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SETTING OF AMPLITUDE WITH MATÉRN PROCESS

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ABSTRACT. All diffusion processes have default amplitudes that cannot be adjusted and we propose an adjustment via the Matérn process in work of Lilly [8]. We use the WASC model of Da Fonseca [2] as a reference model and we introduce in this model the Matérn process. We obtain the adjustment but we have the major problems that we must solve: we must give the sense of the fractional integral with respect Matérn process; we church the law of model which is noised by Matérn process; we church the conditions of market without arbitrage.

1. INTRODUCTION

Let's consider a real datasets of the daily prices CAC40 and SBF120 (see in a site of trading www.boursorama.com or m.fr.investing.com). For each index, the time series start the May 12, 2021 and end the June 15, 2021 and they are presented by the following figure:

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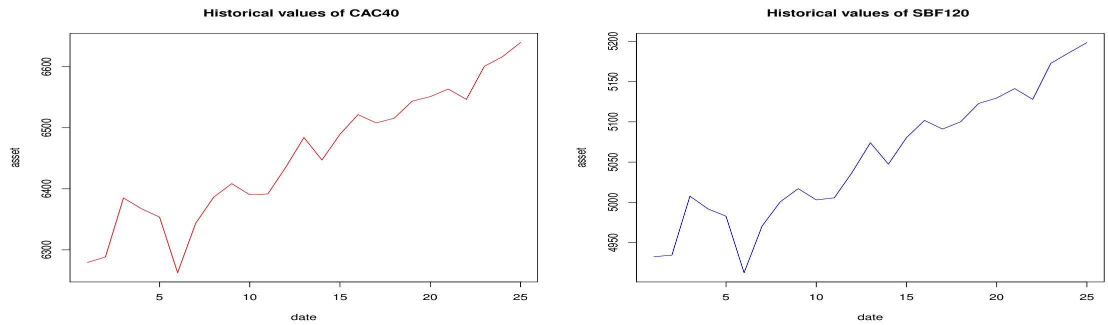

FIGURE 1. Historical Volume of CAC40 and SBF120 Indexes

Using the WASC model of Da Fonseca [2] estimated by the real datasets above and the CGMM method (see in the reference [1]), the course evolution of CAC40 and SBF120 in one daily with frequency 15 second (see in the section 6, the technique to draw the graph) is

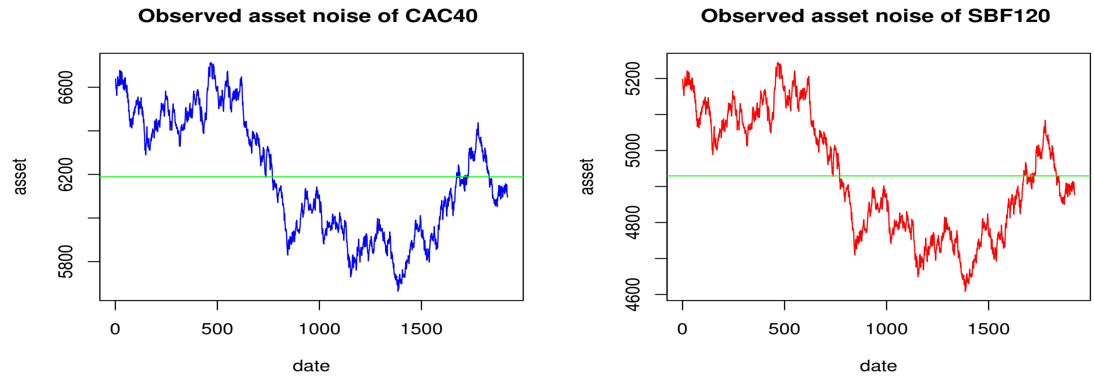

FIGURE 2. The course of WASC model in the one daily with frequency= 15 second

We remark that the graph scale of WASC model is not logic because on the market, the high value of CAC40 (resp SBF120) of June 15, 2021 is 6655.66 (resp 3393.52). However, according to the last graph, the CAC40 (resp SBF120) of WASC model in this time can take a big value greater than 6655.66 (resp 3393.52).

To do this, to adjust the course, we introduce in WASC model the truncated version of the general Matérn process in work of Lilly [8] of the form:

$$M_{t,\alpha,\beta} = \int_0^{+\infty} R(t-s,\alpha,\beta)dW_s, \tag{1.1}$$

where

$$R(t,\alpha,\beta) = \begin{cases} \frac{t^\alpha e^{-\beta t}}{G(\alpha+1)} \text{ if } t \geq 0 \\ 0 \text{ otherwise} \end{cases}, \tag{1.2}$$

$\alpha > \frac{-1}{2}$, $\beta \geq 0$, G is the Gamma function such that $G(x) = \int_0^{+\infty} u^{x-1}e^{-u}du$, where $x > 0$ and W_t is a sBm (standard Brownian motion). We referee the obtained model by Setting of Amplitude with Matérn Process (SAMP) noted $SAMP(\alpha,\beta)$:

$$\begin{cases} d\log S_t = \left(r - \frac{1}{2}vec[tr(e_{ii}\Gamma_{t^-})]\right)dt + \sqrt{\Gamma_t}dZ_t \\ d\Gamma_t = (\nu Q'Q + \Phi\Gamma_t + \Gamma_t\Phi')dt + \sqrt{\Gamma_t}d\tilde{B}_t\sqrt{Q'Q} + \sqrt{Q'Q}(d\tilde{B}_t)'\sqrt{\Gamma_t} \\ dZ_t = \sqrt{1-\rho'\rho}dM_{t,\alpha,\beta} + d\tilde{B}_t\rho \end{cases} \tag{1.3}$$

with

(i) ν is a positive integer nonzero;
(ii) Q and Φ are $n \times n$ dimensional real matrices;
(iii) r is a vector in $\mathbb{R}^n$;
(iv) If $a_1,\ldots,a_n \in \mathbb{R}$, then $vec(a_i) = (a_1,\ldots,a_n)'$ which is a vector in $\mathbb{R}^n$;
(v) e_{ii} is $n \times n$ dimensional matrix defined by $e_{ii} = (\delta_{ijk})_{j,k=1\ldots n}$, where $\delta_{ijk} = \begin{cases} 1 \text{ if } (j,k) = (i,i) \\ 0 \text{ otherwise} \end{cases}$;
(vi) Z_t is the process which defines the stochastic correlation noise between the yield $\log S_t$ and its volatility Γ_t on the continuous path;
(vii) $M_{t,\alpha,\beta}$ is a n–dimensional vector whose components are the truncated Matérn process above;
(viii) $\rho = (\rho_1,\rho_2,\ldots,\rho_n)'$, where $\rho_i \in [-1, 1]$;
(ix) $\tilde{B}_t$ is a $n \times n$ dimensional stochastic matrix whose components are the sBm;
(x) H' is the transpose of the matrix H;
(xi) $tr(H)$ is the trace of the matrix H.

Our work is to prove that this technique will fill this gap and the obtained model will also contribute the pricing option, the hedging problem and even for a wider class other than financial engineering.

2. Property of Matérn process

Let ($\mathbb{R}^n$, $\mathfrak{F}$, $(\mathfrak{F}_t)_{t\geq 0}$, $\mathbb{P}$) be a filtrate and probability space, where

(i) $\mathfrak{F}$ is a tribe on $\mathbb{R}^n$ representing the global information available on the market;

(ii) $(\mathfrak{F}_t)_{t\geq 0}$ is a family of sub–tribe of $\mathfrak{F}$ which is the information available in time t (natural filtration);

(iii) $\mathbb{P}$ is the probability density without risk.

The conditional variance of increment $M(t+h,\alpha,\beta)-M(t,\alpha,\beta)$ is given by

$$Var(M(t+h,\alpha,\beta)-M(t,\alpha,\beta)|\mathfrak{F}_t)=\frac{h^{2\alpha+1}c_{2\alpha,2\beta}(h)}{G(\alpha+1)^2}, \tag{2.1}$$

where

$$c_{x,y}(h)=\int_0^1 u^x e^{-hyu}du, \tag{2.2}$$

which is integrable for all $h\geq 0$, $x>-1$, $y\geq 0$ and

$$0\leq c_{x,y}(h)\leq \frac{1}{x+1} \text{ for all } h\geq 0,\ x>-1 \text{ and } y\geq 0. \tag{2.3}$$

Indeed,

$$\begin{aligned} Var(M(t+h,\alpha,\beta)-M(t,\alpha,\beta)|\mathfrak{F}_t) &= \int_t^{t+h}\frac{(t+h-s)^{2\alpha}e^{-2\beta(t+h-s)}}{G(\alpha+1)^2}ds \\ &= \frac{h^{2\alpha+1}c_{2\alpha,2\beta}(h)}{G(\alpha+1)^2} \text{ through the change of} \\ &\quad\ \text{variable by doing } u=\frac{t+h-s}{h}. \end{aligned}$$

3. Meaning of integral $\int_0^T\sqrt{\Gamma_s}dM_{s,\alpha,\beta}$

In the first equation of model (1.3), we must give the sens of integral $\int_0^T\sqrt{\Gamma_s}dM_{s,\alpha,\beta}$ or $\int_0^T\sigma_{kl,t}dM_{l,s}$, where $\sqrt{\Gamma_t}=(\sigma_{kl,t})_{1\leq k,l\leq n}$ and $M_{l,t}$ are the components of Matérn process $M_{t,\alpha,\beta}$.

Let's consider

$$\sum_{t_i\in\Delta}\sigma_{kl,t_i}(M_{l,t_{i+1}}-M_{l,t_i}), \tag{3.1}$$

with $\Delta = \{0 = t_0 < t_1 < \ldots < t_m = T\}$ and watch what's going when $\Delta \longrightarrow 0$.

If $\alpha = \beta = 0$, then the M_{l,t_i} are sBm which are martingales. Since the σ_{kl,t_i} are adapted continuous, then the sum (3.1) converges to $\int_0^T \sigma_{kl,s} dB_s$ through [9, Proposition 122].

Now, suppose that $\alpha > 0$ and $\beta \geq 0$. The reasoning is based on the work of Ivan Nourdin [11]. Let's admit that for all $t, t' \in [0,T]$ closer, there exist $B > 0$, $\| \sigma_{kl,t'} - \sigma_{kl,t} \| \leq B \mid t' - t \mid^{\frac{1}{2}}$ and put $T = 1$ (to simplify) and $\Delta = \Delta_m = \{k2^{-m}, k = 0, \ldots, 2^{m-1}\}$. If $t \in \Delta_m$, we denote $t' = t + 2^{-m}$ and $\tau = \frac{t+t'}{2}$. Let us $u_m = \sum\limits_{t \in \Delta_m} \sigma_{kl,t}(M_{lt'} - M_{lt})$. So, we have

$$\begin{aligned} u_{m+1} - u_m &= \sum_{t \in \Delta_m} \sigma_{kl,t}(M_{l,\tau} - M_{l,t}) + \sigma_{kl,\tau}(M_{l,t'} - M_{l,\tau}) - \\ &\quad \sum_{t \in \Delta_m} \sigma_{kl,t}(M_{l,t'} - M_{l,\tau}) + \sigma_{kl,t}(M_{l,\tau} - M_{l,t}) \\ &= \sum_{t \in \Delta_m} (\sigma_{kl,\tau} - \sigma_{kl,t})(M_{l,t'} - M_{l,\tau}). \end{aligned}$$

Thus

$$\begin{aligned} \| u_{m+1} - u_m \| &\leq \sum_{t \in \Delta_m} \| \sigma_{kl,\tau} - \sigma_{kl,t} \| \| M_{l,t'} - M_{l,\tau} \| \\ &\leq \sum_{t \in \Delta_m} B \mid \tau - t \mid^{\frac{1}{2}} \left(\mathbb{E}(M_{l,t'} - M_{l,\tau})^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{t \in \Delta_m} \frac{B 2^{-(m+1)\frac{1}{2}} 2^{-(m+1)(\alpha + \frac{1}{2})}}{G(\alpha + 1)} \\ &= \frac{B 2^{-(m+1)\alpha}}{2G(\alpha + 1)} \longrightarrow 0, \text{ as } m \nearrow \infty. \end{aligned}$$

Hence u_m converges to the Young integral in [13] $\int_0^T \sigma_{kl,s} dM_{ls}$ for $\alpha > 0$ and $\beta \geq 0$.

Now, let's prove that for all $t, t' \in [0,T]$ closer, there exist $B > 0$, $\| \sigma_{kl,t'} - \sigma_{kl,t} \| \leq B \mid t' - t \mid^{\frac{1}{2}}$.

Let $U = \left\{ (x_{11,t}, \ldots, x_{1n,t}, x_{22,t}, \ldots, x_{2n,t}, ,, , x_{nn,t}) \in \mathbb{R}^{\frac{n(n+1)}{2}} \right\}$, $x_{ii} > 0$ and the main miners of the symmetric matrix $(x_{kl})_{k,l=1\ldots n}$ are positives. Let $F : U \longrightarrow U$, $F(\sigma_{11,t}, \ldots, \sigma_{1n,t}, \sigma_{22,t}, \ldots, \sigma_{2n,t}, ,, , \sigma_{nn,t}) = (\Gamma_{11,t}, \ldots, \Gamma_{1n,t}, \Gamma_{22,t}, \ldots, \Gamma_{2n,t}, ,, , \Gamma_{nn,t})$, where $\Gamma_{..,t}$ are the components of the volatility matrix Γ_t. Then, F is a global dimorphism. Indeed, for $n = 1$, we have $U =]0, +\infty[$, and $F(\sigma) = \sigma^2$. Thus $DF = 2\sigma > 0$ on U (DF is the derivative of F). The result follows through [3, Global

Inversion Theorem]. For $n = 2$, we have $U = \{(x_{11}, x_{12}, x_{22}) \in \mathbb{R}^3 : x_{11}, x_{22} > 0; x_{11}x_{22} > (x_{12})^2\}$ and $F(\sigma_{11}, \sigma_{12}, \sigma_{22}) = ((\sigma_{11})^2 + (\sigma_{12})^2, (\sigma_{11} + \sigma_{22})\sigma_{12}, (\sigma_{22})^2 + (\sigma_{12})^2)$.

Thus, we have $\det(DF) = \det \begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & 0 \\ \sigma_{12} & \sigma_{11} + \sigma_{22} & \sigma_{12} \\ 0 & 2\sigma_{12} & 2\sigma_{22} \end{bmatrix} = 4(\sigma_{11} + \sigma_{22})(\sigma_{11}\sigma_{22} - (\sigma_{12})^2) > 0$ for all $(\sigma_{11}, \sigma_{12}, \sigma_{22}) \in U$. The result follows also using the Global Inversion Theorem. We remain for $n = 1; 2$ which is the dimension of model that we used.

So, there exist a function $g_{ki} : U \longrightarrow \mathbb{R}$ of $\mathcal{C}^1$ class such that $g_{ki}(\Gamma_t) = \sigma_{ki,t}$ and $d\sigma_{ki,t} = \sum\limits_{s,r=1}^{n} \frac{\partial g_{ki}(\Gamma_t)}{\partial \Gamma_{sr,t}} d\Gamma_{sr,t}$.

Thus, $\parallel \sigma_{kl,t'} - \sigma_{kl,t} \parallel \leq B \mid t' - t \mid^{\frac{1}{2}}$ through to SDE (Stochastic Differential Equation) of Γ_t, where

$$\begin{aligned}
B &= A \sum_{k,l=1}^{n} N_{kl} \\
A &= \sup_{\substack{t\in[0,T] \\ s,r,p,q=1,\dots,n}} \left(\left| \frac{\partial g_{sr}(\Gamma_t)}{\partial \Gamma_{pq,t}} \right| ; |\Gamma_{sr,t}| ; |g_{sr}(\Gamma_t)| \right) < \infty \text{ because the trajectory of} \\
&\quad \Gamma_t \text{ is continuous and the } g_{sr} \text{ are } \mathcal{C}^1(U) \text{ classes.} \\
N_{kl} &= \left| \nu \sum_{j=1}^{n} Q_{kj} Q_{jl} \right| + A \sum_{i=1}^{n} |\Phi_{il}| + |\Phi_{ki}| + A \sum_{i,j=1}^{n} |Q_{jl}| + |Q_{ki}| \\
\Phi &= (\Phi_{kl})_{kl}, \quad \sqrt{Q'Q} = (Q_{kl})_{kl}.
\end{aligned}$$

For $-\frac{1}{2} < \alpha \leq 0$ and $\beta \geq 0$. Through to (2.1) and (2.3), the Matérn process is a–Holder, $a \in (0, \alpha + \frac{1}{2})$. Using [10, Theorem 2.5], the Riemann sum (3.1) converges into $\int_0^T \sqrt{\Gamma_s} d^{A,\eta} M_{s,\alpha,\beta}$, called the η–order Newton–Côtes functional corrected by A of $\sqrt{\Gamma_t}$ with respect to $M_{t,\alpha,\beta}$, which is the meaning of $\int_0^T \sqrt{\Gamma_s} dM_{s,\alpha,\beta}$, where $\eta \in \mathbb{N}^*$ and A is the a–Levy's areas of order $2\eta - 2$. But we do not know if it is possible to construct a Levy area A. Fortunately, through [10, Proposition 3.5], for $\alpha + \frac{1}{2} > \frac{1}{3}$,

$$\int_0^T \sqrt{\Gamma_s} d^{A,1} M_{s,\alpha,\beta} = \int_0^T \sqrt{\Gamma_s} d^{\circ} M_{s,\alpha,\beta}.$$

The right integral is integral's Russo and Vallois [12] defined by

$$\lim_{\epsilon \longrightarrow 0} \epsilon^{-1} \int_0^T \frac{\sqrt{\Gamma_{s+\epsilon}} + \sqrt{\Gamma_s}}{2} (M_{s+\epsilon,\alpha,\beta} - M_{s,\alpha,\beta}) ds. \tag{3.2}$$

4. Laplace transform of model

We will use CGMM method to estimate the model parameter and this method needs Laplace transform.

Let γ be a vector in $\mathbb{R}^n$. The conditional Laplace transform of $\log S_T | \mathfrak{F}_t$ is defined by:

$$\Psi_{\log S}(\gamma, t, T) = \mathbb{E}\{e^{\gamma' \log S_T} | \mathfrak{F}_t\}. \tag{4.1}$$

Proposition 4.1. *We have*

$$\Psi_{\log S}(\gamma, t, T) = e^{tr(A(h)\Gamma_t) + B(h) \log S_t + C(h)} \tag{4.2}$$

with $h = T - t$ *and* $A(h)$, $B(h)$, $C(h)$ *are the functions defined by:*

$$\begin{aligned} A(h) &= F(h)^{-1} G(h), \\ B(h) &= \gamma', \\ C(h) &= tr \left[r\gamma' h + -\frac{\nu}{2} \left(\log F(h) + h \frac{\Phi + \Phi'}{2} \right) \right] \end{aligned}$$

where ($G(h)$*,* $F(h)$*) is defined by:*

$$\begin{aligned} \begin{bmatrix} G(h) \\ F(h) \end{bmatrix} &= exp \left(\begin{bmatrix} \frac{\Phi+\Phi'}{2} h & -2hQ'Q \\ \Upsilon_1(h) & -\frac{\Phi+\Phi'}{2} h \end{bmatrix} \right) \begin{bmatrix} 0 \\ I_n \end{bmatrix}; \\ \Upsilon_1(h) &= -\frac{h}{2} \sum_{i=1}^n \gamma_i e_{ii} + \frac{(1 - \rho'\rho) h^{2\alpha+1} c_{2\alpha,2\beta}(h)}{2G(\alpha+1)^2} \gamma\gamma' - \\ &\quad \frac{h}{2} (\Phi + \Phi') \left(\sqrt{Q'Q} \right)^{-1} (\rho\gamma'). \end{aligned}$$

Proof.

$$\begin{aligned} \Psi_{\log S}(\gamma, t, T) &= \mathbb{E} \left[e^{\gamma' \left[\log S_t + \int_t^T \left(r - \frac{1}{2} vec[tr(e_{ii}\Gamma_s)] \right) ds + \int_t^T \sqrt{\Gamma_s} dZ_s \right]} | \mathfrak{F}_t \right] \\ &= e^{\gamma'(log S_t + r(T-t))} \mathbb{E} \left[e^{\gamma' \left[\int_t^T -\frac{1}{2} vec[tr(e_{ii}\Gamma_s)] ds + \int_t^T \sqrt{\Gamma_s} dZ_s \right]} | \mathfrak{F}_t \right] \end{aligned} \tag{4.3}$$

Since the coefficient of $log S_t$ is only γ', then $B(h) = \gamma' \; \forall h$.

We have

$$\begin{aligned} & \mathbb{E}\left[e^{\gamma'\left[\int_t^T -\frac{1}{2}vec[tr(e_{ii}\Gamma_s)]ds+\int_t^T\sqrt{\Gamma_s}dZ_s\right]}|\mathfrak{F}_t\right] \\ = \quad & \mathbb{E}\left[e^{\int_t^T -\frac{1}{2}\gamma' vec[tr(e_{ii}\Gamma_s)]ds+\sqrt{1-\rho\rho'}\gamma'\int_t^T\sqrt{\Gamma_{s-}}dM_{s,\alpha,\beta}+\gamma'\int_t^T\sqrt{\Gamma_s}d\tilde{B}_s\rho}|\mathfrak{F}_t\right] \\ & \text{through to } dZ_t = \sqrt{1-\rho'\rho}dM_{t,\alpha,\beta}+d\tilde{B}_t\rho \\ = \quad & \mathbb{E}\left[e^{\int_t^T -\frac{1}{2}\gamma' vec[tr(e_{ii}\Gamma_s)]ds+\frac{(1-\rho'\rho)}{2G(\alpha+1)^2}\int_t^T(T-s)^{2\alpha}e^{-2\beta(T-s)}\gamma'\Gamma_s\gamma ds+\int_t^T\gamma'\sqrt{\Gamma_s}d\tilde{B}_s\rho}|\mathfrak{F}_t\right] \end{aligned} \tag{4.4}$$

because $\int_t^T\sqrt{\Gamma_s}dM_{s,\alpha,\beta}$ is Gaussian random variable and Γ_t is independent of $M_{t,\alpha,\beta}$. We notice that $\int_t^T\sqrt{\Gamma_s}d\tilde{B}_s$ remains in function of $\tilde{B}_s$ after conditionally because $\sqrt{\Gamma_t}$ is defined in function of $\tilde{B}_t$.

By integrating the SDE (Stochastic Differential Equation) of Γ_t between (t,T), we have

$$\begin{aligned} \Gamma_T = \Gamma_t + \nu Q'Q(T-t) + \int_t^T(\Phi\Gamma_s+\Gamma_s\Phi')ds \\ + \int_t^T\left(\sqrt{\Gamma_s}d\tilde{B}_s\sqrt{Q'Q}+\sqrt{Q'Q}(d\tilde{B}_s)'\sqrt{\Gamma_s}\right). \end{aligned}$$

So

$$\begin{aligned} \int_t^T\gamma'\sqrt{\Gamma_s}d\tilde{B}_s\rho = \frac{1}{2}\gamma'\left[\Gamma_T-\Gamma_t-\nu Q'Q(T-t)\right. \\ \left. -\int_t^T(\Phi\Gamma_s+\Gamma_s\Phi')ds\right]\left(\sqrt{Q'Q}\right)^{-1}\rho. \end{aligned}$$

By inserting the expression of $\int_t^T\gamma'\sqrt{\Gamma_s}d\tilde{B}_s\rho$ into (4.4), we have

$$(4.4) \quad = \quad e^{-\frac{1}{2}\gamma'\Gamma_t\left(\sqrt{Q'Q}\right)^{-1}\rho-\frac{\nu(T-t)}{2}\gamma'\sqrt{Q'Q}\rho}\mathbb{E}\left[f(\Gamma_T)e^{-\int_t^T g(s,\Gamma_s)ds}|\mathfrak{F}_t\right], \tag{4.5}$$

where

$$\begin{aligned} f(\Gamma_T) \quad &= \quad e^{\frac{1}{2}\gamma'\Gamma_T\left(\sqrt{Q'Q}\right)^{-1}\rho} \\ g(s,\Gamma_s) \quad &= \quad \frac{1}{2}\gamma' vec[tr(e_{ii}\Gamma_s)] - \frac{(1-\rho'\rho)(T-s)^{2\alpha}e^{-2\beta(T-s)}}{2G(\alpha+1)^2}\gamma'\Gamma_s\gamma + \\ & \quad\quad \frac{1}{2}\gamma'(\Phi\Gamma_s+\Gamma_s\Phi')\left(\sqrt{Q'Q}\right)^{-1}\rho. \end{aligned}$$

To obtain the final formula of Ψ, we have need to calculate the following expression

$$X(t,\Gamma_t) \;=\; \mathbb{E}\left[f(\Gamma_T)e^{-\int_t^T g(s,\Gamma_s)ds}|\mathfrak{F}_t\right].$$

According the Feynmann–Kac's argument, $X(t,\Gamma_t)$ should fill the following multidimensional partial differential equation

$$\frac{\partial X(t,\Gamma_t)}{\partial t} + tr\left[2\Gamma_t Q'Q\frac{\partial^2 X(t,\Gamma_t)}{\partial\Gamma^2} + (\nu Q'Q + \Phi'\Gamma_t + \Gamma_t\Phi)\frac{\partial X(t,\Gamma_t)}{\partial\Gamma}\right] - g(t,\Gamma_t)X(t,\Gamma_t) = 0, \tag{4.6}$$

with boundary condition $X(T,\Gamma_t) = f(\Gamma_T) = e^{\frac{1}{2}\gamma'\Gamma_T\left(\sqrt{Q'Q}\right)^{-1}\rho}$.

As Γ_t is affine, then there exists the functions $a(h)$ and $b(h)$ such that

$$X(t,\Gamma_t) = e^{tr[a(h)\Gamma_t]+b(h)}, \tag{4.7}$$

where $h = T - t$, $b(0) = 0$ and $a(0) = \frac{1}{2}\left(\sqrt{Q'Q}\right)^{-1}\rho\gamma'$. Through the expression (4.6) and identifying on the basis vector $(1,\Gamma_t)$, we have

$$\begin{aligned}
\frac{\partial a(h)}{\partial h} &= 2Q'Qa(h)^2 + (\Phi+\Phi')a(h) - \frac{1}{2}\sum_{i=1}^{n}\gamma_i e_{ii} + \frac{(1-\rho'\rho)h^{2\alpha}e^{-2\beta h}}{2G(\alpha+1)^2}\gamma\gamma' \\
&\quad -\frac{1}{2}(\Phi+\Phi')\left(\sqrt{Q'Q}\right)^{-1}\rho\gamma' \\
\frac{\partial b(h)}{\partial h} &= tr[\nu Q'Qa(h)].
\end{aligned}$$

Hence $(4.5) = e^{tr[A(h)\Gamma_t + D(h)]}$, where $A(h)$ is solution of SDE:

$$\begin{aligned}
\frac{\partial A(h)}{\partial h} &= 2A(h)Q'QA(h) + \frac{\Phi+\Phi'}{2}A(h) + A(h)\frac{\Phi+\Phi'}{2} + \Xi(h) \text{ and} \\
D(h) &= tr\left(\int_0^h \nu Q'QA(u)du\right),
\end{aligned}$$

where $\Xi(h) = -\frac{1}{2}\sum\limits_{i=1}^{n}\gamma_i e_{ii} + \frac{(1-\rho'\rho)h^{2\alpha}e^{-2\beta u}}{2G(\alpha+1)^2}\gamma\gamma' - \frac{1}{2}(\Phi+\Phi')(\sqrt{Q'Q})^{-1}(\rho\gamma')$ and $A(0) = 0$.

Let us

$$A(h) = F(h)^{-1}G(h) \text{ with } F(h) \in GL_n(\mathbb{R}) \text{ and } G(h) \in \mathcal{M}_n(\mathbb{R}). \tag{4.8}$$

We have $0 = A(0) = F(0)^{-1}G(0)$. In this case, we take $G(0) = 0$ and $F(0) = I_n$.

Since $\frac{\partial[F(h)A(h)]}{\partial h} = \frac{\partial F(h)}{\partial h}A(h) + F(h)\frac{\partial A(h)}{\partial h}$, we have

$$\begin{aligned}
&\frac{\partial G(h)}{\partial h} - \frac{\partial F(h)}{\partial h}A(h) = F(h)\frac{\partial A(h)}{\partial h} \\
= \;& F(h)\Xi(h) + G(h)\frac{\Phi+\Phi'}{2} + F(h)\frac{\Phi+\Phi'}{2}A(h) + G(h)(2Q'Q)A(h).
\end{aligned}$$

So,

$$\begin{cases}
\frac{\partial G(h)}{\partial h} = G(h)\frac{\Phi+\Phi'}{2} + F(h)\Xi(h) \\
\frac{\partial F(h)}{\partial h} = -2G(h)Q'Q - F(h)\frac{\Phi+\Phi'}{2}.
\end{cases}$$

Thus,

$$\frac{\partial}{\partial h}\begin{bmatrix} G(h) \\ F(h) \end{bmatrix} = \begin{bmatrix} \frac{\Phi+\Phi'}{2} & -2Q'Q \\ \Xi(h) & -\frac{\Phi+\Phi'}{2} \end{bmatrix}\begin{bmatrix} G(h) \\ F(h) \end{bmatrix},$$

and

$$\begin{aligned}
D(h) &= tr\left(\int_0^h \nu Q'QF(u)^{-1}G(u)du\right) \\
&= tr\left(\int_0^h \frac{-\nu}{2}F(u)^{-1}\frac{\partial F(u)}{\partial u} - \frac{\nu}{2}\frac{\Phi+\Phi'}{2}\right) \\
&= tr\left[-\frac{\nu}{2}\left(\log F(h) + h\frac{\Phi+\Phi'}{2}\right)\right].
\end{aligned}$$

□

Let's see now the impacts of Matérn process. Firstly, the covariance between yield and its volatility and between yield and its correlation are the same ones of WASC model in [2]:

$$\begin{aligned}
cov(d\log S_{i,t}, d\Gamma_{ii,t}) &= 2dt\Gamma_{ii,t}\sum_{l=1}^{n} Q_{li}\rho_l \\
cov(d\log S_{p,t}, d\zeta_{pq,t}) &= dt\sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}}\sum_{l=1}^{n} Q_{lq}\rho_l(1-\zeta_{pq,t}^2),
\end{aligned} \tag{4.9}$$

where $i = 1,\dots,n$; $p,q = 1,\dots,n$ and $p \neq q$, the ρ_i are the components of ρ, $S_{i,t}$ is the price of stock i at time t and $\zeta_{pq,t}$ is the correlation between $\Gamma_{pp,t}$ and $\Gamma_{qq,t}$ defined by:

$$\zeta_{pq,t} = \frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}. \tag{4.10}$$

Secondly, the Matérn process has an influence on the option price, see the Tables 4 and 5.

5. THE MARKET WITH RESPECT MATÉRN PROCESS WITHOUT ARBITRAGE

We know that for a market noised by a fractional Brownian motion, Guasoni [5] showed that by paying the transaction at a rate $\epsilon > 0$, the market is without arbitrage. Nothing says that it is valid for the market noised by a Matérn process, hence the following Proposition.

Proposition 5.1. *Let* $S_t = e^{\sigma M_{t,\alpha,\beta}+L_t}$*,* $t \in [0,T]$ *a market with respect Matérn process, where* $\alpha > \frac{-1}{2}$*,* $\beta > 0$*,* $\sigma > 0$ *and* L_t *is a continuous function. Then, the market is arbitrage free with transaction costs* $\epsilon > 0$ *on the finite interval* $[0,T]$.

Proof. Guasoni in [5] showed the equivalence: (a) the absence of arbitrage with general strategies for arbitrarily small transaction costs $\epsilon > 0$, (b) the absence of free lunches with bounded risk for arbitrarily small transaction costs $\epsilon > 0$, and (c) the existence of ϵ–consistent price systems – the analogue of martingale measures under transaction costs – for arbitrarily small $\epsilon > 0$. And Guasoni in [6] showed that if a continuous price process has conditional full support, then it admits consistent price systems for arbitrarily small trans-action costs. Hence, we will show that $S_t = e^{\sigma M_{t,\alpha,\beta}+L_t}$, $t \in [0,T]$ satisfies the conditional full support condition.

According [7, Theorem 3], the topological support of $M_{t,\alpha,\beta}$ which is a continuous Gaussian process with covariance function $\bar{\sigma}(t,s) = cov(M_{t,\alpha,\beta}, M_{s,\alpha,\beta}) = \int_0^{+\infty} R(t-u,\alpha,\beta)R(s-u,\alpha,\beta)du$, is equal to the closure under the uniform norm of the corresponding Reproducing Kernel Hilbert Space (RKHS) defined by

$$\mathcal{H} := \left\{ f \in C_0[0,T], f(t) = \int_0^{+\infty} R(t-s,\alpha,\beta)g(s)ds, g \in L^2[0,T] \right\}, \tag{5.1}$$

where $C_0[0,T]$ is the set of continuous functions f on $[0,T]$, $f(0) = 0$. Thus, it sufficient to show that $\mathcal{H}$ is norm–dense in $C_0[0,T]$.

Let $g(s) = s^{\gamma}e^{\beta(t-s)}$, $\gamma > \frac{-1}{2}$. We have

$$\begin{aligned} f(t) &= \int_0^{+\infty} R(t-s,\alpha,\beta)g(s)ds = \frac{1}{G(\alpha+1)}\int_0^t s^\gamma (t-s)^\alpha ds \\ &= t^{\gamma+\alpha+1}\frac{G(\gamma+1)}{G(\gamma+\alpha+2)}. \end{aligned}$$

Since $\gamma > \frac{-1}{2}$ is arbitrary, then $\mathcal{H}$ contains the algebra of all polynomials with order greater than one. This algebra separates $]0,T]$ and is null only at zero. Through the Stone–Weierstra$\mathcal{B}$ Theorem, $\mathcal{H}$ is dense in $C_0[0,T]$. $\square$

Hence, when we use on the pricing option or hedging, we take $r = \iota\check{1}$ in the model (1.3), where ι is the interest rate on the market and $\check{1}$ is the n–dimensional vector whose components are equal to 1 and the transaction is paid with rate $\epsilon > 0$.

6. Estimating of the parameters of model

In the google site on trading (examples, www.bursorama.com or m.fr.investing.com), we can find the historical values of the course we want (CAC40, SP500, SBF120, Nasdaq,...). We can also have the high and low value for each day, the data frequency, the market opening and closing,...

We use this information to estimate the parameter of our model. To do this, we pass by following two-steps to estimate the parameters of the model:

(i) We estimate the parameters Φ, φ, ρ, ν and $\sqrt{Q'Q}$ using the WASC model, the CGMM method and the real data with step–time T which is the expiration time of option. That is, the type of data we are going to use depends on the expiration time T. For instance, if T= 1 day, we use the daily prices; if T= 1 week, we use the weekly prices and if T=1 month, we use the monthly prices.

(ii) After, we adjust the course with the parameters α and β to correspond the reality on the future period $[0,T]$. In fact, if T is the expiration of the market, we take $t=0$ the time of signing of contract and $t=1$ the time of its expiration. Thus, we draw the price in $[0,1]$ and we regulate by graph the evolution of the price to correspond to the reality. We use the Euler's discretization of the SDE (1.3) to draw the graph of future price of model. To be realistic, this course should not take a value much higher than the high value taken by the current real price and likewise, it should not take a value much lower than the low value taken by the real price. Since we estimate Γ_t with the step–time T according to i), then we can take $\Gamma_t = \Gamma_0$ over $[0, 1[$. What makes it easy to graph the price.

Let's move on the examples to understand this method.

Example 1. *Let's stay on a market CAC40 previously (see in the Introduction) of 16 June, 2021 which open at 9:00 AM and close at 5:35 PM. We want to know the price of June 16, 2021 from the opening until the closing of the market according to our model. To do this, we want the following data: historical daily price of CAC40 and SBF120 (see, Figure 1), high and low value of CAC40 of 15 June, 2021 (high value: 6655.46; low value: 6634.45), frequency of data (15 second). After collecting the data, we move on to estimating the model. We skip step 1 of the estimation. An application of this step can be seen in section 7 above one can consult the theory in the reference [1]. In the step 2, if we take* $\Gamma_t = \Gamma_0$ *in* $[0, 1]$, *then we draw the course graph in the interval* $[0, 1]$ *using the step–time =* $\frac{1}{1922}$ *and the Euler's discretization of return:*

$$\log S_{l,t_{i+1}} - \log S_{l,t_i} = \frac{r - \frac{1}{2}\Gamma_{ll,0}}{1922} + \frac{\sqrt{1-\rho'\rho}\sqrt{c_{2\alpha,2\beta}(\frac{1}{1922})}}{1922^{\alpha}G(\alpha+1)} \left(\sqrt{\Gamma_0}Z\right)_l + \left(\sqrt{\Gamma_0}\tilde{Z}\rho\right)_l,$$

where $l = 1, 2$, $\Delta = \{0 = t_0 < t_1 < \ldots < t_{1922} = 1\}$ *are the subdivisions of* $[0, 1]$, *where* $0 = t_0$ *represents the time 9:00 AM,* t_1 *represents the time 9:00:15 AM,* t_2 *represents the time 9:00:30 AM,... and* $1 = t_{1922}$ *represents the time 5:35 PM;* $\tilde{Z}$ *is the* $n \times n$ *dimensional matrix whose components are the Gaussian random variables* $N(0, \frac{1}{1922})$; *Z is the n-dimensional vector whose components are the Gaussian random variables* $N(0, \frac{1}{1922})$; $\left(\sqrt{\Gamma_0}Z\right)_l$ *is the l-th line of vector* $\sqrt{\Gamma_0}Z$ *and* $\left(\sqrt{\Gamma_0}\tilde{Z}\rho\right)_l$ *is the l-th line of vector* $\sqrt{\Gamma_0}\tilde{Z}\rho$. *After, we adjust the course with the parameters* α *and* β *so that it does not take a false value much greater than 6655.46 and much less than 6634.45 on* $[0, 1]$.

Example 2. *For T= 7 days (1 week), we use the weekly prices to estimate the CGMM estimators. And the procedure is the same as above but instead of 1922, we subdivide the interval* $[0, 1]$ *into* 11454.

Remark 6.1. *As soon as we take* $\Gamma_t = \Gamma_0$ *over* $[0, 1[$, *the hypothesis is open to criticism because it is the same as that of Black and Scholes (constancy volatility). So to avoid* *this, we draw the course from the Euler's discretization of the SDE (1.3) by varying* Γ_t. *It is possible because the obtained Euler's discretization is a recurring iteration.*

7. Application

Here, we will look the course evolution of CAC40 and SBF120 with SAMP model. Thus we will estimate the parameters of model using the real data.

Since our return's dynamic follows the normal distribution, then our model is more efficient if the density of each yield using data should be also Gaussian. To do this, we will test the normality of each yield.

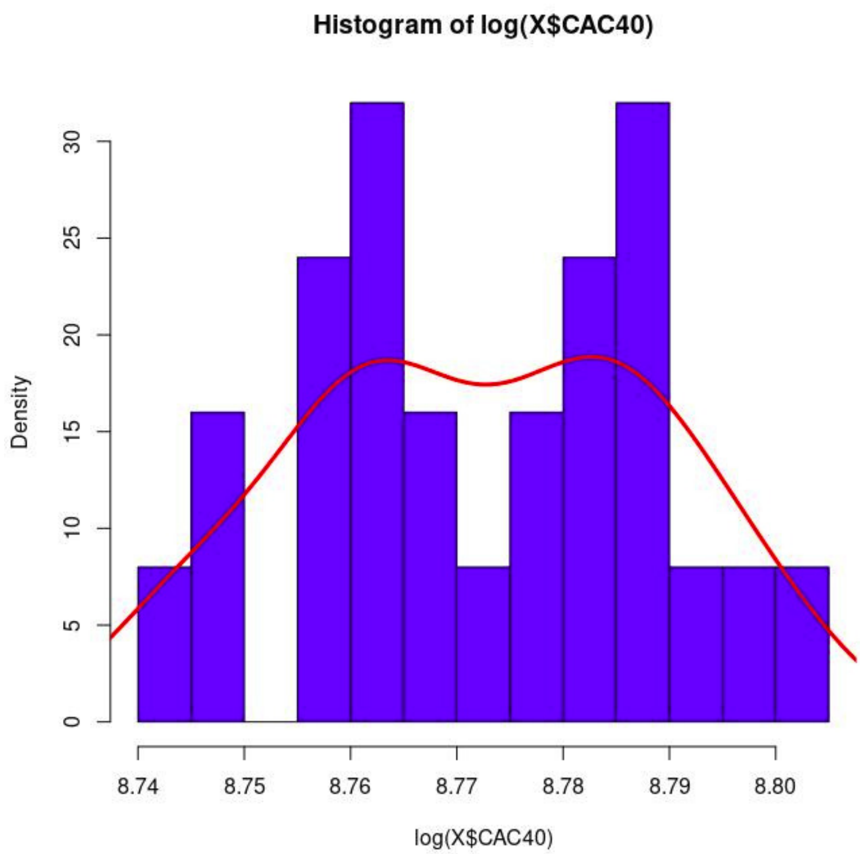

FIGURE 3. Histogram of density of $log(CAC40)$

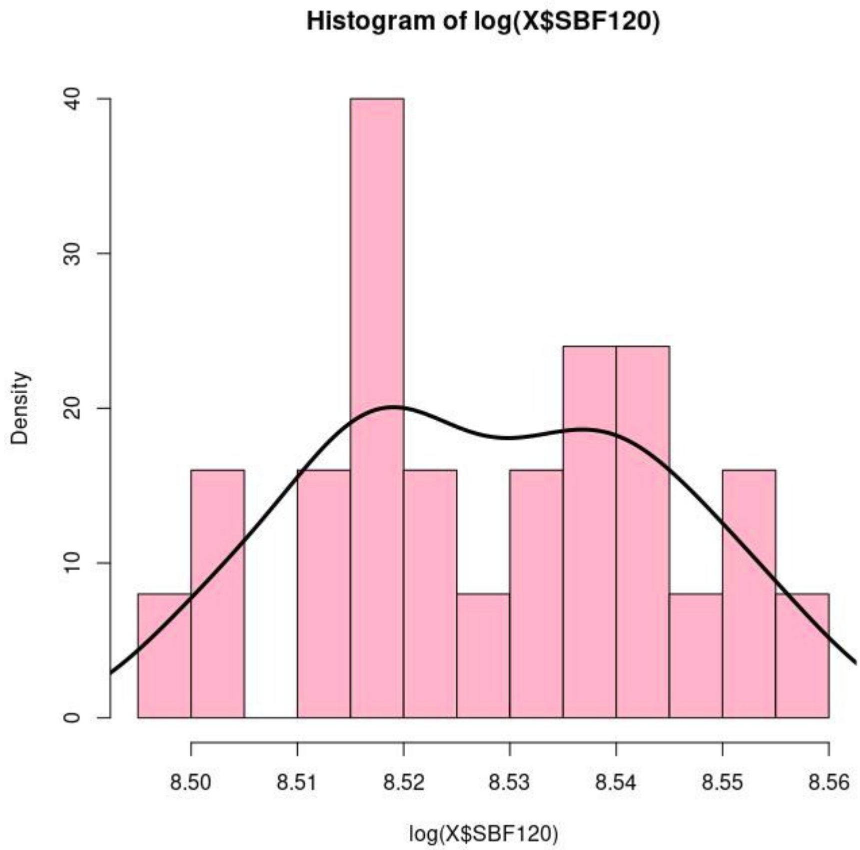

FIGURE 4. Histogram of density of $log(SBF120)$

The normality of sample is represented by these histograms. Now, we will look in the whole.

TABLE 1. Shapiro-Wilk normality test

Indice	Statistic	p-value
log (SBF120)	0.96686	0.5668
log (CAC40)	0.9655	0.5343

We see according this test that the normality hypothesis of variables H_0 is likely at most 50 %.

Step 1: Estimating of Φ, ρ, ν and $\sqrt{Q'Q}$

The details of C.GMM method are in the reference [1].

7.1. **Monte Carlo study.**

The initial parameters used in the simulation are:

$$\Gamma_0 = \begin{bmatrix} 0.0225 & -0.0054 \\ -0.0054 & 0.0144 \end{bmatrix}, \quad \Phi = \begin{bmatrix} -5 & -0.5 \\ -0.5 & -5 \end{bmatrix};$$

$$\rho = (-0.3, -0.4); \quad \nu = 15;$$

$$\sqrt{Q'Q} = \begin{bmatrix} 0.1204 & -0.01097 \\ -0.01097 & 0.09549 \end{bmatrix}.$$

The matrix $\sqrt{Q'Q}$ is obtained by using the stationary relationship of WASC model (cf. [2]):

$$\Gamma_\infty \Phi' + \Phi \Gamma_\infty = -\nu Q'Q.$$

The annual interest rate ι is taken in the range $[0.015, 0.0175]$ which is a daily interest rate 0.000045.

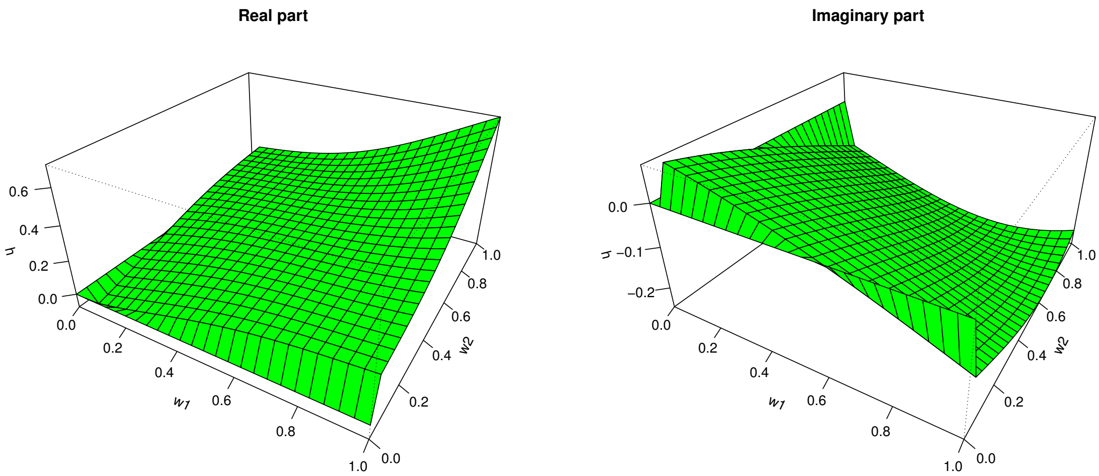

FIGURE 5. C-GMM estimation criterion

The figure show us the values taken by real and imaginary part of the empirical moment of continuum of C-GMM method. It show us that we can minimize its function.

TABLE 2. C-GMM estimator $\hat{\theta}_1$.

parameter	ρ_1	ρ_2	Q_{11}	Q_{12}	Q_{22}
estimator	3.797461×10^{-4}	3.753161×10^{-4}	0.1	0	0.1

Φ_{11}	Φ_{12}	Φ_{21}	Φ_{22}	ν
-131.7092	-0.1	-0.1	-133.1997	2

with objective 1.077699×10^{-5}.

The Table 3 below presents the estimated parameters of model with these standard deviations errors.

TABLE 3. C-GMM estimator $\hat{\theta}$

parameter	estimator	standard deviation error
ρ_1	-0.00294512	2.771415×10^{-20}
ρ_2	-0.00351427	3.204106×10^{-21}
Q_{11}	0.1	0.2273541
$Q_{12} = Q_{21}$	0.079842	0.2298414
Q_{22}	0.1	0.1738339
Φ_{11}	-21.5248763	1.315346
Φ_{12}	-0.1	0.1323577
Φ_{21}	-0.1	0.1074781
Φ_{22}	-20.2548135	0.02805844
ν	2	0.7796776

with objective 1.546725×10^{-5}.

From the expression of correlations, the asset and its volatility (resp correlation) are negatively correlated.

Step 2: Estimating of setting parameters α and β

When we calibrate the course, we found $\alpha = 0.35$ and $\beta = 0$ and the course evolution is

The scale decreases and it is tolerable.

7.2. **European call option of the basket CAC40 and SBF120.** Let be a European call of the basket of indexes (CAC40, SBF120) and note by (K_1, K_2) the strike of index quoted by points. We use the correlation and spread options in the reference [4].

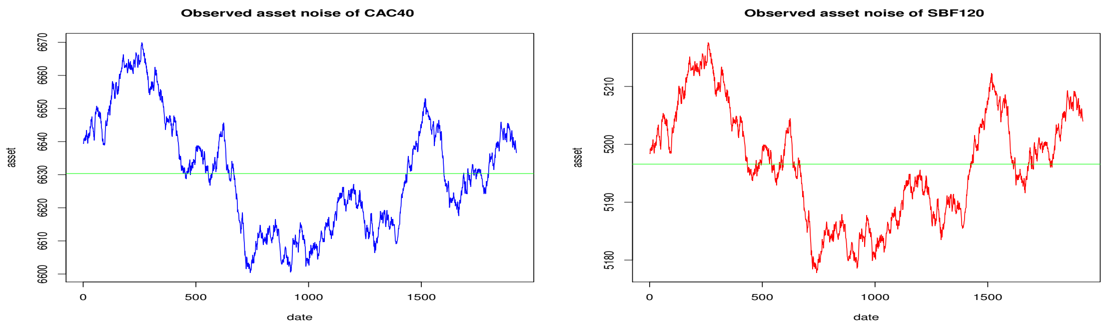

FIGURE 6. The course evolution in the one daily with SAMP$(0, 35;$ 0) and frequency= 15 second

Take $N = 622$; $\epsilon_1 = -3$; $\epsilon_2 = 1$; strike $K = 1409.705$ and $\bar{u} = 38.51832$.

TABLE 4. Spread option

Maturity	T=1 day	2	3	4	5
WASC	141.1157	141.1641	141.1875	141.191	141.1783
SAMP(0.35, 0)	141.0005	141.2334	141.2558	141.1371	140.9325

For correlation option, we take $\bar{u} = 40$; $N = 2^9$; $\alpha_1 = 0.3$ and $\alpha_2 = 0.4$; Strike $(K_1, K_2) = (\ 6610.797, 5223.067)$.

TABLE 5. Correlation option

Maturity	T=1 day	2	3	4	5
WASC	7009.402	12027.28	17294.14	22697.83	28174.37
SAMP(0.35, 0)	5536.392	14967.93	29817.2	49646.92	74155.16

The scale regularization of assets is important because the scale of course is rectified and the price option changes.

8. CONCLUSION

In this article, we have suggested adjusting the price to reality before using for the pricing option or hedging. To do this, we use the WASC model and the Matérn process instead of mBs to do the calibration. So we have a new more realistic model and we get several theories on the stochastic calculus and the market without arbitrage with respect Matérn process.

The suggested reparametrization may be useful for a wider class of models beyond financial engineering e.g. the management of the smart grid and oil production.

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