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ANALYSIS OF APPROXIMATE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS IN THEIR GENERAL FORM WITH GENERALIZED ORDER

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ABSTRACT. In this paper, we dedicate our study on the approximate solutions of van der Pol equation in their general form. First, we prove the approximate analytic solutions to this equation by different perturbation methods, simple perturbation method (SPM), Lindstedt-Poincaré method (PLM) and Averaging method (AM). Then we compare these approximations with each other and with the exact solution. Second, we introduce a new form of generalized Van Der Pol oscillator with fractional-order derivatives. Which is analyzed through phase portraits, Poincaré maps and analytic solutions, we use numerical simulation to illustrate the behavior of the fractional order system.

1. INTRODUCTION

The Van der Pol equation appeared in 1927 by the electrical engineer Balthazar van der Pol, see [1], where he described the oscillations of a triode in electrical circuits. He presented it in its mathematical form as second-order nonlinear oscillatory differential equation. It is one of important useful models in perturbation theory. For understanding of the behavior of this type of equation, many studies have been carried out in different methods to approximate and find the

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behavior of the best approximate solutions of nonlinear equations, see [13, 14]. Researchers have developed many methods and techniques to study approximate analytic solutions with different perturbation methods, see [4–8]. After, many systems display the dynamics of the fractional order, and through previous studies the researchers found that some of these systems display chaotic movements in the fractional order. For more details, we refer the reader to [9–11].

In our work, we have devoted the study of Van der pol equation in their general form. We used the simple perturbation method (SPM), Lindstedt-Poincaré method (PLM), Averaging method (AM) and renormalization group method (RGM). Then, we compared them with the exact solution and with each other to find the best approximation of these methods. After that, we studied van der Pol systems generalized in fractional order in their general form, where we modeled and simulated using Matlab software, and analyzed our results.

This paper is organized as follows. In Section 1, we provide the general framework of our study. In section 2, we define different perturbation methods; simple perturbation method (SPM), Lindstedt-Poincaré method (PLM) and Averaging method (AM) to prove the approximate analytical solutions of the generalized van der Pol equation and then we compare these approximations. In section 3, we demonstrate numerical simulations, such as phase images, Poincare maps and analytical solutions for generalized van der Pol systems in their general form, and arrive at important new results from this study.

2. Approximate solutions methods for the nonlinear differential equation

2.1. **Simple perturbation method (SPM) [7].** Consider the initial value problem

$$\ddot{x} = f\left(t, x, \epsilon\right), \quad \text{with } \ x\left(0\right) \text{ given,}$$

$t \geq 0$, $x \in D \subset \mathbb{R}^n$. If we can expend $f(t, x, \epsilon)$ in a Taylor series with respect to ϵ.

Suppose that the approximate solution is written in the form

$$x\left(t, \epsilon\right) = x_0\left(t\right) + \epsilon x_1\left(t\right) + \cdots + o(\epsilon^n).$$

2.2. **Example.** We consider the differential equation of the VAN der Pol oscillator in their general form is:

$$\ddot{x} + \epsilon \left(b\dot{x}^2 + ax^2 - 1\right) \dot{x} + x = 0. \quad x(0) = A \text{ and } \dot{x}(0) = 0, a, b, A \in \mathbb{R}. \tag{2.1}$$

Suppose that the approximate solution is

$$y_P(t, \epsilon) = x\,(t, \epsilon) = x_0\,(t) + \epsilon x_1\,(t) + \epsilon^2 x_2(t) + o(\epsilon^2). \tag{2.2}$$

To determine $x_0(t)$, $x_1(t)$ and $x_2(t)$, substituting 2.2 into 2.1, and calculating, we find

$$\begin{aligned}
&y_P(t, \epsilon) = A\cos(t) \\
&+ \epsilon \left[(\frac{A^3}{32}[7a + 9b] - \frac{A}{2}) \sin(t) + (\frac{A}{2} - \frac{A^3(a + 3b)}{8})t\cos(t) - \frac{A^3(a - b)}{32} \sin(3t) \right] \\
&+ \epsilon^2((-\frac{A^5}{3072}[9(a + 3b)(a - b) + 9(a - b)(a - 3b) + 24a(a + 3b) + 18a(a - b) \\
&- (5a - 9b)(a - b)] + \frac{2A^3}{256}[7a - 3b]) \cos(t) + (\frac{A^5}{256}[(a - 9b)(a - b) - 9(a - b)^2 \\
&+ 2(a - 9b)(a + 3b) + 6(a - b)] + \frac{3A^3}{64}[7b + a] - \frac{A}{8})t\sin(t) + (\frac{3A^5}{128}(a + 3b)^2 \\
&+ \frac{A}{8} - \frac{A^3}{8}(a + 3b))t^2\cos(t) + (\frac{A^5}{1024}[3(a + 3b)(a - b) + 3(a - b)(a - 3b) \\
&+ 8a(a + 3b) + 6a(a - b)] - \frac{2A^3}{256}[7a - 3b]) \cos(3t) - \frac{A^5}{3072}(5a - 9b)(a - b)\cos(5t) \\
&+ (\frac{3A^5}{256}(a - b)(a + 3b) - \frac{3A^3}{64}(a - b))t\sin(3t)) + o(\epsilon^2).
\end{aligned}$$

2.3. **Lindstedt-Poincaré method [6].** Let be the second order nonlinear differential equations in the following form

$$\ddot{y} + y = \epsilon F\,(y, \dot{y})\,, \quad \epsilon > 0.$$

A new variable $\theta = \omega t$ is introduced, and both y and ω are expanded to powers of as follows

$$\begin{aligned}
y\,(\theta, \epsilon) &= y_0\,(\theta) + \epsilon y_1\,(\theta) + \cdots + \epsilon^n y_n\,(\theta) + \cdots + o(\epsilon^n) \\
\omega\,(\epsilon) &= 1 + \epsilon\omega_1 + \cdots + \epsilon^n\omega_n + \cdots + o(\epsilon^n),
\end{aligned}$$

where, at this point, the ω_j are unknown constants.

Note here that the Lindstedt-Poincaré approximations are periodic due to the following proposition.

Proposition 2.1. *Let the equation*

$$\ddot{y} + y = G(\theta), \qquad y(0) = 0,\ \dot{y}(0) = 0. \tag{2.3}$$

Here $G(\theta) = -2\omega_1 \ddot{y}_0 + F(y_0, \dot{y}_0)$*. The solution to this problem is*

$$y(\theta) = \int_0^\theta sin(\theta - \tau)\, G(\tau)\, d\tau.$$

Moreover, the equation (2.3) has a periodic solution $y_1(\theta)$*, if, and only if,*

$$\begin{cases} \int_0^{2\pi} F(A\,cos\theta, -A\,sin\theta)\, sin\theta\, d\theta = 0, \\ 2\pi\omega_1 A + \int_0^{2\pi} F(A\,cos\theta, -A\,sin\theta)\, cos\theta\, d\theta = 0. \end{cases}$$

2.4. **Example.** We solving the equation 2.1 by Lindstedt-Poincaré method, with $y_L(t,\epsilon) = y(\theta,\epsilon)$, we find

$$\begin{aligned} y_L(t,\epsilon) = {} & \frac{2}{\sqrt{a+3b}}\cos(\theta) + \epsilon\Big(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(\theta) - \sin(3\theta))\Big) \\ & + \epsilon^2\Big(\Big(\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}} - \frac{3(a-b)}{32(a+3b)\sqrt{a+3a}} + \frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}}\Big)\cos(\theta) \\ & + \Big(\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} + \frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} - \frac{9b(a-b)}{8(a+3b)2\sqrt{a+3b}}\Big)\cos(3\theta) \\ & + \Big(-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}} + \frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}}\Big)\cos(5\theta)\Big) + o(\epsilon^2) \\ \omega = {} & 1 - \Big[\frac{a(a-b)}{8(a+3b)^2} - \frac{3(a-b)}{16(a+3b)} + \frac{9b(a-b)}{8(a+3b)^2}\Big]\epsilon^2 + o(\epsilon^2). \end{aligned}$$

The equation $\ddot{y}_1 + y_1 = 2A\omega_1\cos\theta - A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin\theta$ has a periodic solution $y_1(\theta)$, if, and only if,

$$\begin{cases} \int_0^{2\pi} -A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin^2\theta\, d\theta = 0, \\ 2\pi\omega_1 A + \int_0^{2\pi} -A(1 - aA^2\cos^2\theta - bA^2\sin^2\theta)\sin\theta cos\theta\, d\theta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{4}a\pi A^3 + \frac{3}{4}b\pi A^3 - A\pi = 0, \\ 2\omega_1 A\pi = 0. \end{cases} \Rightarrow \begin{cases} A = \frac{2}{\sqrt{a+3b}}, \\ \omega_1 = 0. \end{cases}$$

2.5. **Averaging method [5].** This method applies to equations of the form

$$\ddot{x} + \omega^2 x + \epsilon F(x, \dot{x}) = 0.$$

For $\epsilon \neq 0$ small, Krylov and Boyolinbov posed the solution

$$x(t) = A(t)\ sin(\omega t + \Phi(t)),$$

$$\dot{x}(t) = A(t)\, \omega\, cos(\omega t + \Phi(t)).$$

Let $\theta = \omega t + \Phi$, we find

$$\begin{cases} \dot{A} = -\dfrac{\epsilon}{2\pi} \int_0^{2\pi} cos(\theta)\, f(A\, sin\theta, A\omega\, cos\theta)\, d\theta, \\ \dot{\Phi} = \dfrac{\epsilon}{2\pi A\omega} \int_0^{2\pi} sin(\theta)\, f(A\, sin\theta, A\omega\, cos\theta)\, d\theta. \end{cases}$$

We recall that

$$I_{m,n} = \int_0^{2\pi} \sin^m x \cos^n x dx = 0, \text{ si } m, n \text{ sont impaires.}$$

and further

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}, \qquad I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}.$$

We arrive at $I_{0,0} = 2\pi$.

2.6. **Example.** We solving the equation 2.1 by averaging method. The averaging approximate solution is

$$y_A(t,\epsilon) = \frac{2}{\left[\left(\frac{4}{A_0^2} - (a+3b)\right) e^{-\epsilon t} + 1\right]^{\frac{1}{2}}} \sin(t + \Phi_0),$$

where $y_A(t,\epsilon) = x(t,\epsilon)$.

2.7. **Renormalization group method [4].** The renormalization group method is a method for finding the approximate solution of ordinary differential equations in ($\mathbb{R}^n$) of the form

$$\dot{x} = Fx + g(x, t, \epsilon)$$

(2.4) $$\dot{x} = Fx + \epsilon g_1(x,t) + \epsilon^2 g_2(x,t) + \cdots; \quad x \in \mathbb{R}^n.$$

where ϵ is an infinitely small positive parameter. For this system, we assume that

(1) F be a square matrix $n * n$, diagonalizable with imaginary eigenvalues;

(2) The function $g(x,t,\epsilon)$ is sufficiently differentiable in t,x and ϵ, the power series expansion of ϵ is given by the equation (2.4);
(3) Each $g_i(x,t)$ is periodic in $t \in \mathbb{R}$ and polynomial in x.

Then one might assume there is a similar expansion for the solution

$$x(t,\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 \dot{x_2} + \cdots .$$

2.8. **Example.** We solving the equation 2.1 by renormalization group method. For $x = (z + \bar{z})$ and $y = i(z - \bar{z})$ the equation (2.1) becomes

$$\begin{cases} \dot{z} = iz + \frac{\epsilon}{2}[(z-\bar{z}) - a(z+\bar{z})^2(z-\bar{z}) + b(z-\bar{z})^3] \\ \dot{\bar{z}} = -i\bar{z} - \frac{\epsilon}{2}[(z-\bar{z}) - a(z+\bar{z})^2(z-\bar{z}) + b(z-\bar{z})^3]. \end{cases}$$

It verifies the hypotheses (1-3) with

$$F = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The two equations of the system being identical, the problem amounts to solving one of them, with

$$z(t,\epsilon) = z_0 + \epsilon z_1 + \cdots .$$

$z(t,\epsilon)$ is the solution searched by the renormalization group method which diverges for t long because of the last term.

To zero order we have

$$z_0 = z(t,0) = qe^{it} = qZ(t),$$

with q the integration constant By setting $q = re^{i\theta(\zeta)}$ with $x = (z+\bar{z})$ and $y = i(z - \bar{z})$, we find

$$\begin{aligned} x(t,\epsilon) &= 2r\cos(\theta(\zeta)+t) - \frac{r\epsilon}{2}\sin(\theta(\zeta)+t) \\ &\quad + \frac{\epsilon}{2}\left[\left(\tfrac{b-a}{2}\right) r^3 \sin 3(\theta(\zeta)+t) + (a+3b)r^3\sin(\theta(\zeta)+t)\right] + O(\epsilon^2). \end{aligned}$$

Suppose that $y_R(t,\epsilon)$ solution of (2.1) by renormalization group method, then $y_R(t,\epsilon) = x(t,\epsilon)$.

The renormalization group equation becomes:

$$\begin{cases} \frac{dr}{d\zeta} = \frac{\epsilon r}{2}(1-(a+3b)r^2), \\ \frac{d\theta(\zeta)}{d\zeta} = 0. \end{cases}$$

It is easy to prove that EGR has a stable periodic orbit (the limit cycle) of radius $r_s = \sqrt{\frac{1}{a+3b}}$ with $a+3b>$0.

3. Comparison of approximate solutions

We compare $y_E(t,0)$ the exact solution, $y_P(t,\epsilon)$ the approximate solution with simple perturbation method, $y_L(\theta,\epsilon)$ the approximate solution with Lindestedt method and $y_A(t,\epsilon)$ the approximate solution with Averaging method of Ven Der Pol equation.

3.1. **Comparison of approximate solutions to order ϵ^2.**

$g(t)$ is the Taylor series expansion of $y_L((1+[\frac{a(a-b)}{8(a+3b)^2}-\frac{3(a-b)}{16(a+3b)}+\frac{9b(a-b)}{8(a+3b)^2}]\epsilon^2)t,\epsilon)$ in the order ϵ^2 in the neighbourhood of $\epsilon=0$.

$$\begin{aligned} g(t) = & \frac{2}{\sqrt{a+3b}}\cos(t)+\epsilon(\frac{1}{4(a+3b)}\frac{1}{\sqrt{a+3b}}(a-b)(3\sin(t)-\sin(3t))) \\ & +\epsilon^2((\frac{-4a(a-b)}{96(a+3b)^2\sqrt{a+3b}}-\frac{3(a-b)}{32(a+3b)\sqrt{a+3b}} \\ & +\frac{45b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(t)+(\frac{3a(a-b)}{32(a+3b)^2\sqrt{a+3b}} \\ & +\frac{3(a-b)}{32(a+3b)\sqrt{a+3b}}-\frac{9b(a-b)}{8(a+3b)^2\sqrt{a+3b}})\cos(3t) \\ & +(-\frac{5a(a-b)}{96(a+3b)^2\sqrt{a+3b}}+\frac{9b(a-b)}{48(a+3b)^2\sqrt{a+3b}})\cos(5t) \\ & -[\frac{a(a-b)}{2(a+3b)^2\sqrt{a+3b}}+\frac{3(a-b)}{4(a+3b)\sqrt{a+3b}}+\frac{9b(a-b)}{2(a+3b)^2\sqrt{a+3b}}]t\sin(t)) \end{aligned}$$

$h(t)$ is the Taylor series expansion of $y_A(t,\epsilon)$ in the order ϵ^2 in the neighbourhood of $\epsilon=0$,

$$h(t) = (A+(\frac{A}{2}-\frac{A^3(a+3b)}{8})\epsilon t+(\frac{3A^5}{128}(a+3b)^2+\frac{A}{8}-\frac{A^3}{8}(a+3b))\cos(t).$$

$y_E(t,0) = A\cos(t)$.

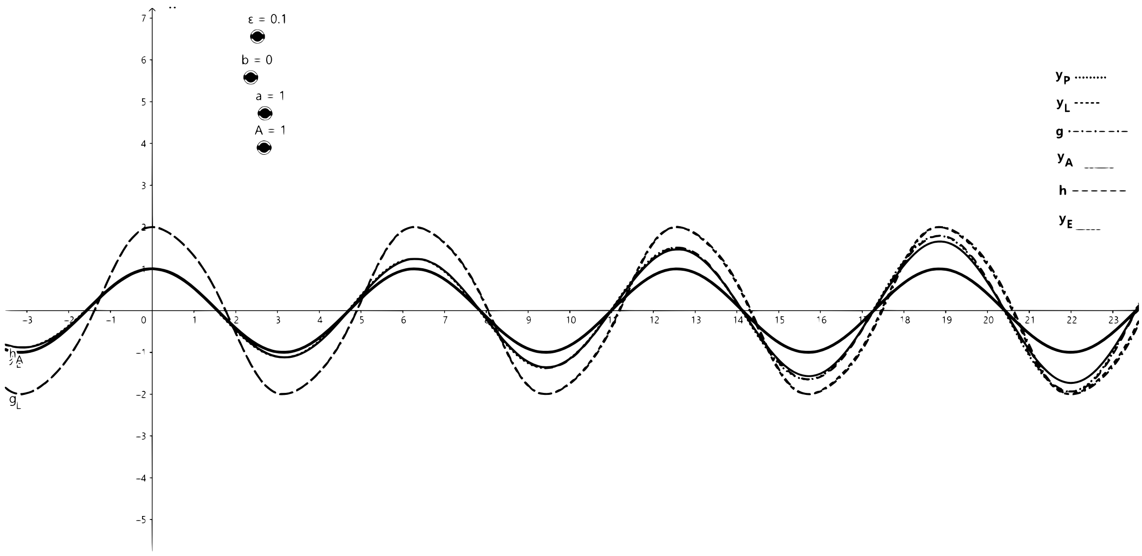


FIGURE 1. Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.1$ and $A = 1$

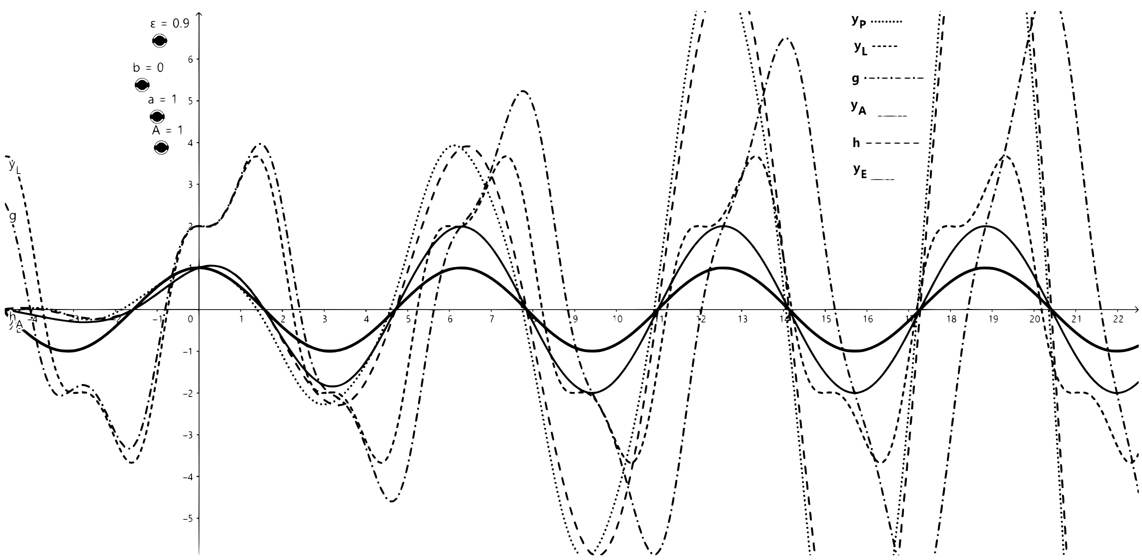


FIGURE 2. Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.9$ and $A = 1$

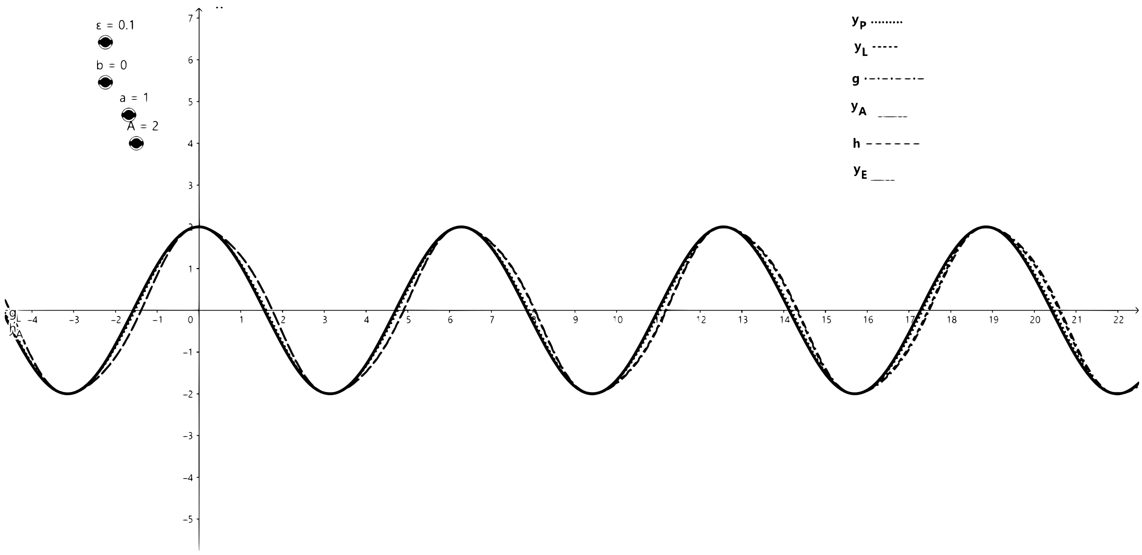


FIGURE 3. Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.1$ and $A = 2$

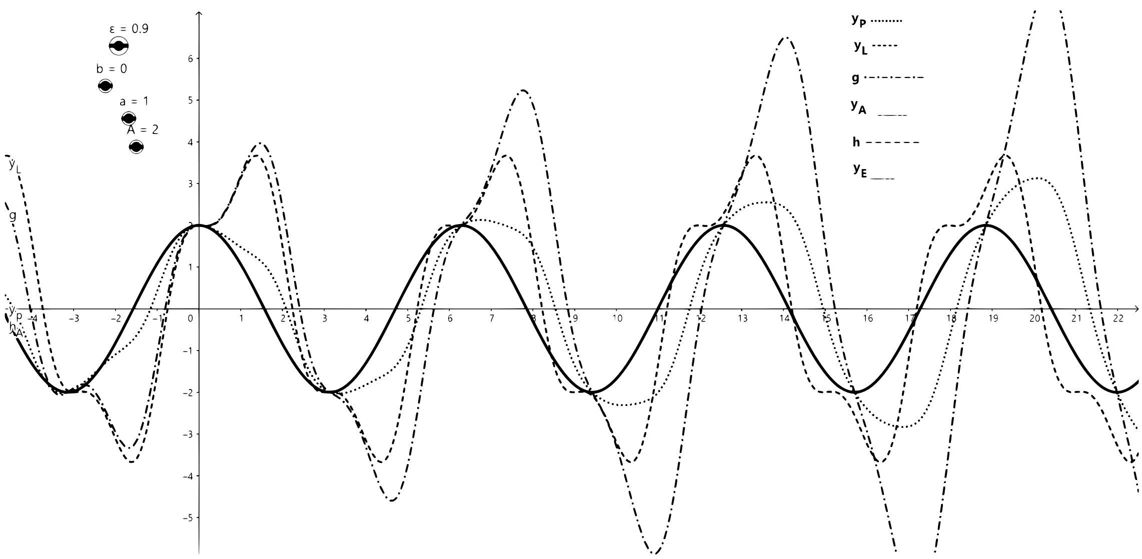


FIGURE 4. Comparison of the SPM solution, LPM solution and AM solution for $\epsilon = 0.9$ and $A = 2$

Remark 3.1. *Figures 1- 4 show the analytic approximate solutions to order ϵ^2, obtained by different methods SPM, LPM and AM at different values of ϵ and A.*

3.2. **Comparison of approximate solutions to order ϵ.** To compare the approximate solutions of the first order, we used another approximate solution, which is the solution by the renormalization group method, and denotes it by $y_R(t,\epsilon)$.

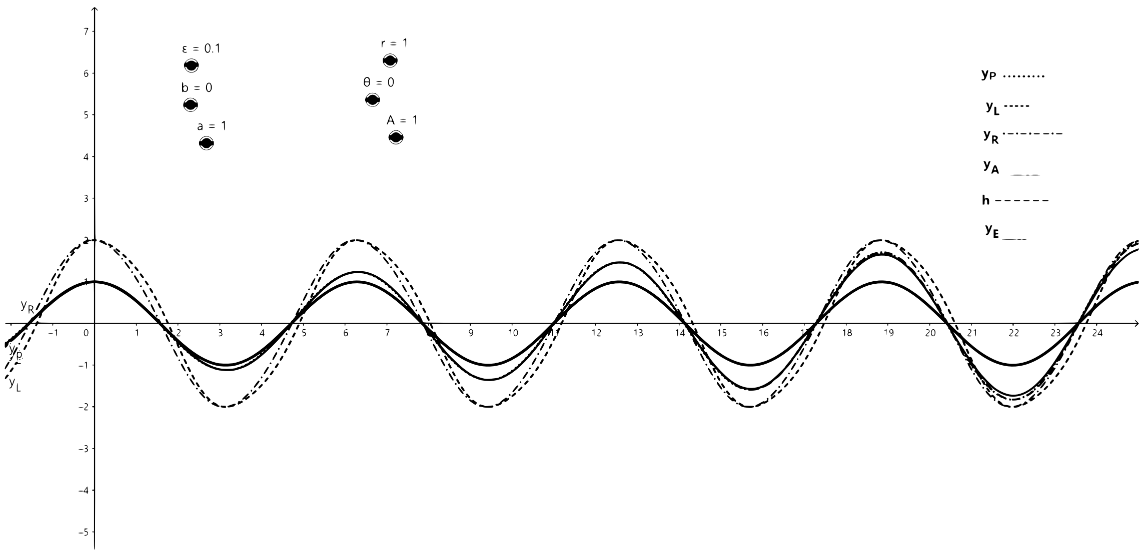


FIGURE 5. Comparison of the SPM solution, LPM solution, AM and solution RGM solution for $\epsilon = 0.1$, $r = 1$ and $A = 1$

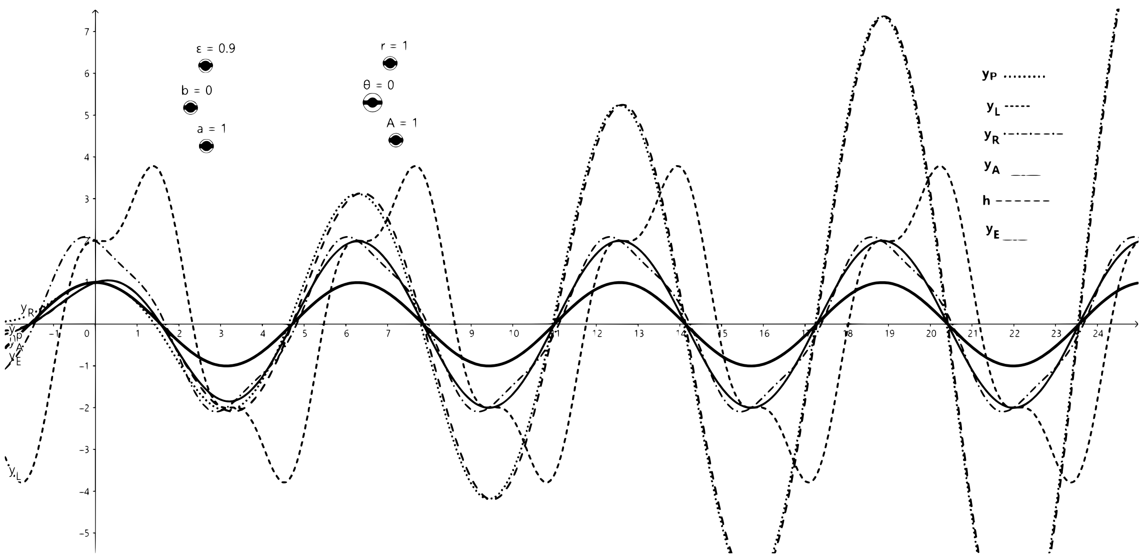


FIGURE 6. Comparison of the SPM solution, LPM solution, AM solution and RGM solution for $\epsilon = 0.9$, $r = 1$ and $A = 1$

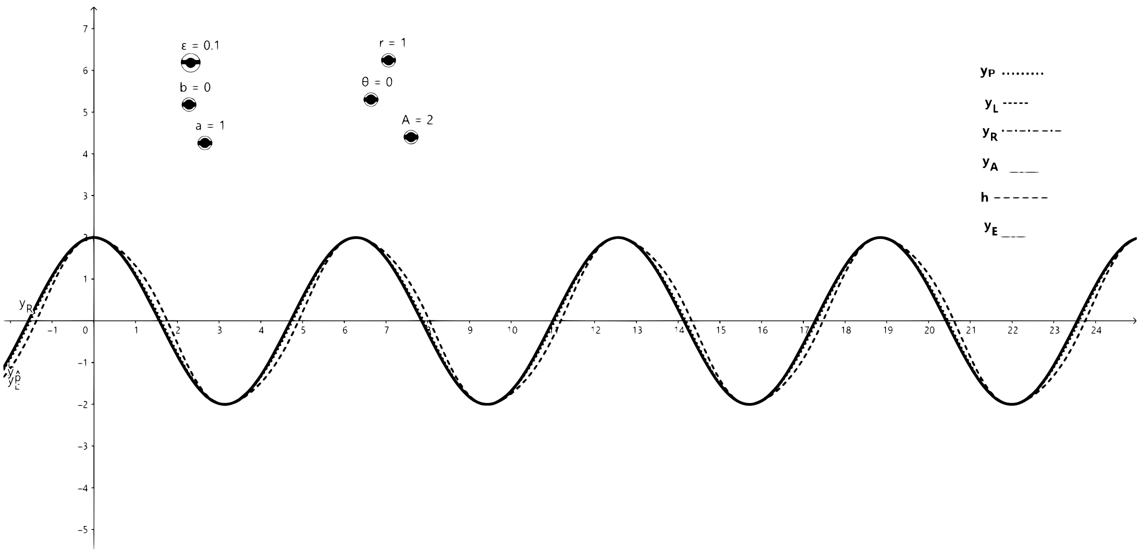


FIGURE 7. Comparison of the SPM solution, LPM solution, AM and solution RGM solution for $\epsilon = 0.1$, $r = 1$ and $A = 2$

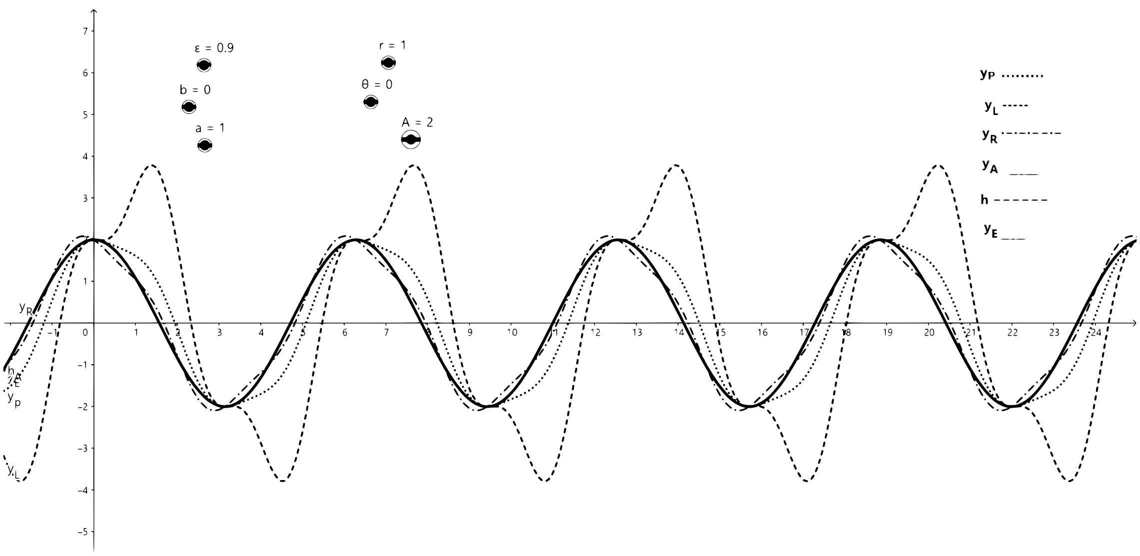


FIGURE 8. Comparison of the SPM solution, LPM solution, AM solution and RGM solution for $\epsilon = 0.9$, $r = 1$ and $A = 2$

Remark 3.2. *Figures 5- 8 show the approximate analytic solutions up to the order of ϵ. Obtained by different methods SPM, LPM, AM and RGM at different values of ϵ, r and A.*

4. Study of the generalized VAN der Pol oscillator with Fractional Derivatives

4.1. **Definition and approximation of fractional order operators.** Fractional derivative has many definitions [2]. We use the Riemann-Liouville definition of the fractional-order derivative

$$aD_t^\alpha f(t) = \frac{d^n}{dt^n} D^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad \alpha > 0.$$

where $\Gamma(.)$ is a gamma function and n is an integer such that $n-1 \leq \alpha < n$.

For better control and analysis of dynamic systems, we used an alternative definition through the Laplace transform. We present the definition as follows

$$L\{\frac{d^\alpha f(t)}{dt^\alpha}\} = s^\alpha L\{f(t)\} = F(s).$$

In our study we used the Charef's approximation frequency method [3] which is based on transfer functions of $1/s^\alpha$ with different fractional orders, $\alpha = 0.1 - 0.9$ steps, giving a maximum error of $2dB$ in the calculations and this is to approximate the behavior of the fractional system based on the frequency domain arguments.

4.2. **Generalized VAN der Pol oscillator with fractional derivatives.** The van der Pol equation is presented in standard form by a second-order nonlinear differential equation of the type:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0, \tag{4.1}$$

where ϵ is a parameter. The equivalent state space formulation has the form

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -x_1 - \epsilon(x_1^2 - 1)x_2. \end{cases} \tag{4.2}$$

The equation (4.1) has undergone several modifications due to the application of some fractional powers to the dependent variable x and/or its derivatives. These nonlinear differential equations are called fractional van der Poel equations.

Barbosa and al. [12] also suggested the introduction of a fractional-order time derivative in the state-space equations (4.2) of the standard Ven der pol in the form

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = x_2 \\ \frac{dx_2}{dt} = -x_1 - \epsilon(x_1^2 - 1)x_2, \end{cases}$$

where α is fractional number. This system is analyzed by Barbosa and al. [9].

The generalized van der Pol system which is written as

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -x_1 - \epsilon(ax_1^2 + bx_2^2 + cx + d)x_2, \end{cases}$$

where ϵ, a, b, c, d are parameters. The corresponding fractional order system is

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = x_2 \\ \frac{dx_2}{dt} = -x_1 - \epsilon(ax_1^2 + bx_2^2 + cx + d)x_2, \end{cases} \tag{4.3}$$

where α is fractional number.

A modified version of Eq. (4.3) is now proposed. The generalized fractional order van der Pol system (4.3) is transformed into an generalized fractional order van der Pol system with the degree of its polynomials,

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = x_2 \\ \frac{dx_2}{dt} = -x_1 - \epsilon(ax_1^2 + bx_2^2 + cx + d)(x_2)^n, \end{cases} \tag{4.4}$$

where $n \in \mathbb{N}$, $0 < \alpha < 1$ and $\epsilon > 0$.

Note that the system (4.4) reduces to the classical van der Pol system (7) when $\alpha = 1, n = 1, a = 1, b = 0, c = 0, d = -1$ and that the total system order is changed to $\alpha + 1 < 2$. The differential equation of system (4.4) is given by

$$x^{(1+\alpha)} + \epsilon(ax^2 + b(x^{(\alpha)})^2 + cx + d)(x^{(\alpha)})^n + x = 0. \tag{4.5}$$

In this section, we analyse and present simulation results of the chaotic dynamics produced from a new generalized fractional van der Pol system (4.5).

4.3. **Numerical simulations for the fractional order generalized van der Pol systems.**

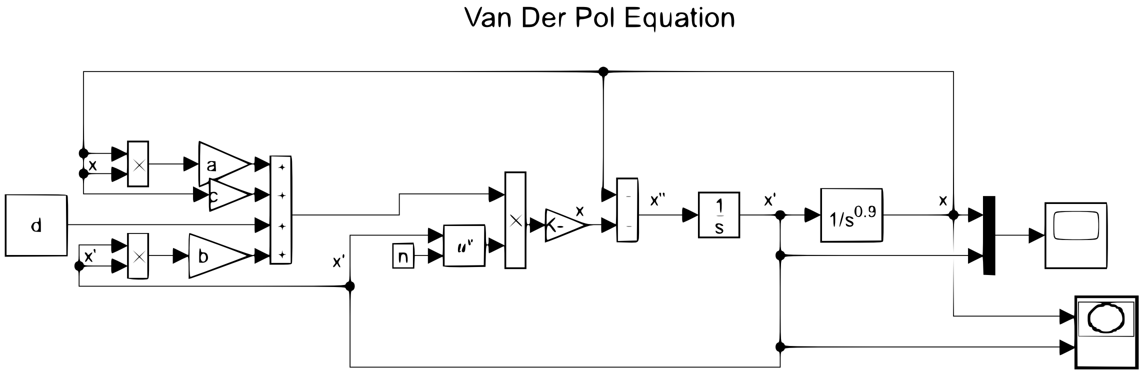
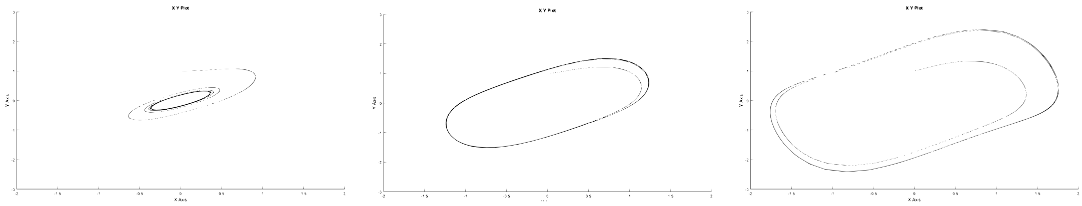


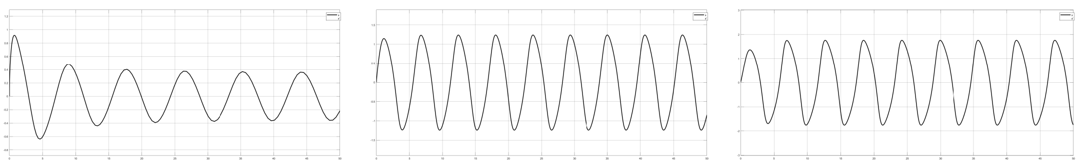
FIGURE 9. Block diagram of the the generalized fractional Van der Pol system under study.

Figure 9 shows The block diagram representation of system (4.5).



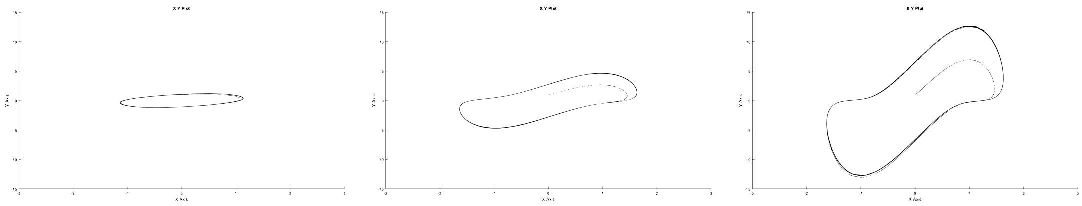
$\alpha = 0.4$ $\alpha = 0.7$ $\alpha = 0.9$

FIGURE 10. Phase portraits of (4.5) $\alpha = \{0.4, 0.7, 0.9\}, n = 1$ and $\epsilon = 1$.



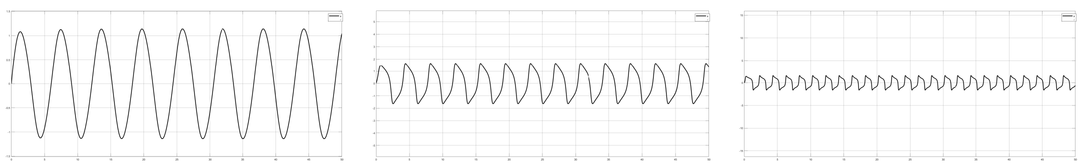
$\alpha = 0.4$ $\alpha = 0.7$ $\alpha = 0.9$

FIGURE 11. Analytical solution of VPO (4.5) such that: $\alpha = \{0.4, 0.7, 0.9\}, n = 1$ and $\epsilon = 1$.



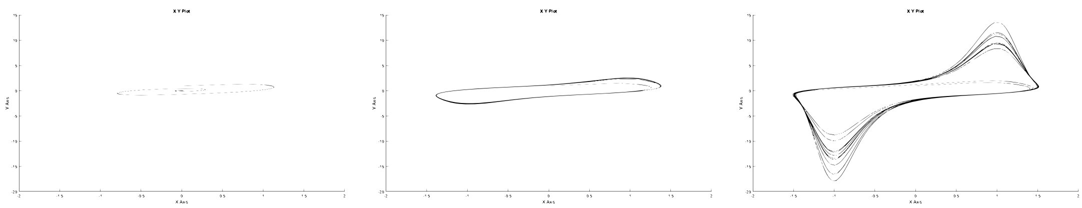
$\epsilon = 0.5$ $\epsilon = 4$ $\epsilon = 16$

FIGURE 12. Phase portraits of (4.5): $\epsilon = \{0.5, 4, 16\}, n = 1$ and $\alpha = 0.8$.



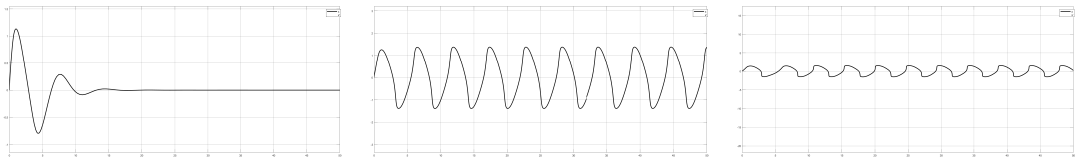
$\epsilon = 0.5$ $\epsilon = 4$ $\epsilon = 16$

FIGURE 13. Analytical solution of VPO (4.5) such that: $\epsilon = \{0.5, 4, 16\}, n = 1$ and $\alpha = 0.8$.



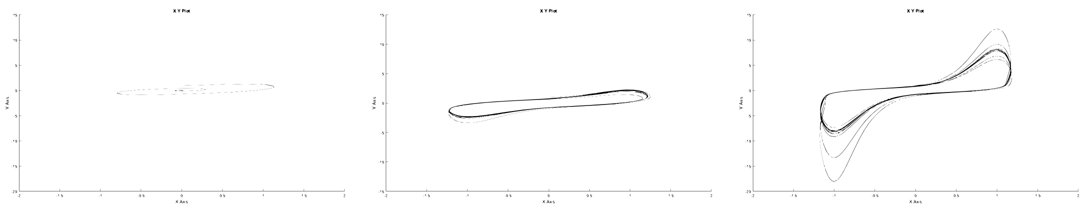
$\alpha = 0.6$ $\alpha = 0.7$ $\alpha = 0.8$

FIGURE 14. Phase portraits of (4.5) $\alpha = \{0.6, 0.7, 0.8\}, n = 3$ and $\epsilon = 1$.



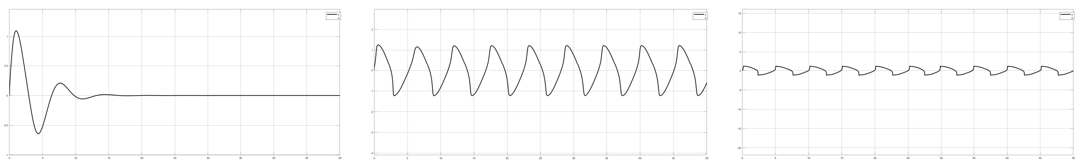
$\alpha = 0.6$ $\alpha = 0.7$ $\alpha = 0.8$

FIGURE 15. Analytical solution of VPO (4.5) such that: $\alpha = \{0.6, 0.7, 0.8\}, n = 3$ and $\epsilon = 1$.



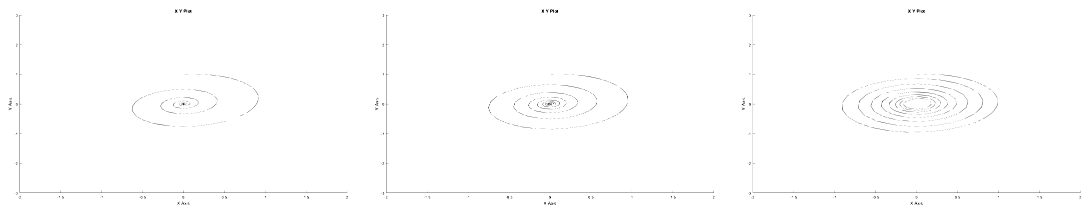
$\epsilon = 0.9$ $\epsilon = 2$ $\epsilon = 4$

FIGURE 16. Phase portraits of (4.5): $\epsilon = \{0.9, 2, 4\}, n = 3$ and $\alpha = 0.6$.



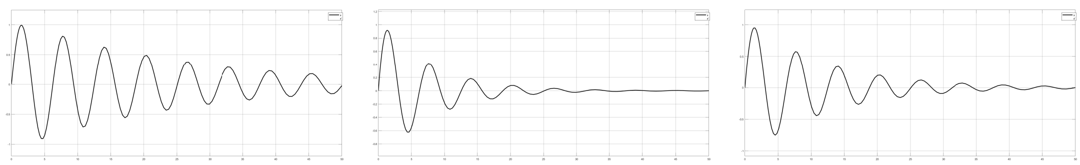
$\epsilon = 0.9$ $\epsilon = 2$ $\epsilon = 4$

FIGURE 17. Analytical solution of VPO (4.5) such that: $\epsilon = \{0.9, 2, 4\}, n = 3$ and $\alpha = 0.6$.



$\alpha = 0.85$ $\alpha = 0.9$ $\alpha = 0.95$

FIGURE 18. Phase portraits of (4.5) $\alpha = \{0.85, 0.9, 0.95\}, n = 15$ and $\epsilon = 0.1$.



$\alpha = 0.85$ $\alpha = 0.9$ $\alpha = 0.95$

FIGURE 19. Analytical solution of VPO (4.5) such that: $\alpha = \{0.85, 0.9, 0.95\}, n = 15$ and $\epsilon = 0.1$.

Figures 10 to 19 show phase space, Poincaré maps, and analytic solutions at different values of ϵ, α and n. We investigate important differences in the limit cycle, revealing a significant influence of ϵ, α and n on system dynamics.

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