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A 3D-2D ASYMPTOTIC ANALYSIS OF ELASTIC PROBLEM WITH NONLINEAR DISSIPATIVE AND SOURCE TERMS

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ABSTRACT. In this paper, we consider the asymptotic analysis for the elasticity problem with a dissipative and source terms in a three dimensional thin domain Ω^ε. Firstly, we obtain the variational formulation of the problem. Then we establish some estimates independent of the parameter ε. Finally, we give a specific Reynolds equation associated and prove the uniqueness of the limit problem.

1. INTRODUCTION

In this paper, we study the asymptotic analysis of a problem of a linear elasticity with a dissipative and source terms in a thin dimain $\Omega^\varepsilon \subset \mathbb{R}^3$ with Tresca and Dirichlet boundary conditions. The boundary of the domain is decomposed as $\partial\Omega^\varepsilon = \Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$, where ω is the bottom of the domain, Γ_1^ε is the upper surface and Γ_L^ε is the lateral surface. Similar studies have been made by several authors but with the usual boundary conditions, we cite for exemple: The asymptotic analysis of the solutions of a linear viscoelastic problem with a dissipative and source terms in a three-dimensional thin domain was studied in [1]. In [2], The

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authors studied the asymptotic behavior of an elasticity problem with a nonlinear dissipative term in a bidimensional thin domain. The study of the asymptotic analysis of a frictionless contact betwenn two elastic bodies in a three-dimensional thin domain has been considered in [3]. The authors In [10] worked the asymptotic convergence of a dynamical problem of a non isothermal linear elasticity with friction of Tresca type.

Recently, the asymptotic analysis of an incompressible fluid in a three-dimensional thin domain has attracted the attention of many researchers, when one dimension of the fluid domain tends to zero, (see e.g., [5,7,8]) and the references cited therein. Also, some authors have studied the asymptotic analysis of a dynamical problem of isothermal elasticity with non linear friction of Tresca type but without the intervention of the nonlinear term see for instance [9].

The work is organized as follows. In Section 2 we present some notations and give the problem statement and variational formulation. In section 3, by a scale change $z = \dfrac{x_3}{\varepsilon}$, we transform the initial problem posed in the domain Ω^ε into a new problem posed in a fixed domain Ω independent of the parameter ε. Then, we find some estimates on the displacement. In section 4, the limit problem with a specific weak form of the Reynolds equation are studied.

2. Problem statement and variational formulation

Let Ω^ε be a bounded domain of $\mathbb{R}^3$, where ε is a small parameter that will tend to zero. The boundary of Ω^ε will be denote by $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$ with $\bar{\Gamma}_1^\varepsilon$ is the upper surface for equation $x_3 = \varepsilon h(x') = \varepsilon h(x_1, x_2)$, Γ_L^ε is the lateral boundary and ω is a bounded domain of $\mathbb{R}^3$ of equation $x_3 = 0$ which constitutes the bottom of the domain Ω^ε. We suppose that h is a function of class C^1 defined on ω such that $0 < h_* = h_{\min} \leq h(x') \leq h_{\max} = h^*,\ \forall (x', 0) \in \omega$. The domain Ω^ε is given by

$$\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3, (x', 0) \in \omega,\ \ 0 < x_3 < \varepsilon h(x')\}.$$

Let $T > 0$. In the time interval $]0, T[$, the law of elastic behavior is given by

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + \lambda d_{kk}(u^\varepsilon)\,\delta_{ij},$$

where μ and λ are the Lamé coefficients, u^ε is the displacement field, σ^ε the stress tensor, δ_{ij} is the Krönecker symbol and

$$d_{ij}\left(u^{\varepsilon}\right)=\frac{1}{2}\left(\frac{\partial u_i^{\varepsilon}}{\partial x_j}+\frac{\partial u_j^{\varepsilon}}{\partial x_i}\right),\ 1\leq i,j\leq 3,$$

is the symetric deformation tensor. We denote by $n=(n_1,n_2,n_3)$ the unit outward normal vector on Γ^{ε}. The normal and the tangential components of u^{ε} on the boundary ω are

$$u_n^{\varepsilon}=u^{\varepsilon}.n \text{ and } u_{\tau}^{\varepsilon}=u^{\varepsilon}-(u_n^{\varepsilon})n.$$

Also, for a regular function σ^{ε}, we define its normal and tangential components of σ^{ε} on the boundary ω given by

$$\sigma_n^{\varepsilon}=(\sigma^{\varepsilon}.n).n \ \text{ and } \sigma_{\tau}^{\varepsilon}=\sigma^{\varepsilon}.n-(\sigma_n^{\varepsilon})n.$$

The complete problem consists to find the displacement field $u^{\varepsilon}:\Omega^{\varepsilon}\times\,]0,T[\,\to\mathbb{R}^3$ such that

(1) $$\frac{\partial^2 u^{\varepsilon}}{\partial t^2}-\text{div}(\sigma^{\varepsilon}\left(u^{\varepsilon}\right))+\alpha^{\varepsilon}\left(1+\left|\frac{\partial u^{\varepsilon}}{\partial t}\right|\right)\frac{\partial u^{\varepsilon}}{\partial t}=f^{\varepsilon}-|u^{\varepsilon}|u^{\varepsilon} \text{ in } \Omega^{\varepsilon}\times\,]0,T[$$

(2) $$\sigma_{ij}^{\varepsilon}\left(u^{\varepsilon}\right)=2\mu d_{ij}\left(u^{\varepsilon}\right)+\lambda d_{kk}\left(u^{\varepsilon}\right)\delta_{ij}\ i,j=1,2,3 \text{ in } \Omega^{\varepsilon}\times\,]0,T[$$

(3) $$u^{\varepsilon}=0 \text{ on } \Gamma_1^{\varepsilon}\times\,]0,T[$$

(4) $$u^{\varepsilon}=0 \text{ on } \Gamma_L^{\varepsilon}\times\,]0,T[$$

(5) $$\frac{\partial u^{\varepsilon}}{\partial t}.n=0 \text{ on } \omega\times\,]0,T[$$

(6) $$\left.\begin{array}{l}|\sigma_{\tau}^{\varepsilon}|<k^{\varepsilon}\implies\left(\dfrac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau}=0\\ |\sigma_{\tau}^{\varepsilon}|=k^{\varepsilon}\implies\exists\beta\geq 0 \text{ such that } \left(\dfrac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau}=-\beta\sigma_{\tau}^{\varepsilon}\end{array}\right\}\text{ on } \omega\times\,]0,T[$$

(7) $$u^{\varepsilon}(x,0)=u_0(x),\ \frac{\partial u^{\varepsilon}}{\partial t}(x,0)=u_1(x),\ \forall x\in\Omega^{\varepsilon}$$

The equation (1) represents the deformations of elastic body with a dissipative and source terms in the dynamic regime, f^{ε} represents a force density and α^{ε} is positive constant. The equation (5) and (6) represents the Tresca friction law on

$\omega \times]0, T[$, where k^ε is the friction coefficient. The initial conditions of the problem are given in (7).

We recall that Tresca's boundary condition (6) is equivalent to

$$\left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau .\sigma_\tau^\varepsilon + k^\varepsilon \left|\left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau\right| = 0 \text{ on } \omega \times]0, T[\tag{8}$$

To get a weak formulation, we introduce the closed convex set

$$K^\varepsilon = \left\{v \in H^1 (\Omega^\varepsilon)^3 : v = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon,\ v.n = 0 \text{ on } \omega\right\}.$$

By standard calculations, the variational formulation of problem (1)-(7) is given by

Problem P: Find a displacement field $u^\varepsilon \in K^\varepsilon$ where $\dfrac{\partial u^\varepsilon}{\partial t} \in K^\varepsilon$, $\forall t \in [0, T]$, such that

$$\left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) + a\left(u^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) + \left(|u^\varepsilon| u^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) \tag{9}$$

$$+\alpha^\varepsilon \left(\left(1 + \left|\frac{\partial u^\varepsilon}{\partial t}\right|\right) \frac{\partial u^\varepsilon}{\partial t}, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) + j^\varepsilon(\varphi) - j^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t}\right) \geq \left(f^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right), \forall \varphi \in K^\varepsilon$$

$$u^\varepsilon(x, 0) = u_0(x)\ \frac{\partial u^\varepsilon}{\partial t}(x, 0) = u_1(x),$$

where

$$a(u, v) = 2\mu \int_{\Omega^\varepsilon} d(u)d(v)dx + \lambda \int_{\Omega^\varepsilon} \text{div}(u)\text{div}(v)dx$$

$$j^\varepsilon(v) = \int_\omega k^\varepsilon |v_\tau| \, dx'$$

$$(f, v) = \int_{\Omega^\varepsilon} fv dx$$

Theorem 2.1. *Under the assumptions*

$$f^\varepsilon, \frac{\partial f^\varepsilon}{\partial t} \in L^2\left(0, T; L^2(\Omega^\varepsilon)^3\right),$$

$$k^\varepsilon \in L^\infty(\omega), k^\varepsilon > 0 \textit{ does not depend of } t,$$

$$u_0 \in H^1(\Omega^\varepsilon)^3,\ u_1 \in H^1(\Omega^\varepsilon)^3,\ (u_1)_\tau = 0, \tag{10}$$

there exists a unique solution u^ε *of* (9) *such that*

$$u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \in L^\infty\left(0, T; H^1(\Omega^\varepsilon)^3\right),$$

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} \in L^\infty\left(0,T;L^2\left(\Omega^\varepsilon\right)^3\right) \cap L^2\left(0,T;H^1\left(\Omega^\varepsilon\right)^3\right).$$

The proof of this theorem is similar in [4, 6].

3. Change of the domain and some estimates

For the asymptotic analysis of problem (1)-(7) we use the approach which consist in transporing the initially posed problem in the domain Ω^ε which depends on a small parameter ε to an equivalent problem with a fixed domain Ω which is independent of ε. For that, we introduce the change of the variable $z = \dfrac{x_3}{\varepsilon}$, which changes (x', x_3) in Ω^ε to (x', z) in Ω where

$$\Omega = \{(x', z) \in \mathbb{R}^3, (x', 0) \in \omega \text{ and } 0 < z < h(x')\}.$$

and we denote by $\Gamma = \bar{\omega} \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_L$ its boundary, then we define the following functions in Ω

$$\begin{cases} \hat{u}_i^\varepsilon(x', z, t) = u_i^\varepsilon(x', x_3, t), \ i = 1, 2 \\ \hat{u}_3^\varepsilon(x', z, t) = \varepsilon^{-1} u_3^\varepsilon(x', x_3, t). \end{cases}$$

For the data of the problem (1)-(7), we assume that they depend of ε as follows

$$\begin{cases} \hat{f}(x', z, t) = \varepsilon^2 f(x', x_3, t), \\ \hat{k} = \varepsilon k, \\ \hat{\alpha} = \varepsilon^2 \alpha^\varepsilon. \end{cases}$$

with $\hat{f}$, $\hat{k}$ and $\hat{\alpha}$ independent of ε.

Moreover, we define some functions spaces on Ω

$$K = \left\{\varphi \in H^1(\Omega)^3 : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \text{ and } \varphi.n = 0 \text{ on } \omega\right\}.$$

$$\Pi(K) = \left\{\varphi = (\varphi_1, \varphi_2) \in H^1(\Omega)^2 : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L\right\}.$$

$$V_z = \left\{v = (v_1, v_2) \in L^2(\Omega)^2 : \frac{\partial v_i}{\partial z} \in L^2(\Omega), i = 1, 2 \text{ and } v = 0 \text{ on } \Gamma_1\right\}.$$

V_z is the Banach space with norm

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2}\left(\|v_i\|^2_{L^2(\Omega)} + \left\|\frac{\partial v_i}{\partial z}\right\|^2_{L^2(\Omega)}\right)\right)^{\frac{1}{2}}.$$

We multiply (1) by ε and passing to the fixed domain Ω, by injecting the new data and the unknown, we obtain:

Find $\hat{u}^\varepsilon \in K$, with $\dfrac{\partial \hat{u}^\varepsilon}{\partial t} \in K$, $\forall t \in [0,T]$, such that

$$\sum_{i=1}^{2} \varepsilon^2 \left(\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2}, \hat{\varphi}_i - \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) + \varepsilon^4 \left(\frac{\partial^2 \hat{u}_3^\varepsilon}{\partial t^2}, \hat{\varphi}_3 - \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right) + \hat{a}\left(\hat{u}^\varepsilon, \hat{\varphi} - \frac{\partial \hat{u}^\varepsilon}{\partial t} \right)$$

(11)

$$+ \varepsilon \sum_{i=1}^{2} \left(|\hat{u}_i^\varepsilon| \hat{u}_i^\varepsilon, \hat{\varphi}_i - \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) + \varepsilon \sum_{i=1}^{2} \left(|\hat{u}_3^\varepsilon| \hat{u}_3^\varepsilon, \hat{\varphi}_3 - \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right)$$

$$+ \hat{\alpha} \sum_{i=1}^{2} \left(\left(1 + \left| \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right| \right) \frac{\partial \hat{u}_i^\varepsilon}{\partial t}, \hat{\varphi}_i - \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) + \varepsilon^2 \hat{\alpha} \left(\left(1 + \left| \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right| \right) \frac{\partial \hat{u}_3^\varepsilon}{\partial t}, \hat{\varphi}_3 - \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right)$$

$$+ \hat{j}\left(\hat{\varphi}\right) - \hat{j}\left(\frac{\partial \hat{u}^\varepsilon}{\partial t} \right) \geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) + \varepsilon \left(\hat{f}_3, \hat{\varphi}_3 - \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right) \ \forall \hat{\varphi} \in K$$

$$\hat{u}^\varepsilon(0) = \hat{u}_0 \ \frac{\partial \hat{u}^\varepsilon}{\partial t}(0) = \hat{u}_1,$$

where

$$\hat{a}\left(\hat{\psi}, \hat{\varphi}\right) = \mu \varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial \hat{\psi}_i}{\partial x_j} + \frac{\partial \hat{\psi}_j}{\partial x_i} \right) \frac{\partial \hat{\varphi}_i}{\partial x_j} dx' dz$$

$$\mu \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial \hat{\psi}_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{\psi}_3}{\partial x_i} \right) \left(\frac{\partial \hat{\varphi}_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{\varphi}_3}{\partial x_i} \right) dx' dz$$

$$+ 2\mu\varepsilon^2 \int_\Omega \frac{\partial \hat{\psi}_3}{\partial z} \frac{\partial \hat{\varphi}_3}{\partial z} dx' dz + \lambda \varepsilon^2 \int_\Omega \operatorname{div}\left(\hat{\psi}\right) \operatorname{div}\left(\hat{\varphi}\right) dx' dz,$$

and

$$\hat{j}(\hat{\varphi}) = \int_\omega \hat{k} \left| \hat{\varphi}_\tau \right| dx'.$$

In the next, we will obtain estimates on $\hat{u}^\varepsilon$. These estimates will be useful in proving the convergence of $\hat{u}^\varepsilon$ toward the expected function.

Theorem 3.1. *Under the hypotheses of Theorem 2.1, there exists a constant c independent of ε such that*

$$(12)\qquad \sum_{i=1}^{2}\left(\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon^{\frac{2}{3}}\hat{u}_i^\varepsilon\right\|_{L^3(\Omega)}^3\right)$$
$$+\sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon^{\frac{5}{3}}\hat{u}_3^\varepsilon\right\|_{L^3(\Omega)}^3\leq c$$

$$(13)\qquad \sum_{i=1}^{2}\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right\|_{L^3(0,T;L^3(\Omega))}^3+\left\|\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right\|_{L^3(0,T;L^3(\Omega))}^3\leq c$$

$$(14)\qquad \sum_{i=1}^{2}\left(\left\|\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial z\partial t}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon^2\frac{\partial^2 \hat{u}_3^\varepsilon}{\partial x_i\partial t}\right\|_{L^2(\Omega)}^2\right)$$
$$+\sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial x_j\partial t}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon\frac{\partial^2 \hat{u}_3^\varepsilon}{\partial z\partial t}\right\|_{L^2(\Omega)}^2+\left\|\varepsilon^2\frac{\partial^2 \hat{u}_3^\varepsilon}{\partial t^2}\right\|_{L^2(\Omega)}^2\leq c$$

Proof. First, we recall some inequalities.
- Korn's inequalty [11]:

$$\|d(u^\varepsilon)\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2\geq C_K\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}}^2;$$

- Poincaré inequality:

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)^3}\leq \varepsilon h^*\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}};$$

- Young's inequality:

$$ab\leq \eta^2\frac{a^2}{2}+\eta^{-2}\frac{b^2}{2},\ \forall(a,b)\in\mathbb{R}^2,\ \forall\eta>0,$$

where h^* and C_K are constants independent of ε.

Let u^ε be a solution of the problem (9). We take $\varphi=0$, then

$$\left(\frac{\partial^2 u^\varepsilon}{\partial t^2},\frac{\partial u^\varepsilon}{\partial t}\right)+a\left(u^\varepsilon,\frac{\partial u^\varepsilon}{\partial t}\right)+\left(|u^\varepsilon|u^\varepsilon,\frac{\partial u^\varepsilon}{\partial t}\right)$$
$$+\alpha^\varepsilon\left(\left(1+\left|\frac{\partial u^\varepsilon}{\partial t}\right|\right)\frac{\partial u^\varepsilon}{\partial t},\frac{\partial u^\varepsilon}{\partial t}\right)\leq\left(f^\varepsilon,\frac{\partial u^\varepsilon}{\partial t}\right),$$

whence

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial u^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+a\left(u^\varepsilon,u^\varepsilon\right)+\frac{2}{3}\left\|u^\varepsilon\right\|^3_{L^3(\Omega^\varepsilon)^3}\right]+\alpha^\varepsilon\left\|\frac{\partial u^{\varepsilon}}{\partial t}\right\|^3_{L^3(\Omega^\varepsilon)^3}\leq\left(f^\varepsilon,\frac{\partial u^{\varepsilon}}{\partial t}\right).$$

For $s\in[0,t]$ by integration, and using the Korn inequality, we get

(15) $$\left\|\frac{\partial u^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+2\mu C_K\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+\frac{2}{3}\left\|u^\varepsilon\right\|^3_{L^3(\Omega^\varepsilon)^3}+\alpha^\varepsilon\int_0^t\left\|\frac{\partial u^{\varepsilon}}{\partial t}(s)\right\|^3_{L^3(\Omega^\varepsilon)^3}ds$$

$$\leq 2\int_0^t\left(f^\varepsilon(s),\frac{\partial u^{\varepsilon}}{\partial t}(s)\right)ds+\left[\left\|u_1\right\|^2_{L^2(\Omega^\varepsilon)^3}+(2\mu+3\lambda)\left\|\nabla u_0\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+\frac{2}{3}\left\|u_0\right\|^3_{L^3(\Omega^\varepsilon)^3}\right].$$

On the other hand, we have

$$2\int_0^t\left(f^\varepsilon(s),\frac{\partial u^{\varepsilon}}{\partial t}(s)\right)ds=2\left(f^\varepsilon(t),u^\varepsilon(t)\right)-2\left(f^\varepsilon(0),u_0\right)-2\int_0^t\left(\frac{\partial f^{\varepsilon}}{\partial t}(s),u^\varepsilon(s)\right)ds.$$

Using the Cauchy-Schwarz, Poincaré inequality and the Young inequality, we obtain

(16) $$\left|2\int_0^t\left(f^\varepsilon(s),\frac{\partial u^{\varepsilon}}{\partial t}(s)\right)ds\right|\leq\mu C_K\left\|\nabla u^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}$$

$$+\frac{\varepsilon^2h^{*2}}{\mu C_K}\left\|f^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)^3}+\varepsilon^2h^{*2}\left\|f^\varepsilon(0)\right\|^2_{L^2(\Omega^\varepsilon)^3}+\left\|\nabla u_0\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}$$

$$+\mu C_K\int_0^t\left\|\nabla u^\varepsilon(s)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}ds+\frac{\varepsilon^2h^{*2}}{\mu C_K}\int_0^t\left\|\frac{\partial f^{\varepsilon}}{\partial t}(s)\right\|^3_{L^3(\Omega^\varepsilon)^3}ds$$

Using (15) and (16), we get

(17)

$$\left[\left\|\frac{\partial u^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\mu C_K\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]+\frac{2}{3}\left\|u^\varepsilon\right\|^3_{L^3(\Omega^\varepsilon)^3}+\alpha^\varepsilon\int_0^t\left\|\frac{\partial u^{\varepsilon}}{\partial t}(s)\right\|^3_{L^3(\Omega^\varepsilon)^3}ds$$

$$\leq\left\|u_1\right\|^2_{L^2(\Omega^\varepsilon)^3}+(1+2\mu+3\lambda)\left\|\nabla u_0\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+\frac{\varepsilon^2h^{*2}}{\mu C_K}\left\|f^\varepsilon(t)\right\|^2_{L^2(\Omega^\varepsilon)^3}$$

$$+\varepsilon^2h^{*2}\left\|f^\varepsilon(0)\right\|^2_{L^2(\Omega^\varepsilon)^3}+\frac{\varepsilon^2h^{*2}}{\mu C_K}\int_0^t\left\|\frac{\partial f^{\varepsilon}}{\partial t}(s)\right\|^2_{L^2(\Omega^\varepsilon)^3}ds$$

$$+\int_0^t\left[\left\|\frac{\partial u^{\varepsilon}}{\partial t}(s)\right\|^2_{L^2(\Omega^\varepsilon)^3}+\mu C_K\left\|\nabla u^\varepsilon(s)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\right]ds.$$

As $\varepsilon^2\left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^3}=\varepsilon^{-1}\left\|\hat{f}\right\|^2_{L^2(\Omega)^3}$, multiplying (17) by ε we deduce that

$$\varepsilon\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\mu C_K\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\right]$$
$$+\frac{2}{3}\varepsilon\left\|u^\varepsilon\right\|^3_{L^3(\Omega^\varepsilon)^3}+\varepsilon\alpha^\varepsilon\int_0^t\left\|\frac{\partial u^\varepsilon}{\partial t}(s)\right\|^3_{L^3(\Omega^\varepsilon)^3}ds$$
$$\leq A+\int_0^t\varepsilon\left[\left\|\frac{\partial u^\varepsilon}{\partial t}(s)\right\|^2_{L^2(\Omega^\varepsilon)^3}+\mu C_K\left\|\nabla u^\varepsilon(s)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\right]ds,$$

where A is a constant that does not depend of ε with

$$A=\left\|\hat{u}_1\right\|^2_{L^2(\Omega)^3}+(1+2\mu+3\lambda)\left\|\nabla\hat{u}_0\right\|^2_{L^2(\Omega)^{3\times 3}}+h^{*2}\left\|\hat{f}(0)\right\|^2_{L^2(\Omega)^3}$$
$$+\frac{h^{*2}}{\mu C_K}\left\|\hat{f}\right\|^2_{L^\infty\left(0,T;L^2(\Omega)^3\right)}+\frac{h^{*2}}{\mu C_K}\left\|\frac{\partial\hat{f}}{\partial t}\right\|^2_{L^2\left(0,T;L^2(\Omega)^3\right)}.$$

Now using Gronwall's lemma, we have

$$\varepsilon\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\mu C_K\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\right]\leq C.$$

Thus, we conclude (12) and (13).

The functional $j^\varepsilon(.)$ is convex but nondifferentiable. The overcome this difficulty, we shall use the following approach. Let $j^\varepsilon_\zeta(.)$ be a functional defined by

$$j^\varepsilon_\zeta(v)=\int_\omega k_\varepsilon(x')\,\phi_\zeta\left(|v_\tau|^2\right)dx',$$

where

$$\phi_\zeta(\lambda)=\frac{1}{1+\zeta}|\lambda|^{1+\zeta},\ \zeta>0.$$

To show the a priori estimate (14), we consider the approximate equation

$$\left(\frac{\partial^2 u^\varepsilon_\zeta}{\partial t^2},\varphi\right)+a\left(u^\varepsilon_\zeta,\varphi\right)+\left(\left(j^\varepsilon_\zeta\right)'\left(\frac{\partial u^\varepsilon_\zeta}{\partial t}\right),\varphi\right)\tag{18}$$
$$+\alpha^\varepsilon\left(\left(1+\left|\frac{\partial u^\varepsilon_\zeta}{\partial t}\right|\right)\frac{\partial u^\varepsilon_\zeta}{\partial t},\varphi\right)+2\left(|u^\varepsilon_\zeta|u^\varepsilon_\zeta,\varphi\right)=(f^\varepsilon,\varphi)$$
$$u^\varepsilon_\zeta(0)=u_0\ \frac{\partial u^\varepsilon_\zeta}{\partial t}(0)=u_1.$$

We differentiate (18) in t and we take $\varphi = \dfrac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}$ we get

$$\left(\frac{\partial^3 u_\zeta^\varepsilon}{\partial t^3}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right) + a\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right) + 2\alpha^\varepsilon\left(\left(\frac{1}{2} + \left|\frac{\partial u_\zeta^\varepsilon}{\partial t}\right|\right)\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right)$$
$$+2\left(|u_\zeta^\varepsilon|\frac{\partial u_\zeta^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right) + \left(\frac{\partial}{\partial t}\left(j_\zeta^\varepsilon\right)'\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}\right), \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right) = \left(\frac{\partial f^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right),$$

as $\left(\dfrac{\partial}{\partial t}\left(j_\zeta^\varepsilon\right)'\left(\dfrac{\partial u_\zeta^\varepsilon}{\partial t}\right), \dfrac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right) \geq 0$; we have

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Omega^\varepsilon)^3} + a\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}, \frac{\partial u_\zeta^\varepsilon}{\partial t}\right)\right] \leq \left(\frac{\partial f^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right) - 2\left(|u_\zeta^\varepsilon|\frac{\partial u_\zeta^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right).$$

Integrating this inequality over $(0, t)$ and use Korn's inequality, we obtain

(19)
$$\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Omega^\varepsilon)^3} + 2\mu C_K\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq \left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|^2_{L^2(\Omega^\varepsilon)^3}$$
$$+ (2\mu + 3\lambda + \mu C_K)\left\|\nabla u_1\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \mu C_K\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}$$
$$+\frac{\varepsilon^2 h^{*2}}{\mu C_K}\left\|\frac{\partial f^\varepsilon}{\partial t}(0)\right\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{\varepsilon^2 h^{*2}}{\mu C_K}\left\|\frac{\partial f^\varepsilon}{\partial t}(t)\right\|^2_{L^2(\Omega^\varepsilon)^3}$$
$$+\frac{\varepsilon^2 h^{*2}}{\mu C_K}\int_0^t\left\|\frac{\partial^2 f^\varepsilon}{\partial t^2}(s)\right\|^2_{L^2(\Omega^\varepsilon)^3} ds + \mu C_K\int_0^t\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}(s)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds$$
$$-4\int_0^t\int_{\Omega^\varepsilon}|u_\zeta^\varepsilon(s)|\frac{\partial u_\zeta^\varepsilon}{\partial t}(s)\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(s)dx'dx_3ds.$$

Using the Holder inequality, the Young inequality and the Sobolev embedding, we get

$$\left|-4\int_0^t\int_{\Omega^\varepsilon}|u_\zeta^\varepsilon(s)|\frac{\partial u_\zeta^\varepsilon}{\partial t}(s)\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(s)dx'dx_3ds\right|$$
$$\leq 4\int_0^t\left\|u_\zeta^\varepsilon(s)\right\|_{L^4(\Omega^\varepsilon)^3}\left\|\frac{\partial u^\varepsilon}{\partial t}(s)\right\|_{L^4(\Omega^\varepsilon)^3}\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(s)\right\|_{L^2(\Omega^\varepsilon)^3} ds$$
$$\leq 4C_*^2 T + \int_0^t\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(s)\right\|^2_{L^2(\Omega^\varepsilon)^3} ds,$$

where C_* independent of ζ and ε, thus

$$\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Omega^\varepsilon)^3} + \mu C_K \left\|\nabla \frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq \left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|^2_{L^2(\Omega^\varepsilon)^3}$$
$$+ (2\mu + 3\lambda + \mu C_K)\left\|\nabla u_1\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + 4C_*^2 T + \frac{\varepsilon^2 h^{*2}}{\mu C_K}\left\|\frac{\partial f^\varepsilon}{\partial t}(0)\right\|^2_{L^2(\Omega^\varepsilon)^3}$$
$$+ \frac{\varepsilon^2 h^{*2}}{\mu C_K}\left\|\frac{\partial f^\varepsilon}{\partial t}(t)\right\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{\varepsilon^2 h^{*2}}{\mu C_K}\int_0^t \left\|\frac{\partial^2 f^\varepsilon}{\partial t^2}(s)\right\|^2_{L^2(\Omega^\varepsilon)^3} ds$$
$$+ \int_0^t \left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(s)\right\|^2_{L^2(\Omega^\varepsilon)^3} ds + \mu C_K \int_0^t \left\|\nabla \frac{\partial u_\zeta^\varepsilon}{\partial t}(s)\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} ds. \tag{20}$$

Now let us estimate $\dfrac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)$, from (18) and (10) we deduce

$$\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0), \varphi\right) = (f^\varepsilon(0), \varphi) - a(u_0, \varphi) - \alpha^\varepsilon\left((1 + |u_1|)\, u_1, \varphi\right) - (|u_0| u_0, \varphi)\ ,$$

for all $\varphi \in K^\varepsilon$. Therefore

$$\left|\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0), \varphi\right)\right|$$
$$\leq \varepsilon h^* \left\|f^\varepsilon(0)\right\|_{L^2(\Omega^\varepsilon)^3} \left\|\nabla\varphi\right\|_{L^2(\Omega^\varepsilon)^{3\times 3}} + (2\mu + 3\lambda)\left\|u_0\right\|_{H^1(\Omega^\varepsilon)^3} \left\|\varphi\right\|_{H^1(\Omega^\varepsilon)^3}$$
$$+ \varepsilon^2 h^{*2} \alpha^\varepsilon \left\|\nabla u_1\right\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \left\|\nabla\varphi\right\|_{L^2(\Omega^\varepsilon)^{3\times 3}} + \varepsilon h^* \alpha^\varepsilon \sqrt{\varepsilon}\left(\int_\Omega |\hat{u}_1|^4\, dx' dz\right)^{\frac{1}{2}} \left\|\nabla\varphi\right\|_{L^2(\Omega^\varepsilon)^{3\times 3}}$$
$$+ \varepsilon h^* \left(\int_\Omega |\hat{u}_0|^4\, dx' dz\right)^{\frac{1}{2}} \left\|\nabla\varphi\right\|_{L^2(\Omega^\varepsilon)^{3\times 3}}.$$

As $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we get

$$\left|\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0), \varphi\right)\right| \leq \varepsilon h^* \left\|f^\varepsilon(0)\right\|_{L^2(\Omega^\varepsilon)^3} \left\|\varphi\right\|_{H^1(\Omega^\varepsilon)^3} + (2\mu + 3\lambda)\left\|u_0\right\|_{H^1(\Omega^\varepsilon)^3} \left\|\varphi\right\|_{H^1(\Omega^\varepsilon)^3}$$
$$+ \varepsilon^2 h^{*2} \alpha^\varepsilon \left\|u_1\right\|_{H^1(\Omega^\varepsilon)^3} \left\|\varphi\right\|_{H^1(\Omega^\varepsilon)^3} + \varepsilon^{\frac{3}{2}} h^* \alpha^\varepsilon c_s \left\|\hat{u}_1\right\|_{H^1(\Omega)^3} \left\|\varphi\right\|_{H^1(\Omega^\varepsilon)^3}$$
$$+ \varepsilon h^* c_s \left\|\hat{u_0}\right\|_{H^1(\Omega)^3} \left\|\varphi\right\|_{H^1(\Omega^\varepsilon)^3}.$$

We multiply this last inequality by $\sqrt{\varepsilon}$, we obtain

$$\sqrt{\varepsilon}\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|_{L^2(\Omega^\varepsilon)^3} \leq C',$$

where

$$C' = h^* \left\| \hat{f}(0) \right\|_{L^2(\Omega)^3} + (2\mu + 3\lambda) \left\| \hat{u_0} \right\|_{H^1(\Omega)^3} + h^* c_s \left\| \hat{u_0} \right\|^2_{H^1(\Omega)^3}$$
$$+ \hat{\alpha} h^{*2} \left\| \hat{u_1} \right\|_{H^1(\Omega)^3} + \hat{\alpha} h^* c_s \left\| \hat{u_1} \right\|^2_{H^1(\Omega)^3},$$

does not depend of ε. Passing to the limit in (20) as ζ tends to zero, we find

$$\left[\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2} \right\|^2_{L^2(\Omega^\varepsilon)^3} + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \right] \leq \left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(0) \right\|^2_{L^2(\Omega^\varepsilon)^3} \tag{21}$$
$$+ (2\mu + 3\lambda + \mu C_K) \left\| \nabla u_1 \right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + 4C_*^2 T + \frac{\varepsilon^2 h^{*2}}{\mu C_K} \left\| \frac{\partial f^\varepsilon}{\partial t}(0) \right\|^2_{L^2(\Omega^\varepsilon)^3}$$
$$+ \frac{\varepsilon^2 h^{*2}}{\mu C_K} \left\| \frac{\partial f^\varepsilon}{\partial t}(t) \right\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{\varepsilon^2 h^{*2}}{\mu C_K} \int_0^t \left\| \frac{\partial^2 f^\varepsilon}{\partial t^2}(s) \right\|^2_{L^2(\Omega^\varepsilon)^3} ds$$
$$+ \int_0^t \left[\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(s) \right\|^2_{L^2(\Omega^\varepsilon)^3} + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon}{\partial t}(s) \right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \right] ds.$$

Multiplying now (21) by ε, we obtain

$$\varepsilon \left[\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2} \right\|^2_{L^2(\Omega^\varepsilon)^3} + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \right] \leq B$$
$$+ \int_0^t \varepsilon \left[\left\| \frac{\partial^2 u^\varepsilon}{\partial t^2}(s) \right\|^2_{L^2(\Omega^\varepsilon)^3} + \mu C_K \left\| \nabla \frac{\partial u^\varepsilon}{\partial t}(s) \right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \right] ds,$$

where B is a constant that does not depend of ε with

$$B = (2\mu + 3\lambda + \mu C_K) \left\| \nabla \hat{u_1} \right\|^2_{L^2(\Omega)^{3\times 3}} + (C')^2 + \frac{h^{*2}}{\mu C_K} \left\| \frac{\partial \hat{f}}{\partial t}(0) \right\|^2_{L^2(\Omega)^3}$$
$$+ \frac{h^{*2}}{\mu C_K} \left\| \frac{\partial \hat{f}}{\partial t} \right\|_{L^\infty\left(0,T;L^2(\Omega)^3\right)} + \frac{h^{*2}}{\mu C_K} \left\| \frac{\partial^2 \hat{f}}{\partial t^2} \right\|^2_{L^2\left(0,T;L^2(\Omega)^3\right)}.$$

By the Gronwall's lemma, there exists a constant C that does not depend of ε such that

$$\varepsilon \left\| \frac{\partial^2 u^\varepsilon}{\partial t^2} \right\|^2_{L^2(\Omega^\varepsilon)^3} + \varepsilon \left\| \nabla \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq C,$$

we conclude (14). □

4. Convergence results and the limit problem

Theorem 4.1. *Under the assumptions of Theorem 3.1, there exists* $u_i^* \in L^2(0,T;V_z) \cap L^\infty(0,T;V_z)$, $i=1,2$ *such that*

$$\left.\begin{array}{c} \hat{u}_i^\varepsilon \rightharpoonup u_i^*,\ i=1,2 \\ \dfrac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup \dfrac{\partial u_i^*}{\partial t},\ i=1,2 \end{array}\right\} \tag{22}$$

weakly in $L^2(0,T;V_z)$ *and weakly* $*$ *in* $L^\infty(0,T;V_z)$*;*

$$\left.\begin{array}{c} \dfrac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup \dfrac{\partial u_i^*}{\partial t},\ i=1,2 \\ \varepsilon\dfrac{\partial \hat{u}_3^\varepsilon}{\partial t} \rightharpoonup 0 \end{array}\right\} \tag{23}$$

weakly in $L^3(0,T;L^3(\Omega))$*;*

$$\left.\begin{array}{c} \varepsilon^{\frac{2}{3}}\hat{u}_i^\varepsilon \rightharpoonup 0,\ i=1,2 \\ \varepsilon^{\frac{5}{3}}\hat{u}_3^\varepsilon \rightharpoonup 0 \end{array}\right\} \tag{24}$$

weakly in $L^3(0,T;L^3(\Omega))$ *and weakly* $*$ *in* $L^\infty(0,T;L^3(\Omega))$*;*

$$\left.\begin{array}{c} \varepsilon\dfrac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup 0,\ i=1,2 \\ \varepsilon\dfrac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0,\ i,j=1,2 \\ \varepsilon^2\dfrac{\partial \hat{u}_3^\varepsilon}{\partial t} \rightharpoonup 0 \\ \varepsilon^2\dfrac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0,\ i=1,2 \\ \varepsilon\dfrac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \end{array}\right\} \tag{25}$$

weakly in $L^2(0,T;L^2(\Omega))$ *and weakly* $*$ *in* $L^\infty(0,T;L^2(\Omega))$*;*

$$\left.\begin{array}{c} \varepsilon\dfrac{\partial^2 \hat{u}_i^\varepsilon}{\partial x_j \partial t} \rightharpoonup 0,\ i,j=1,2 \\ \varepsilon^2\dfrac{\partial^2 \hat{u}_3^\varepsilon}{\partial z \partial t} \rightharpoonup 0 \\ \varepsilon^2\dfrac{\partial^2 \hat{u}_3^\varepsilon}{\partial x_i \partial t} \rightharpoonup 0,\ i=1,2 \\ \varepsilon\dfrac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2} \rightharpoonup 0,\ i=1,2 \\ \varepsilon^2\dfrac{\partial^2 \hat{u}_3^\varepsilon}{\partial t^2} \rightharpoonup 0 \end{array}\right\} \tag{26}$$

weakly in $L^2(0,T;L^2(\Omega))$ *and weakly* $*$ *in* $L^\infty(0,T;L^2(\Omega))$.

Proof. According to Theorem 3.1 there exists a constant c independent of ε, such that

$$\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 \leq c \; i=1,2.$$

Using this estimate with the Poincaré inequality in the domain, we obtain

$$\|\hat{u}_i^\varepsilon\|_{V_z}^2 \leq \left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 \leq c \; i=1,2.$$

So $(\hat{u}_1^\varepsilon,\hat{u}_2^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;V_z)\cap L^\infty(0,T;V_z)$, which implies the existence of an element (u_1^*,u_2^*) in $L^2(0,T;V_z)\cap L^\infty(0,T;V_z)$ such that $(\hat{u}_1^\varepsilon,\hat{u}_2^\varepsilon)_\varepsilon$ converges weakly to (u_1^*,u_2^*) in $L^2(0,T;V_z)\cap L^\infty(0,T;V_z)$; thus, we obtain (22). For (23) to (26) through to (14) and (22). □

Theorem 4.2. *Under the hypotheses of Theorem 4.1, the limit* $u^*=(u_1^*,u_2^*)$ *satisfies*

(4.1)

$$\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right)dx'dz+\hat{j}(\hat{\varphi})-\hat{j}\left(\frac{\partial u^*}{\partial t}\right)$$
$$+\hat{\alpha}\sum_{i=1}^{2}\int_\Omega\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\left(\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right)dx'dz \geq \sum_{i=1}^{2}\int_\Omega \hat{f}_i\left(\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right)dx'dz,$$

fo rall $\hat{\varphi}\in\Pi(K)^2$*;*

$$(28)\qquad \begin{cases} -\mu\dfrac{\partial^2 u_i^*}{\partial z^2}(t)+\hat{\alpha}\left(1+\left|\dfrac{\partial u_i^*}{\partial t}(t)\right|\right)\dfrac{\partial u_i^*}{\partial t}(t)=\hat{f}_i(t)\; i=1,2 \text{ in } L^2(\Omega) \\ u_i^*(x',z,0)=u_{0,i}^*(x',z),\; i=1,2 \end{cases}.$$

Proof. Choosing $\hat{\varphi}=\left(\hat{\varphi}_1,\hat{\varphi}_2,\dfrac{\partial u_3^*}{\partial z}\right)$ in (11) and passing to the limit when ε tends to zero and using the convergence results of Theorem 4.1, we deduce

$$(29)\qquad \mu\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right)dx'dz+\hat{j}(\hat{\varphi})-\hat{j}\left(\frac{\partial u^*}{\partial t}\right)$$
$$+\hat{\alpha}\sum_{i=1}^{2}\int_\Omega\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\left(\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right)dx'dz \geq \sum_{i=1}^{2}\left(\hat{f}_i,\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right).$$

We now choose in (29):

$$\hat{\varphi}_i = \frac{\partial u_i^*}{\partial t} \pm \psi_i,\ \psi_i \in H_0^1(\Omega)\ \ i = 1, 2,$$

and using Green's formula. Taking $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Omega)$, then $\psi_2 = 0$ and $\psi_1 \in H_0^1(\Omega)$, we obtain

$$-\mu \int_\Omega \frac{\partial^2 u_i^*}{\partial z^2} \psi_i dx' dz + \hat{\alpha} \int_\Omega \left(1 + \left|\frac{\partial u_i^*}{\partial t}\right|\right) \frac{\partial u_i^*}{\partial t} \psi_i dx' dz = \int_\Omega \hat{f}_i \psi_i dx' dz.$$

Thus

$$-\mu \frac{\partial^2 u_i^*}{\partial z^2} + \hat{\alpha} \left(1 + \left|\frac{\partial u_i^*}{\partial t}\right|\right) \frac{\partial u_i^*}{\partial t} = \hat{f}_i\ i = 1, 2 \text{ in } H^{-1}(\Omega), \tag{30}$$

as $\hat{f}_i \in L^2(\Omega)$ then (3.35) is valid in $L^2(\Omega)$. □

Theorem 4.3. *Under the same assumptions of Theorem 4.1, the traces*

$$s^* = u^*(x', 0, t),\ \tau^* = \frac{\partial u^*}{\partial z}(x', 0, t),$$

satisfy

$$\int_\omega \hat{k} \left(\left|\psi + \frac{\partial s^*}{\partial t}\right| - \left|\frac{\partial s^*}{\partial t}\right|\right) dx' - \int_\omega \mu \tau^* \psi dx' \geq 0,\ \forall \psi \in L^2(\omega)^2 \tag{31}$$

and the following limit form of the Tresca boundary conditions

$$\left.\begin{array}{r} \mu|\tau^*| < \hat{k} \implies \dfrac{\partial s^*}{\partial t} = 0 \\ \mu|\tau^*| = \hat{k} \implies \exists\beta \geq 0 \text{ such that } \dfrac{\partial s^*}{\partial t} = \beta\tau^* \end{array}\right\} \text{a.e on } \omega \times]0, T[. \tag{32}$$

Moreover u^* and s^* satisfies the following weak form of the Reynolds equation

$$\int_\omega \left(\tilde{F} - \mu \frac{h}{2} s^* + \int_0^h \mu u^*(x', z, t)\, dz + \tilde{U}_t\right) \nabla\psi(x')\, dx' = 0, \tag{33}$$

for all $\psi \in H^1(\omega)$, where

$$\tilde{F}(x', h, t) = \int_0^h F(x', z, t)\, dz - \frac{h}{2} F(x', h, t)$$

$$F(x', z, t) = \int_0^z \int_0^\zeta \hat{f}(x', \eta, t)\, d\eta d\zeta$$

$$\tilde{U}_t(x', h, t) = -\hat{\alpha} \int_0^h U_t(x', z, t)\, dz + \frac{\hat{\alpha} h}{2} U_t(x', h, t)$$

$$U_t\left(x',z,t\right)=\int_0^z\int_0^\zeta\left(1+\left|\frac{\partial u^*}{\partial t}\right|\right)\frac{\partial u^*}{\partial t}\left(x',\eta,t\right)d\eta d\zeta.$$

Proof. We choose $\varphi_i=\dfrac{\partial u_i^*}{\partial t}+\psi_i, i=1,2$ in (27) where $\psi\in\Pi\left(K\right)$, we find

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{\partial u_i^*}{\partial z}\frac{\partial\psi_i}{\partial z}dx'dz+\hat{j}\left(\psi+\frac{\partial u^*}{\partial t}\right)-\hat{j}\left(\frac{\partial u^*}{\partial t}\right)$$
$$+\hat{\alpha}\sum_{i=1}^{2}\int_{\Omega}\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\psi_i dx'dz\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i dx'dz,\ \forall\psi_i\in\Pi\left(K\right).$$

Using Green's formula, we get

$$-\mu\sum_{i=1}^{2}\int_{\Omega}\frac{\partial^2 u_i^*}{\partial z^2}\psi dx'dz+\hat{\alpha}\sum_{i=1}^{2}\int_{\Omega}\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\psi_i dx'dz$$
$$-\mu\sum_{i=1}^{2}\int_{\omega}\tau_i^*\psi_i dx'+\int_{\omega}\hat{k}\left(\left|\psi+\frac{\partial s^*}{\partial t}\right|-\left|\frac{\partial s^*}{\partial t}\right|\right)dx'\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i dx'dz.$$

From (28), we obtain

$$\int_{\omega}\hat{k}\left(\left|\psi+\frac{\partial s^*}{\partial t}\right|-\left|\frac{\partial s^*}{\partial t}\right|\right)dx'-\int_{\omega}\mu\tau^*\psi dx'\geq 0,\ \psi\in D\left(\omega\right)^2.$$

The density of $D\left(\omega\right)$ in $L^2\left(\omega\right)$ we deduce (31). We obtain also (32) as in another study [7]. To prove (33) we integrate twice (28) between 0 and z we obtain

$$-\mu u_i^*\left(x',z,t\right)+\mu s_i^*+\mu z\tau_i^*+\hat{\alpha}\int_0^z\int_0^\zeta\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\left(x',\eta,t\right)d\eta d\zeta$$
$$=\int_0^z\int_0^\zeta\hat{f}_i\left(x',\eta,t\right)d\eta d\zeta. \quad (34)$$

In particular for $z=h$, we get

$$\mu s_i^*+\mu z\tau_i^*+\hat{\alpha}\int_0^h\int_0^\zeta\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\left(x',\eta,t\right)d\eta d\zeta$$
$$=\int_0^h\int_0^\zeta\hat{f}_i\left(x',\eta,t\right)d\eta d\zeta. \quad (35)$$

Integrating (34) from 0 to h, we obtain

$$-\mu\int_0^h u_i^*\left(x',z,t\right)dz+\mu s_i^* h+\mu\frac{h^2}{2}\tau_i^*$$

(36)
$$+\hat{\alpha}\int_0^h\int_0^z\int_0^\zeta\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\left(x',\eta,t\right)d\eta d\zeta dz$$

$$=\int_0^h\int_0^z\int_0^\zeta \hat{f}_i\left(x',\eta,t\right)d\eta d\zeta dz.$$

From (35) and (36), we deduce

$$\tilde{F}-\mu\frac{h}{2}s^*+\int_0^h\mu u^*\left(x',z,t\right)dz+\tilde{U}_t=0.$$

Therefore

$$\int_\omega\left(\tilde{F}-\mu\frac{h}{2}s^*+\int_0^h\mu u^*\left(x',z,t\right)dz+\tilde{U}_t\right)\nabla\psi\left(x'\right)dx'=0.$$

□

Theorem 4.4. *The limit solution u^* is unique in $L^2\left(0,T;V_z\right)\cap L^\infty\left(0,T;V_z\right)$.*

Proof. Suppose that there exist two solution u^* and u^{**} of the variational inequality (27), we have

(37)
$$\mu\sum_{i=1}^2\int_\Omega\frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right)dx'dz+\hat{j}\left(\hat{\varphi}\right)-\hat{j}\left(\frac{\partial u^*}{\partial t}\right)$$
$$+\hat{\alpha}\sum_{i=1}^2\int_\Omega\left(1+\left|\frac{\partial u_i^*}{\partial t}\right|\right)\frac{\partial u_i^*}{\partial t}\left(\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right)dx'dz\geq\sum_{i=1}^2\left(\hat{f}_i,\hat{\varphi}_i-\frac{\partial u_i^*}{\partial t}\right),$$

and

(38)
$$\mu\sum_{i=1}^2\int_\Omega\frac{\partial u_i^{**}}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varphi}_i-\frac{\partial u_i^{**}}{\partial t}\right)dx'dz+\hat{j}\left(\hat{\varphi}\right)-\hat{j}\left(\frac{\partial u^{**}}{\partial t}\right)$$
$$+\hat{\alpha}\sum_{i=1}^2\int_\Omega\left(1+\left|\frac{\partial u_i^{**}}{\partial t}\right|\right)\frac{\partial u_i^{**}}{\partial t}\left(\hat{\varphi}_i-\frac{\partial u_i^{**}}{\partial t}\right)dx'dz\geq\sum_{i=1}^2\left(\hat{f}_i,\hat{\varphi}_i-\frac{\partial u_i^{**}}{\partial t}\right).$$

We take $\hat{\varphi} = \dfrac{\partial u^{**}}{\partial t}$ in (37) then $\hat{\varphi} = \dfrac{\partial u^{*}}{\partial t}$ in (38) and by summing the two inequalities, we obtain

$$\mu \sum_{i=1}^{2} \int_{\Omega} \frac{\partial}{\partial z} \left(u_i^{*} - u_i^{**}\right) \frac{\partial}{\partial z} \left(\frac{\partial u_i^{*}}{\partial t} - \frac{\partial u_i^{**}}{\partial t} \right) dx' dz$$
$$+ \hat{\alpha} \sum_{i=1}^{2} \int_{\Omega} \left(\frac{\partial u_i^{*}}{\partial t} - \frac{\partial u_i^{**}}{\partial t} \right) \left(\frac{\partial u_i^{*}}{\partial t} - \frac{\partial u_i^{**}}{\partial t} \right) dx' dz$$
$$+ \hat{\alpha} \sum_{i=1}^{2} \int_{\Omega} \left(\left| \frac{\partial u_i^{*}}{\partial t} \right| \frac{\partial u_i^{*}}{\partial t} - \left| \frac{\partial u_i^{**}}{\partial t} \right| \frac{\partial u_i^{**}}{\partial t} \right) \left(\frac{\partial u_i^{*}}{\partial t} - \frac{\partial u_i^{**}}{\partial t} \right) dx' dz \leq 0.$$

If we put $\tilde{W}(t) = u^{*}(t) - u^{**}(t)$, this implies

$$\mu \frac{d}{dt} \left\| \frac{\partial \tilde{W}}{\partial z} \right\|_{L^2(\Omega)^2}^2 + \hat{\alpha} \left\| \frac{\partial \tilde{W}}{\partial t} \right\|_{L^2(\Omega)^2}^2 + \frac{\hat{\alpha}}{4} \left\| \frac{\partial \tilde{W}}{\partial t} \right\|_{L^3(\Omega)^2}^3 \leq 0.$$

As $\tilde{W}(0) = 0$ then

$$\left\| \frac{\partial \tilde{W}}{\partial t} \right\|_{L^2(\Omega)^2}^2 = 0.$$

Using Poincaré's inequality, we conclude

$$\left\| \tilde{W} \right\|_{L^2(0,T;V_z)} = \left\| \tilde{W} \right\|_{L^\infty(0,T;V_z)} = 0.$$

□

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