

THE $Q_{N_0^1}$ -MATRIX COMPLETION PROBLEM FOR DIGRAPHS 5×5 MATRICES

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ABSTRACT. An $n \times n$ matrix is called a $Q_{N_0^1}$ matrix if for every $k = \{1, 2, 3, \dots, n\}$ the sum of all $k \times k$ principal minors is non-positive. A digraph D is said to have $Q_{N_0^1}$ -completion if every partial $Q_{N_0^1}$ -matrix specifying D can be completed to a $Q_{N_0^1}$ -matrix. In this paper $Q_{N_0^1}$ -matrix completion problem is considered. It is shown that partial non-positive $Q_{N_0^1}$ -matrices representing all digraphs of order 5 with $q = 0$ to 7 have $Q_{N_0^1}$ completion.

1. INTRODUCTION

A partial matrix is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A partial matrix M is fully specified if all entries of M are specified. Let $\langle n \rangle = \{1, 2, \dots, n\}$ and M be an $n \times n$ partial matrix, i.e., one with n rows and n columns. For a subset α of n , the principal partial sub matrix $M(\alpha)$ is the partial matrix obtained from M by deleting all rows and columns not indexed by α .

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A principal minor of M is the determinant of a fully specified principal sub matrix of M . A partial non-positive (negative) minor is a partial matrix whose specified entries are non-positive (negative).

A real $n \times n$ matrix is called an N_0^1 -matrix if all its principal minors are non-positive and each entry is non-positive. Obviously, the diagonal entries of N_0^1 -matrix are non-positive [5].

A real $n \times n$ matrix A is a $Q_{N_0^1}$ -matrix if for every $k = \{1, 2, \dots, n\}$, $S_k(A) < 0$, where $S_k(A)$ denotes the sum of all $k \times k$ principal minors. A non-positive Q -matrix is a Q -matrix whose all entries are non-positive [4]. $Q_{N_0^1}$ matrices arise in multivariate analysis, in linear complementary problems, in the theory of global univalence of functions, and in completion problems. The property of being N , N_0 or Q -matrix is invariant under permutation similarity.

We can define a partial $Q_{N_0^1}$ -matrix in a similar way: A partial matrix B is a $Q_{N_0^1}$ -matrix if $S_k(B) < 0$ for every $k = \{1, 2, \dots, n\}$ for which all $k \times k$ principal submatrices are fully specified. A more useful characterization of a partial $Q_{N_0^1}$ -matrix is given in Proposition 3.1. A completion of a partial matrix is a specific choice of values for the unspecified entries. A $\square$ -completion of a partial $\square$ -matrix M is a completion of M which is a $\square$ -matrix. Matrix completion problems for several classes of matrices including the classes of N and N_0 -matrices have been studied.

In 2009, DeAlba et al. [4] considered the Q -matrix completion problem. Since the property of being a Q -matrix is not inherited by principal sub matrices, the authors observed that the Q -matrix completion problem is substantially different from the completion problems studied earlier and attracts special attention. Further, the authors classified all digraphs of order 5 as to $Q_{N_0^1}$ -matrix completion.

2. PRELIMINARIES

The following graph theoretic terms may be found in most standard reference works, for example in [2, 3].

Definition 2.1. A graph $G = [V(G), E(G)]$ is a finite non-empty set of positive integers $V(G)$ whose members are called vertices and a set $E(G)$ of unordered pairs (V, U) of vertices called edges of G . If (v, u) is an edge of G , then we say that v and u are adjacent in G and (v, u) is incident vertices both v and u . The order of G is

the number of vertices of G . A Sub graph of a graph $G = [V(G), E(G)]$ is a graph $H = [V(H), E(H)]$, where $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$.

Definition 2.2. A Digraph $D = [V(D), E(D)]$ is a finite set of positive integers $V(D)$, whose members are called vertices, and a set $E(D)$ of ordered pairs (v, u) of vertices called arcs (also called directed edges). In the arc (V, U) , Where V is the tail and U is the head.

Definition 2.3. A subdigraph of the digraph $D = [V(D), E(D)]$ in a digraph $H = [V(H), E(H)]$ where $V(H)$ is a subset of $V(D)$ and $E(H)$ is a subset of $E(D)$ whose end point are in $V(H)$. A Subdigraph is complete if it contains all possible arcs between its vertices [2].

Definition 2.4. Two digraph are said to be isomorphic if one can be obtained from the other by relabeling the vertices, that is, if there is a one to one correspondence between the vertices of the two digraphs. Two or more arcs joining the same pair of vertices are called multiple edges and an edge joining a vertex to itself is called a loop.

Definition 2.5. Let π be a permutation of a non-empty finite set V . The digraph $D_\pi = (V, A_\pi)$, where $A_\pi = (v, \pi(v)) : v \in V$, is called a permutation digraph. Clearly, each component of a permutation digraph is a cycle. The digraph D_π is said to be negative if π is an odd permutation.

Definition 2.6. A permutation subdigraph H (of order k) of a digraph D is a permutation digraph that is a subdigraph of D (of order k).Further, H is negative if the corresponding permutation is odd. A digraph D is negatively stratified if D has a negative permutation subdigraph of order k for every $k = \{2, 3, \dots |D|\}$. By P_k we denote the collection of all permutation subdigraphs of order k of k_n^* .Further, we denote by P_k^+ (resp. P_k^-) the collection of all positive (resp. negative) permutation subdigraphs of order k of k_n^* .

Definition 2.7. Let $B = [b_{ij}]$ be an $n \times n$ matrix. We have

$$\det B = \sum (sgn \pi) b_{1\pi(1), \dots, n\pi(n)},$$

where the sum is taken over all permutation π of $< n >$. For a permutation digraph P of k_n^* we denote the product $\prod \{b_{ij} : (i, j) \in B(P)\}$ by $w(P, B)$. For $k \in \{1, 2, \dots, n\}$ we denote the sum of $k \times k$ principal minors of B by $S_k(B)$.we have $S_k(B) = \sum_{P \in P_k^+} w(P, B) - \sum_{P \in P_k^-} w(P, B)$.

Notation 2.1. Let A be an $Q_{N_0^1}$ -matrix. Then

- (i) If P is a permutation matrix, then PAP^T is an $Q_{N_0^1}$ -matrix;
- (ii) If D is a positive diagonal matrix, then DA, AD is an $Q_{N_0^1}$ -matrix;
- (iii) Any principal submatrix of A is an $Q_{N_0^1}$ -matrix. the set of $Q_{N_0^1}$ -matrices is closed under permutation similarity and left and right positive diagonal multiplication. In this paper, we may take all diagonal entries to be -1 [5].

3. DIGRAPH AND PARTIAL NON-POSITIVE Q -MATRICES COMPLETIONS

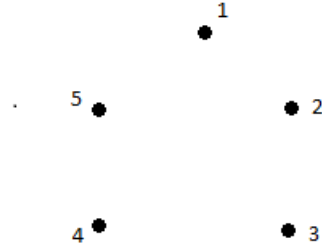
Recollect that a partial Q -matrix M is a partial matrix such that $S_k(M) < 0$ for every $k = 1, 2, 3, \dots, n$ for which all $k \times k$ principal submatrices are fully specified. Let M be a partial non positive Q -matrix. If all 1×1 principal submatrices (i.e., all diagonal entries) in M are specified, then their sum $S_1(M)$ (the trace of M) must be negative. If all $k \times k$ principal submatrices are fully specified for some $k \geq 2$, then M is fully specified and, therefore, is a non-positive Q -matrix. Thus, a partial non-positive Q -matrix is characterized as follows.

Proposition 3.1. Every $n \times n$ partial $Q_{N_0^1}$ -matrix with all specified off-diagonal entries negative has $Q_{N_0^1}$ -matrix completion.

Theorem 3.1. Let $M = [m_{ij}]$ be a partial non-positive $Q_{N_0^1}$ -matrix and $\alpha = \{i : m_{ij}\}$ is specified. If the principal partial sub matrix $M(\alpha)$ of M has a non-positive $Q_{N_0^1}$ -completion, then M has non-positive $Q_{N_0^1}$ -completion.

4. ANALYSIS OF 5 X 5 MATRICES SPECIFYING DIGRAPHS WITH 5 VERTICES:

In this section, the partial non-positive Q -matrices are extracted from the digraphs as follows: A specified entry a_{ij} will be used to represent an arc in the digraph, an unspecified entry x_{ij} will be used to represent a missing arc in the digraph while d_{ii} will specify the diagonal entries to be -1 . All possible digraph with 5 vertices and 0 to 7 edges are considered and 5×5 matrices specifying these digraphs extracted. Throughout the paper, p represents the number of vertices and q represents the number of edges of the digraph.

4.1. Deliberate the digraph below: $p = 5, q = 0$.

Let

$$A = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q – matrix representing the digraph. For $t < 0$, By definition of the completion, $d_{ii} = -1, x_{ij} = -t$

$$A(t) = \begin{bmatrix} -1 & -t & -t & -t & -t \\ -t & -1 & -t & -t & -t \\ -t & -t & -1 & -t & -t \\ -t & -t & -t & -1 & -t \\ -t & -t & -t & -t & -1 \end{bmatrix}$$

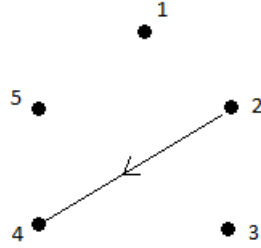
of A . Then

$$\begin{aligned} S_1(A(t)) &= -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55}); \\ S_2(A(t)) &= -10t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1; \\ S_3(A(t)) &= -18t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2; \\ S_4(A(t)) &= -15t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3; \\ S_5(A(t)) &= -4t^5 + f_5(t) \text{ where } \deg f_5(t) \leq 4. \end{aligned}$$

Here $f_i(t)$ is a polynomial in t of degree at most $i, i = 2, 3, 4, 5$.

Consequently, $A(t)$ is a non-positive Q -matrix for sufficiently large $t, S_k(A) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence A has $Q_{N_0^1}$ completion. This digraph above is a null graph and this shows that a null graph has $Q_{N_0^1}$ completion.

4.2. Deliberate the digraph below: $p = 5, q = 1$.



Let

$$B = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & a_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q – matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$, $a_{24} \leq 0$

$$B(t) = \begin{bmatrix} -1 & -t & -t & -t & -t \\ -t & -1 & -t & a_{24} & -t \\ -t & -t & -1 & -t & -t \\ -t & -t & -t & -1 & -t \\ -t & -t & -t & -t & -1 \end{bmatrix}$$

of B . Then

$$S_1(B(t)) = -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55});$$

$$S_2(B(t)) = -9t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1;$$

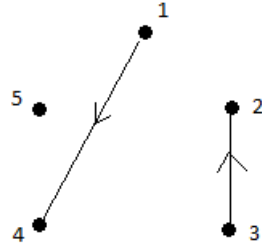
$$S_3(B(t)) = -15t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2;$$

$$S_4(B(t)) = -12t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3;$$

$$S_5(B(t)) = -3t^5 + f_5(t) \text{ where } \deg f_5(t) \leq 4.$$

Here $f_i(t)$ is a polynomial in t of degree at most i , $i = 2, 3, 4, 5$.

Consequently, $B(t)$ is a non-positive Q -matrix for sufficiently large t , $S_k(B) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence B has $Q_{N_0^1}$ completion.

4.3. Deliberate the digraph below: $p = 5, q = 2$.

Let

$$C = \begin{bmatrix} d_{11} & x_{12} & x_{13} & a_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q – matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$, $a_{14} \leq 0$, $a_{32} \leq 0$,

$$C(t) = \begin{bmatrix} -1 & -t & -t & a_{14} & -t \\ -t & -1 & -t & -t & -t \\ -t & a_{32} & -1 & -t & -t \\ -t & -t & -t & -1 & -t \\ -t & -t & -t & -t & -1 \end{bmatrix}$$

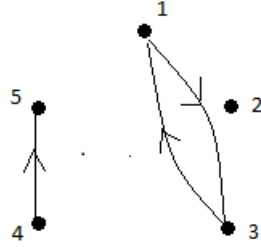
of C . Then

$$\begin{aligned} S_1(C(t)) &= -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55}); \\ S_2(C(t)) &= -8t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1; \\ S_3(C(t)) &= -13t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2; \\ S_4(C(t)) &= -9t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3; \\ S_5(C(t)) &= -2t^5 + f_5(t) \text{ where } \deg f_5(t) \leq 4. \end{aligned}$$

Here $f_i(t)$ is a polynomial in t of degree at most i , $i = 2, 3, 4, 5$.

Consequently, $C(t)$ is a non-positive Q -matrix for sufficiently large t , $S_k(C) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence C has $Q_{N_0^1}$ completion.

4.4. Deliberate the digraph below: $p = 5, q = 3$.



Let

$$D = \begin{bmatrix} d_{11} & x_{12} & a_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ a_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q – matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$, $a_{13} \leq 0$, $a_{31} \leq 0$, $a_{45} \leq 0$;

$$D(t) = \begin{bmatrix} -1 & -t & a_{13} & -t & -t \\ -t & -1 & -t & -t & -t \\ a_{31} & -t & -1 & -t & -t \\ -t & -t & -t & -1 & a_{45} \\ -t & -t & -t & -t & -1 \end{bmatrix}$$

of D . Then

$$S_1(D(t)) = -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55});$$

$$S_2(D(t)) = -9t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1;$$

$$S_3(D(t)) = -12t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2;$$

$$S_4(D(t)) = -6t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3;$$

$$S_5(D(t)) = 3t^4[d_{11} - a_{13} - a_{31} + d_{33}] + f_5(t) \text{ where } \deg f_5(t) \leq 4.$$

Here $f_i(t)$ is a polynomial in t of degree at most i , $i = 2, 3, 4, 5$.

Consequently, $D(t)$ is a non-positive Q -matrix for sufficiently large t , $S_k(D) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence D has $Q_{N_0^1}$ completion.

4.5. Example. We show that D have $Q_{N_0^1}$ - completion by a counter example as shown below.

Let

$$D = \begin{bmatrix} d_{11} & x_{12} & a_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ a_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q -matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$ where $t = 3 \leq t \leq 7$, $a_{13} = -6$, $a_{31} = -4$, $a_{45} = 5$

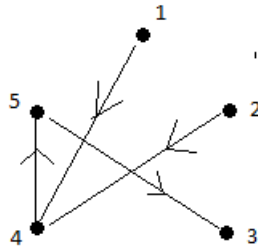
$$D(t) = \begin{bmatrix} -1 & -3 & -6 & -3 & -3 \\ -3 & -1 & -3 & -3 & -3 \\ -4 & -3 & -3 & -3 & -3 \\ -3 & -3 & -3 & -1 & -5 \\ -3 & -3 & -3 & -3 & -1 \end{bmatrix}$$

of D . Then

$$S_1(D(t)) = -5; \quad S_2(D(t)) = -93; \quad S_3(D(t)) = -339; \quad S_4(D(t)) = -52; \quad S_5(D(t)) = -232.$$

This implies that D is $Q_{N_0^1}$ -matrix. Hence there is $Q_{N_0^1}$ -completion for the digraph ($3 \leq t \leq 7$). Otherwise there D is not a $Q_{N_0^1}$ -completion for the digraph.

4.6. Deliberate the digraph below: $p = 5$, $q = 4$.



Let

$$E = \begin{bmatrix} d_{11} & x_{12} & x_{13} & a_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & a_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ x_{51} & x_{52} & a_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q – matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$, $a_{14} \leq 0$, $a_{24} \leq 0$, $a_{45} \leq 0$, $a_{53} \leq 0$

$$E(t) = \begin{bmatrix} -1 & -t & -t & a_{14} & -t \\ -t & -1 & -t & a_{24} & -t \\ -t & -t & -1 & -t & -t \\ -t & -t & -t & -1 & a_{45} \\ -t & -t & a_{53} & -t & -1 \end{bmatrix}$$

of E . Then

$$S_1(E(t)) = -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55});$$

$$S_2(E(t)) = -6t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1;$$

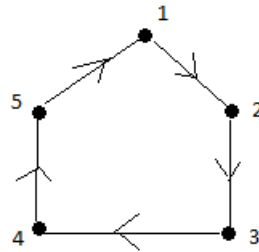
$$S_3(E(t)) = -11t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2;$$

$$S_4(E(t)) = -7t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3;$$

$S_5(E(t)) = -t^5 + f_5(t)$ where $\deg f_5(t) \leq 4$. Here $f_i(t)$ is a polynomial in t of degree at most i , $i = 2, 3, 4, 5$.

Consequently, $E(t)$ is a non-positive Q -matrix for sufficiently large t , $S_k(E) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence E has $Q_{N_0^1}$ completion.

4.7. Deliberate the digraph below: $p = 5$, $q = 5$.



Let

$$F = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q -matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$, $a_{12} \leq 0$, $a_{23} \leq 0$, $a_{34} \leq 0$, $a_{45} \leq 0$, $a_{51} \leq 0$

$$F(t) = \begin{bmatrix} -1 & a_{12} & -t & -t & -t \\ -t & -1 & a_{23} & -t & -t \\ -t & -t & -1 & a_{34} & -t \\ -t & -t & -t & -1 & a_{45} \\ a_{51} & -t & -t & -t & -1 \end{bmatrix}$$

of F . Then

$$S_1(F(t)) = -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55});$$

$$S_2(F(t)) = -5t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1;$$

$$S_3(F(t)) = -9t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2;$$

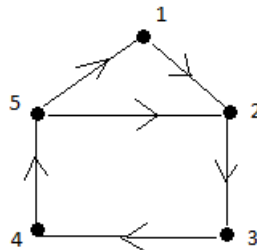
$$S_4(F(t)) = -5t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3;$$

$$S_5(F(t)) = -3t^5 + f_5(t) \text{ where } \deg f_5(t) \leq 4.$$

Here $f_i(t)$ is a polynomial in t of degree at most i , $i = 2, 3, 4, 5$.

Consequently, $F(t)$ is a non-positive Q -matrix for sufficiently large t , $S_k(F) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence F has $Q_{N_0^1}$ completion.

4.8. Deliberate the digraph below: $p = 5$, $q = 6$.



Let

$$G = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & a_{52} & x_{53} & x_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q – matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$, $a_{12} \leq 0$, $a_{23} \leq 0$, $a_{34} \leq 0$, $a_{45} \leq 0$, $a_{51} \leq 0$, $a_{52} \leq 0$

$$G(t) = \begin{bmatrix} -1 & a_{12} & -t & -t & -t \\ -t & -1 & a_{23} & -t & -t \\ -t & -t & -1 & a_{34} & -t \\ -t & -t & -t & -1 & a_{45} \\ a_{51} & a_{52} & -t & -t & -1 \end{bmatrix}$$

of G . Then

$$S_1(G(t)) = -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55});$$

$$S_2(G(t)) = -4t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1;$$

$$S_3(G(t)) = -8t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2;$$

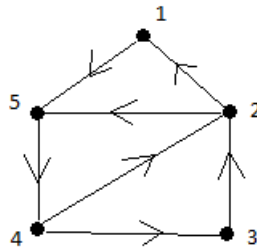
$$S_4(G(t)) = -4t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3;$$

$$S_5(G(t)) = -t^5 + f_5(t) \text{ where } \deg f_5(t) \leq 4.$$

Here $f_i(t)$ is a polynomial in t of degree at most i , $i = 2, 3, 4, 5$.

Consequently, $G(t)$ is a non-positive Q -matrix for sufficiently large t , $S_k(G) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence G has $Q_{N_0^1}$ completion.

4.9. Deliberate the digraph below: $p = 5$, $q = 7$.



Let

$$H = \begin{bmatrix} d_{11} & x_{12} & x_{13} & x_{14} & a_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & a_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & a_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & a_{54} & d_{55} \end{bmatrix}$$

be the partial non-positive Q -matrix representing the digraph.

For $t < 0$, By definition of the completion, $d_{ii} = -1$, $x_{ij} = -t$, $a_{15} \leq 0$, $a_{21} \leq 0$, $a_{25} \leq 0$, $a_{32} \leq 0$, $a_{42} \leq 0$, $a_{43} \leq 0$, $a_{54} \leq 0$

$$H(t) = \begin{bmatrix} -1 & -t & -t & -t & a_{15} \\ a_{21} & -1 & -t & -t & a_{25} \\ -t & a_{32} & -1 & -t & -t \\ -t & a_{42} & a_{43} & -1 & -t \\ -t & -t & -t & a_{54} & -1 \end{bmatrix}$$

of H . Then

$$\begin{aligned} S_1(H(t)) &= -(d_{11} + d_{22} + d_{33} + d_{44} + d_{55}); \\ S_2(H(t)) &= -3t^2 + f_2(t) \text{ where } \deg f_2(t) \leq 1; \\ S_3(H(t)) &= -7t^3 + f_3(t) \text{ where } \deg f_3(t) \leq 2; \\ S_4(H(t)) &= -5t^4 + f_4(t) \text{ where } \deg f_4(t) \leq 3; \\ S_5(H(t)) &= -t^5 + f_5(t) \text{ where } \deg f_5(t) \leq 4. \end{aligned}$$

Here $f_i(t)$ is a polynomial in t of degree at most i , $i = 2, 3, 4, 5$.

Consequently, $H(t)$ is a non-positive Q -matrix for sufficiently large t , $S_k(H) < 0$ for all $k = 1, 2, 3, 4, 5$. Hence H has $Q_{N_0^1}$ completion.

5. CONCLUSION

Hence we conclude that a null graph of order five has completion. It has also been shown that digraphs of order five with one to seven arcs which are either cyclic or acyclic have completion. In spite of this, any digraph for a 5×5 matrix in which unspecified entries (arcs) change to specified entries within a t value which has already been considered above (the counter example) does not have a non-positive $Q_{N_0^1}$ -completion. Research results from these studies can be used to fill in part of the missing data in retail surveys so that market trends can be predicted.

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