1(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big) - j\big(\dot{\mathbf{u}}_2(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big) \\
&= \big(\eta_2(t) - \eta_1(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big)_{V' \times V}, \quad \forall t \in [0,T]\,.
\end{aligned} \tag{4.11}$$

Keeping in mind (3.16) and (3.18)(b) we deduce that

$$\begin{aligned}
&\Big(\mathcal{A}(\varepsilon(\mathbf{w}_1(t))) - \mathcal{A}(\varepsilon(\mathbf{w}_2(t))), \varepsilon(\mathbf{w}_1(t)) - \varepsilon(\mathbf{w}_2(t))\Big)_{\mathcal{H}} \\
&\geq C\,|\mathbf{w}_1(t) - \mathbf{w}_2(t)|_V^2\,, \quad \forall t \in [0,T]\,.
\end{aligned} \tag{4.12}$$

From (3.33), (3.23) and (3.24), we find

$$j(\mathbf{w}_1(t), \mathbf{w}_1(t) - \mathbf{w}_2(t)) - j(\mathbf{w}_2(t), \mathbf{w}_1(t) - \mathbf{w}_2(t)) \geq 0, \quad \forall t \in [0,T]. \tag{4.13}$$

Moreover, using Cauchy–Schwartz inequality we obtain

$$\big(\eta_2(t) - \eta_1(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big)_{V' \times V} \leq |\eta_2(t) - \eta_1(t)|_{V'}\,|\mathbf{w}_1(t) - \mathbf{w}_2(t)|_V\,. \tag{4.14}$$

Combining (4.11)–(4.14) with some algebraic manipulations leads to

$$|\mathbf{w}_1(t) - \mathbf{w}_2(t)|_V \leq C\,|\eta_1(t) - \eta_2(t)|_{V'}\,. \tag{4.15}$$

Since $\mathbf{u}_i(t) = \int_0^t \mathbf{w}_i(s)ds + \mathbf{u}_0$ and $\mathbf{u}_1(0) = \mathbf{u}_2(0)$ we have

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V \leq \int_0^t |\mathbf{w}_1(s) - \mathbf{w}_2(s)|_V\,ds.$$

From the two previous inequalities we deduce (4.3), which concludes the proof. □

Let $\chi \in L^2(0, T, V')$ be given. In the second step we consider the following intermediate variational problem.

Problem $\boldsymbol{PV_\chi}$. Find the temperature $\theta_\chi : [0, T] \to V$ such that

$$(\dot{\theta}_\chi(t), \mathbf{v})_{V'\times V} + a_0(\theta_\chi(t), \mathbf{v}) = \big(\chi(t) + q(t), \mathbf{v}\big)_{V'\times V} \quad \forall \mathbf{v} \in V, \quad a.e.\ t \in (0, T)\,, \tag{4.16}$$

$$\theta_\chi(0) = \theta_0. \tag{4.17}$$

Lemma 4.2. *There exists a unique solution θ_χ to the auxiliary problem PV_χ and it has the regularity expressed in (3.45). Moreover, if θ_i represents the solution of problem PV_{χ_i} for $\chi_i \in L^2(0, T; V')$, $i = 1, 2$ then there exists $C > 0$ such that*

$$|\theta_1(t) - \theta_2(t)|_V^2 \leq C \int_0^t |\chi_1(s) - \chi_2(s)|_{V'}^2\,ds. \tag{4.18}$$

Proof. Using the definition (3.31) of the bilinear form a_0, leads to

$$a_0(\xi, \xi) = k_0 \int_\Omega |\nabla \xi|^2\,dx + B \int_\Gamma |\xi|^2\,d\gamma.$$

By an application of the Friedrichs–Poincaré inequality, we can find a constant $F > 0$ such that

$$\int_\Omega |\nabla \xi|^2\,dx + \frac{B}{k_0} B \int_\Gamma |\xi|^2\,d\gamma \geq F \int_\Omega |\xi|^2\,dx.$$

Thus, there exists a constant $C > 0$ which satisfies

$$a_0(\xi, \xi) \geq C\,|\xi|_{V'}^2\,, \quad \forall \xi \in V,$$

which implies that a_0 is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (4.16) has a unique solution θ_χ satisfying (3.45). It remains to show estimate (4.18), consider $\chi_1, \chi_2 \in L^2(0,T;V')$ and denote $\theta_\chi = \theta_i$ for $i = 1, 2$. If we take the substitution $\chi = \chi_1, \chi = \chi_2$ in (4.16) and subtracting the two obtained equations we obtain, by choosing $\mathbf{v} = \theta_1 - \theta_2$ as test function

$$(\dot\theta_1 - \dot\theta_2, \theta_1 - \theta_2)_{V'\times V} + a_0(\theta_1 - \theta_2, \theta_1 - \theta_2) = (\chi_1 - \chi_2, \theta_1 - \theta_2)_{V'\times V}, \quad a.e.\ t \in (0,T). \tag{4.19}$$

Keeping in mind the inequality $a_0(\theta_1 - \theta_2, \theta_1 - \theta_2) \geq 0$ we find that

$$(\dot\theta_1 - \dot\theta_2, \theta_1 - \theta_2)_{V'\times V} \leq (\chi_1 - \chi_2, \theta_1 - \theta_2)_{V'\times V}. \tag{4.20}$$

Using Cauchy–Schwartz inequality we obtain

$$(\dot\theta_1 - \dot\theta_2, \theta_1 - \theta_2)_{V'\times V} \leq |\chi_1 - \chi_2|_{V'} \, |\theta_1 - \theta_2|_V \, . \tag{4.21}$$

We integrate the previous inequality with respect to time and use the initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$ to find that

$$\frac{1}{2} |\theta_1(t) - \theta_2(t)|_V^2 \leq \int_0^t |\chi_1(s) - \chi_2(s)|_{V'} \, |\theta_1(s) - \theta_2(s)|_V \, ds. \tag{4.22}$$

Multiplying the members of the previous inequality by 2, and Applying the inequality

$$2ab \leq a^2 + b^2 \quad \forall a, b \in \mathbb{R},$$

allow us to find

$$|\theta_1(t) - \theta_2(t)|_V^2 \leq \int_0^t |\chi_1(s) - \chi_2(s)|_{V'}^2 \, ds + \int_0^t |\theta_1(s) - \theta_2(s)|_V^2 \, ds.$$

It follows now from a Gronwall-type argument that

$$|\theta_1(t) - \theta_2(t)|_V^2 \leq C \int_0^t |\chi_1(s) - \chi_2(s)|_{V'}^2 \, ds. \tag{4.23}$$

□

Let $\mu \in L^2(0,T;L^2(\Omega)$ be given. In the third step we consider the following variational problem for the damage field.

Problem $\boldsymbol{PV_\mu}$. Find the damage field $\beta_\mu : [0,T] \to H^1(\Omega)$ such that

$$\beta_\mu(t) \in K, \quad \big(\dot{\beta}_\mu(t), \xi - \beta_\mu(t)\big)_{L^2(\Omega)} + a_1\big(\beta_\mu(t), \xi - \beta_\mu(t)\big) \geq \big(\mu(t), \xi - \beta(t)\big)_{L^2(\Omega)} \quad \forall \xi \in V,\ a.e.\ t \in (0,T)\,, \tag{4.24}$$

$$\beta_\mu(0) = \beta_0. \tag{4.25}$$

We apply Theorem 2.1 to problem PV_μ.

Lemma 4.3. *There exists a unique solution β_μ to the auxiliary problem PV_μ satisfying (3.46), Moreover, if β_i represents the solution of problem PV_{μ_i} for $\mu_i \in L^2(0,T;L^2(\Omega))$, $i = 1,2$ then there exists $C > 0$ such that*

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} \leq C \int_0^t |\mu_1(s) - \mu_2(s)|^2_{L^2(\Omega)}\, ds. \tag{4.26}$$

Proof. We have $H^1(\Omega)$ is dense in $L^2(\Omega)$, and the inclusion map is continuous, $L^2(\Omega)$ is identified with $(L^2(\Omega))'$ and it is identified with a subspace of $(H^1(\Omega))'$, where $(L^2(\Omega))'$ and $(H^1(\Omega))'$ represent the dual of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. The notation $(.,.)_{(H^1(\Omega))'\times H^1(\Omega)}$ denote the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$, we can write

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'$$
$$(\beta,\xi)_{(H^1(\Omega))'\times H^1(\Omega)} = (\beta,\xi)_{L^2(\Omega)} \quad \forall \xi \in H^1(\Omega),$$

and we not that K is a closed convex set in $H^1(\Omega)$. Then, using (3.32) the definition of the bilinear form a_1, and the fact that $\beta_\mu \in K$ in (3.28), it is easy to see that lemma 4.3 is a straight consequence of Theorem 2.1. Consider now $\mu_1, \mu_2 \in L^2(0,T;L^2(\Omega))$ and denote $\beta_\mu = \beta_i$ for $i = 1,2$. If we take the substitution $\mu = \mu_1,\ \mu = \mu_2$ in (4.25) and subtracting the two obtained inequalities we obtain, by choosing $\xi = \beta_1 - \beta_2$ as test function

$$(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \leq (\mu_1 - \mu_2, \beta_1 - \beta_2)_{L^2(\Omega)} \quad a.e.\ t \in (0,T).$$

Taking into account that $a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \geq 0$, we obtain

$$(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} \leq (\mu_1 - \mu_2, \beta_1 - \beta_2)_{L^2(\Omega)}. \tag{4.27}$$

Integrating the previous inequality with respect to time and using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$, allow us to find

$$\frac{1}{2}\left|\beta_1(t)-\beta_2(t)\right|^2 L^2(\Omega) \leq \int_0^t (\mu_1-\mu_2, \beta_1-\beta_2)_{L^2(\Omega)}ds. \tag{4.28}$$

Employing Hölder's and Young's inequalities, we deduce that

$$|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)} \leq \int_0^t |\mu_1(s)-\mu_2(s)|^2_{L^2(\Omega)}\,ds + \int_0^t |\beta_1(s)-\beta_2(s)|^2_{L^2(\Omega)}\,ds.$$

The previous inequality combined with Gronwall's inequality lead to

$$|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)} \leq C\int_0^t |\mu_1(s)-\mu_2(s)|^2_{L^2(\Omega)}\,ds. \tag{4.29}$$

□

In the next step we use $\mathbf{u}_\eta, \theta_\chi$ and β_μ obtained above in lemma 4.1, lemma 4.2 and lemma 4.3, respectively to construct the following problem for the stress field.

Problem $PV_{\eta\chi\mu}$. Find the stress field $\boldsymbol{\sigma}_{\eta\chi\mu} : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta\chi\mu}(t) = \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta\chi\mu}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big)ds, \quad \forall t\in[0,T]. \tag{4.30}$$

In the study of problem $PV_{\eta\chi\mu}$ we have the following result.

Lemma 4.4. *There exists a unique solution of problem $PV_{\eta\chi\mu}$ and it satisfies $\boldsymbol{\sigma}_{\eta\chi\mu} \in W^{1,2}(0,T;\mathcal{H})$. Moreover, if $\mathbf{u}_i$, θ_i, β_i and $\boldsymbol{\sigma}_i$, represent the solutions of problem PV_{η_i}, PV_{χ_i}, PV_{μ_i} and $PV_{\eta_i\chi_i\mu_i}$, respectively, for $(\eta_i,\chi_i,\mu_i) \in L^2(0,T;V'\times V'\times L^2(\Omega))$, $i=1,2$, then there exists $C>0$ such that $\forall t\in[0,T]$*

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} \leq C\bigg(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V + \int_0^t (|\mathbf{u}_1(s)-\mathbf{u}_2(s)|^2_V + |\theta_1(s)-\theta_2(s)|^2_V + |\beta_1(s)-\beta_2(s)|^2_{L^2(\Omega)})ds\bigg). \tag{4.31}$$

Proof. Let $\Lambda_{\eta\chi\mu} : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}(t) = \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big)ds, \quad \forall\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H}),\ \forall t\in(0,T). \tag{4.32}$$

For $\boldsymbol{\sigma}_i \in L^2(0,T;\mathcal{H})$, $i=1,2$ we us (4.32), hypothesis (3.20) and Holder's inequality to obtain for all $t \in (0,T)$

$$|\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_1(t) - \Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \leq L_{\mathcal{G}} \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}\, ds.$$

It follows that

$$\begin{aligned}
&|\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_1(t) - \Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} \\
&\leq \left(L_{\mathcal{G}} \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}\, ds\right)^2 \\
&\leq (L_{\mathcal{G}})^2 T \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}}\, ds \\
&\leq C \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}}\, ds.
\end{aligned}$$

It follows from lemma 2.1 that there exists a unique element $\boldsymbol{\sigma}_{\eta\chi\mu} \in L^2(0,T;\mathcal{H})$ such that $\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_{\eta\chi\mu} = \boldsymbol{\sigma}_{\eta\chi\mu}$. Moreover, $\boldsymbol{\sigma}_{\eta\chi\mu}$ is the unique solution of problem $PV_{\eta\chi\mu}$ and, using (4.32), (3.19), (3.20) and the regularity of $\mathbf{u}_\eta, \theta_\chi, \beta_\mu$, it follows that $\boldsymbol{\sigma}_{\eta\chi\mu} \in W^{1,2}(0.T;\mathcal{H})$. Consider now $(\eta_1,\chi_1,\mu_1),(\eta_2,\chi_2,\mu_2) \in L^2(0,T;V' \times V' \times L^2(\Omega))$, and for $i=1,2$ denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\theta_{\chi_i} = \theta_i$, $\beta_{\mu_i} = \beta_i$ and $\boldsymbol{\sigma}_{\eta_i\chi_i\mu_i} = \sigma_i$. Thus we have

$$\boldsymbol{\sigma}_i = \mathcal{B}(\varepsilon(\mathbf{u}_i(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_i(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}_i(s))\big), \varepsilon(\mathbf{u}_i(s)), \theta_i(s), \beta_i(s)\Big) ds, \quad \forall t \in (0,T)\,,$$

and using the properties (3.19) and (3.20) of the operators $\mathcal{B}$ and $\mathcal{G}$, we find

$$\begin{aligned}
|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} \leq C\Bigg(&|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V \\
&+ \int_0^t (|\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V + |\theta_1(s) - \theta_2(s)|^2_V) ds \\
&+ \int_0^t (|\beta_1(s) - \beta_2(s)|^2_{L^2(\Omega)} + |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}}) ds\Bigg), \quad \forall t \in [0,T]\,.
\end{aligned} \tag{4.33}$$

Using now a Gronwall argument in the previous inequality we deduce (4.31), which concludes the proof. □

Finally as a consequence of these results and using the properties of the operators $\mathcal{B}$ and $\mathcal{G}$, and of the functions ψ and ϕ, we may consider the operator

$$\mathcal{L} : L^2\big(0,T;V' \times V' \times L^2(\Omega)\big) \to L^2\big(0,T;V' \times V' \times L^2(\Omega)\big)$$

$$\mathcal{L}(\eta,\chi(,\mu)(t) = \big(\mathcal{L}_1(\eta,\chi,\mu)(t), \mathcal{L}_2(\eta,\chi,\mu)(t), \mathcal{L}_3(\eta,\chi,\mu)(t)\big), \tag{4.34}$$

where, for all $t \in [0,T]$ the mappings $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are given by

$$\begin{aligned}\big(\mathcal{L}_1(\eta,\chi,\mu)(t),\mathbf{v}\big)_{V'\times V} = \bigg(&\mathcal{B}\big(\varepsilon(\mathbf{u}_\eta(t))\big)\\ &+ \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta\chi\mu}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big)ds, \varepsilon(\mathbf{v})\bigg)_{\mathcal{H}},\end{aligned} \tag{4.35}$$

$$\mathcal{L}_2(\eta,\chi,\mu)(t) = \psi\big(\boldsymbol{\sigma}_{\eta\chi\mu}(t), \varepsilon(\dot{\mathbf{u}}_\eta(t)), \theta_\chi(t), \beta_\mu(t)\big). \tag{4.36}$$

$$\mathcal{L}_3(\eta,\chi,\mu)(t) = \phi\big(\boldsymbol{\sigma}_{\eta\chi\mu}(t), \varepsilon(\mathbf{u}_\eta(t)), \theta_\chi(t), \beta_\mu(t)\big). \tag{4.37}$$

Here, for every $(\eta,\chi,\mu) \in L^2\big(0,T;V' \times V' \times L^2(\Omega)\big)$, $u_\eta, \theta_\chi, \beta_\mu$ and $\boldsymbol{\sigma}_{\eta\chi\mu}$ represent the displacement field, the temperature, the damage field and the stress field obtained in Lemmas 4.1, 4.2, 4.3 and 4.4 respectively and we have the following result.

Lemma 4.5. *The operator $\mathcal{L}$ has a unique fixed point, in other words, there exists a unique element $(\eta^*,\chi^*,\mu^*) \in L^2\big(0,T;V' \times V' \times L^2(\Omega)\big)$ such that $\mathcal{L}(\eta^*,\chi^*,\mu^*) = (\eta^*,\chi^*,\mu^*)$.*

Proof. In view of Lemmas 4.1, 4.2, 4.3 and 4.4, it is clear that the operator $\mathcal{L}$ is well defined and takes values in $L^2\big(0,T;V' \times V' \times L^2(\Omega)\big)$.

Let $t \in |0,T|$ and $(\eta_1,\chi_1,\mu_1), (\eta_2,\chi_2,\mu_2) \in L^2\big(0,T;V' \times V' \times L^2(\Omega)\big)$, and for the sake of simplicity, we write $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\theta_{\chi_i} = \theta_i$, $\beta_{\mu_i} = \beta_i$ and $\boldsymbol{\sigma}_{\eta_i\chi_i\mu_i} = \boldsymbol{\sigma}_i$, for $i = 1,2$.

Using (3.17), (3.19), (3.20), and elementary algebraic manipulations, we find

$$\begin{aligned}\big|\mathcal{L}_1(\eta_1,\chi_1,\mu_1)(t) - \mathcal{L}_1(\eta_2,\chi_2,\mu_2)(t)\big|_{V'}^2 \le&\\ C\bigg(&|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2\\ &+ \int_0^t \big(|\mathbf{u}_1(s) - \mathbf{u}_2(s)|_V^2 + |\theta_1(s) - \theta_2(s)|_V^2\big)ds\\ &+ \int_0^t \big(|\beta_1(s) - \beta_2(s)|_{L^2(\Omega)}^2 + |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}^2\big)ds\bigg).\end{aligned} \tag{4.38}$$

Furthermore, from definition (4.36), assumptions (3.21) on ψ we obtain

$$\begin{aligned}
&\big|\mathcal{L}_2(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}_2(\eta_2,\chi_2,\mu_2)(t)\big|_{V'}^2\\
&\leq \big|\psi\big(\boldsymbol{\sigma}_1(t),\varepsilon(\dot{\mathbf{u}}_1(t)),\theta_1(t),\beta_1(t)\big)-\psi\big(\boldsymbol{\sigma}_2(t),\varepsilon(\dot{\mathbf{u}}_2(t)),\theta_2(t),\beta_2(t)\big)\big|_{V'}^2\\
&\leq C\bigg(\ |\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2+|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)|_V^2+|\theta_1(t)-\theta_2(t)|_V^2\\
&\qquad\qquad +|\beta_1(t)-\beta_2(t)|_{L^2(\Omega)}^2\bigg).
\end{aligned}\tag{4.39}$$

Similarly, using definition (4.37), assumptions (3.22) on ϕ we find

$$\begin{aligned}
&\big|\mathcal{L}_3(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}_3(\eta_2,\chi_2,\mu_2)(t)\big|_{L^2(\Omega)}^2\\
&\leq \big|\phi\big(\boldsymbol{\sigma}_1(t),\varepsilon(\mathbf{u}_1(t)),\theta_1(t),\beta_1(t)\big)-\phi\big(\boldsymbol{\sigma}_2(t),\varepsilon(\mathbf{u}_2(t)),\theta_2(t),\beta_2(t)\big)\big|_{L^2(\Omega)}^2,\\
&\leq C\bigg(\ |\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2+|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2\\
&\qquad\qquad +|\theta_1(t)-\theta_2(t)|_V^2+|\beta_1(t)-\beta_2(t)|_{L^2(\Omega)}^2\bigg).
\end{aligned}\tag{4.40}$$

It follows from (4.38)–(4.40) and (4.34) that

$$\begin{aligned}
&\bigg|\mathcal{L}(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}(\eta_2,\chi_2,\mu_2)(t)\bigg|_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)}^2\leq C\bigg(\ |\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2\\
&\qquad +|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)|_V^2+|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2\\
&\qquad +|\theta_1(t)-\theta_2(t)|_V^2+|\beta_1(t)-\beta_2(t)|_{L^2(\Omega)}^2\\
&\qquad +\int_0^t(|\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2+|\theta_1(s)-\theta_2(s)|_V^2)ds\\
&\qquad +\int_0^t(|\beta_1(s)-\beta_2(s)|_{L^2(\Omega)}^2+|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}^2)ds\bigg).
\end{aligned}\tag{4.41}$$

Inserting (4.31) in (4.41) yields

(4.42)

$$\begin{aligned}
&\bigg|\mathcal{L}(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}(\eta_2,\chi_2,\mu_2)(t)\bigg|_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)}^2\\
&\leq C\bigg(\ |\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)|_V^2+|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2+|\theta_1(t)-\theta_2(t)|_V^2+|\beta_1(t)-\beta_2(t)|_{L^2(\Omega)}^2
\end{aligned}$$

$$+\int_0^t (|\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2 + |\theta_1(s)-\theta_2(s)|_V^2 + |\beta_1(s)-\beta_2(s)|_{L^2(\Omega)}^2)ds\Big).$$

On the other hand, using (4.3) we can infer that there exists $C>0$ such that

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2 \le C\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\, ds. \tag{4.43}$$

Taking into account that $|\mathbf{w}_1(t)-\mathbf{w}_2(t)|_V \le C\,|\eta_1(t)-\eta_2(t)|_{V'}$, applying the estimates (4.18), (4.26), (4.43) and substituting in (4.42) we obtain

(4.44)

$$\begin{aligned}&\Big|\mathcal{L}(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}(\eta_2,\chi_2,\mu_2)(t)\Big|^2_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)}\\ &\le C\Big(\int_0^t \big(|\eta_1(s)-\eta_2(s)|_{V'}^2+|\chi_1(s)-\chi_2(s)|_{V'}^2+|\mu_1(s)-\mu_2(s)|_{L^2(\Omega)}^2\big)ds\\ &+\int_0^t\int_0^s \big(|\eta_1(s)-\eta_2(s)|_{V'}^2+|\chi_1(s)-\chi_2(s)|_{V'}^2+|\mu_1(s)-\mu_2(s)|_{L^2(\Omega)}^2\big)dr\,ds\Big),\end{aligned}$$

which implies with some algebraic manipulations, that there exists $C>0$ such that

$$\begin{aligned}&\Big|\mathcal{L}(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}(\eta_2,\chi_2,\mu_2)(t)\Big|^2_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)}\\ &\le C\int_0^t \Big|(\eta_1,\chi_1,\mu_1)(t)-(\eta_2,\chi_2,\mu_2)(t)\Big|^2_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)}ds.\end{aligned} \tag{4.45}$$

Applying Lemma 2.1, we deduce that The operator $\mathcal{L}$ has a unique fixed point in $L^2\big(0,T;V'\times V'\times L^2(\Omega)\big)$, i.e. there exists a unique element $(\eta^*,\chi^*,\mu^*)\in L^2\big(0,T;V'\times V'\times L^2(\Omega)\big)$ such that $\mathcal{L}(\eta^*,\chi^*,\mu^*)=(\eta^*,\chi^*,\mu^*)$. □

We have all the ingredients to prove the Theorem 3.1 which we complete now.

Proof of Theorem 3.1. Let $(\eta^*,\chi^*,\mu^*)\in L^2\big(0,T;V'\times V'\times L^2(\Omega)\big)$ be the fixed point of the operator $\mathcal{L}$ defined by (4.34)-(4.37) and let $\mathbf{u}_{\eta^*}$, θ_{χ^*}, β_{μ^*}, $\boldsymbol{\sigma}_{\eta^*\chi^*\mu^*}$ be the solutions of the problems PV$_\eta$, PV$_\chi$, PV$_\mu$ and PV$_{\eta\chi\mu}$ respectively, for $\eta=\eta^*,\chi=\chi^*$, and $\mu=\mu^*$, and we denote

$$\mathbf{u}=\mathbf{u}_{\eta^*},\quad \boldsymbol{\sigma}=\mathcal{A}\varepsilon(\dot{\mathbf{u}})+\boldsymbol{\sigma}_{\eta^*\chi^*\mu^*}, \tag{4.46}$$

$$\theta = \theta_{\chi^*}, \quad \beta = \beta_{\mu^*}. \tag{4.47}$$

We shall prove that the quadruple $\{\mathbf{u}, \boldsymbol{\sigma}, \theta, \beta\}$ is the unique solution of Problem PV. Indeed, we write (4.30) for $\eta = \eta^*, \chi = \chi^*$, and $\mu = \mu^*$ and use (4.46) to obtain that (3.38) is satisfied. Then we use (4.1) for $\eta = \eta^*$ and use the first equality in (4.46) to find

$$\begin{aligned}\big(\mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))), \varepsilon(\mathbf{v})\big)_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) + (\eta^*(t), \mathbf{v})_{V'\times V} = (f(t), \mathbf{v})_{V'\times V},\\ \forall \mathbf{v} \in V \quad a.e.\ t \in (0,T).\end{aligned} \tag{4.48}$$

Using equality $\mathcal{L}_1(\eta^*, \chi^*, \mu^*) = \eta^*$ combined with (4.35) leads to

$$\begin{aligned}(\eta^*(t), \mathbf{v})_V &= \mathcal{B}\big(\varepsilon(\mathbf{u}_\eta(t))\big)_{\mathcal{H}}\\ &+ \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta\chi\mu}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}_\eta(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big)ds.\end{aligned} \tag{4.49}$$

We substitute (4.49) in (4.48) and use (3.38) to conclude that (3.39) is satisfied. Moreover, from equalities $\mathcal{L}_2(\eta^*, \chi^*, \mu^*) = \chi^*$ and $\mathcal{L}_3(\eta^*, \chi^*, \mu^*) = \mu^*$ combined with (4.36) and (4.37), we find that

$$\chi^* = \psi\big(\boldsymbol{\sigma}(t), \varepsilon(\dot{\mathbf{u}}(t)), \theta(t), \beta(t)\big). \tag{4.50}$$

$$\mu^* = \phi\big(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \theta(t), \beta(t)\big). \tag{4.51}$$

We write (4.16) for $\chi = \chi^*$ and use the first equality in (4.47) we obtain

$$\begin{aligned}(\dot{\theta}(t), \mathbf{v})_{V'\times V} + a_0(\theta(t), \mathbf{v}) = (\chi^*(t) + q(t), \mathbf{v})_{V'\times V},\\ \forall \mathbf{v} \in V \quad a.e.\ t \in (0,T)\,.\end{aligned} \tag{4.52}$$

Substituting (4.50) in (4.52) implies that (3.40) is satisfied.

Next, we write (4.25) for $\mu = \mu^*$ and use the second equality in (4.47) to find

$$\begin{aligned}\big(\dot{\beta}(t), \xi - \beta(t)\big)_{L^2(\Omega)} + a_1\big(\beta(t), \xi - \beta(t)\big) \geq \big(\mu^*(t), \xi - \beta(t)\big)_{L^2(\Omega)},\\ \forall \xi \in V,\ a.e.\ t \in (0,T)\,.\end{aligned} \tag{4.53}$$

We combine now (4.53) and (4.51) to see that (3.41) is satisfied, which concludes the existence part of Theorem 3.1. The uniqueness part of Theorem 3.1

is a consequence of the uniqueness of the fixed point of the operator $\mathcal{L}$ and the unique solvability of the problems PV_{η}, PV_{χ}, PV_{μ} and $\mathrm{PV}_{\eta\chi\mu}$ which concludes the proof. □

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