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CONVERGENCE OF URYSOHN-TYPE NONLINEAR INTEGRAL OPERATORS AT p–LEBESGUE POINT

Sevgİ Esen Almali[1] and Davut Kayabaşi

Dedicated to the memory of A.D. Gadjiev

ABSTRACT. In this paper, a general form of the family of Urysohn-type nonlinear integral operators with kernel $K_\lambda(x,t,g)$ is discussed and theorems about the point convergence of this family at p–Lebesgue points of a function in L_p are given. Here, λ is the accumulation point and is a positive parameter that changes in the real numbers. Kernel function $K_\lambda(x,t,g(t))$ is an entire analytic function with respect to its third variable.

1. INTRODUCTION

Representing a given function f as the limit of a set of functions enriched with better properties to obtain an easy computation is a problem of Approximation Theory for longer than a century. After the importance of the problem was understood, the first answer was given by the German mathematician K. Weierstrass [23] in 1885.

In his convergence theorem, K. Weierstrass showed that "every continuous function f defined in the interval $[a,b]$ has a uniformly convergent polynomial sequence

[1]*corresponding author*
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in that interval". Many mathematicians have worked on a shorter and more convenient proof of this theorem, known as the Weierstrass convergence theorem. The most obvious and influential proof was given by the Russian mathematician S. N. Bernstein [8] in 1912.

In the process that followed, many mathematicians tackled the approximation problem for functions of various classes. One of these classes, the class of integrable functions, is a larger class than the class of continuous functions that Weierstrass worked with. The integral in Lebesgue sense of a function from the class of integrable functions is defined outside the set whose Lebesgue measure is zero. Therefore, when the approximation problem is considered in this class, it has been seen that it is more appropriate to approximate a function of this class with a sequence or a family of integral operators.

The family of linear integral operators, which is a special case of linear positive operators can be given as

$$L_\lambda(f;x) = \int_D f(t)\, K_\lambda(t,x)\, dt, \quad x \in D, \quad \lambda \in \Lambda,$$

for each real number $\lambda \in \Lambda$, where $K_\lambda(t,x)$ is a kernel family and Λ is a non-empty set of indices. Examples of studies on such integral operators are given by Mamedov [17] and Esen [10].

Linear positive operators are easy to construct and examining their properties is uncomplicated compared to nonlinear operators. The simplest constructions for approximating functions can often be described by linear positive operators (see [14]). For this reason, studies on linear positive operators have had a very important place in approximation theory since the 1950s. In addition to these studies on linear operators, which have a wide place in the literature, there are also many studies on the family of non-linear integral operators and their approximation properties.

One of these studies was given by Vainberg [22] in 1953. In this study, Vainberg examined the existence of characteristic functions for nonlinear integral operators with a nonpositive kernels and for the product of self-adjoint and potential operators [22].

In 1966, Gadjiev [11] examined approximation by the family of Hammerstein-type nonlinear integral operators.

Later, in 1981, Musielak introduced an approximation problem using nonlinear integral operators defined in [18] as

$$T_\omega f(s) = \int_G K_\omega(t-s, f(t))\,dt, \quad s \in G.$$

Also, Swiderski and Wachnicki [21], Bardaro and Vinti [5], Bardaro et al. [6], Bardaro et al. [7], Almali and Gadjiev [1], Almali [2] and Guller and Uysal [12] studied on nonlinear integral operators.

Another study on nonlinear integral operators is Anar's work [4] in 2019. In the indicated work, Anar examined the approximation of the integral funnel of the closed ball of $L_p\,(p>1)$ space using the Urysohn-type integral operator.

Guller [13] presented the more recent study on this subject. In this study, Guller gave Fatou-type (pointwise) convergence theorems for infinite sum of nonlinear integral operators represented as

$$\Psi_v(f;x) = \sum_{m=1}^{\infty} \int_a^b f^m(t)\, H_{v,m}(t-x)\,dt, \quad v \in \Lambda, \quad x \in (a,b)$$

at the generalized type characteristic points of the function $f \in L_p(a,b)$.

Another study on nonlinear integral operators was given by Almali [3] in 2019. In this study, Almali considered a general form of the family of Urysohn-type nonlinear integral operators defined as

$$L_\lambda(u,x) = \int_{\mathbb{R}} K_\lambda(x,t,u(t))\,dt,$$

where $K_\lambda(x,t,u)$ is the kernel function, $\mathbb{R}$ denotes the real axis and λ is a parameter varying in the set of positive real numbers. When we dive into the the details of this study, we see that the Taylor series representation of the kernel of the operator is used and the theorems and proofs about the pointwise convergence of this operator at Lebesgue points of a function in L_1 are given under some assumptions satisfied by the kernel function K_λ.

The present study is a continuation of [3]. In this study, unlike the Almali's study, theorems about the pointwise convergence of the family of Urysohn-type nonlinear integral operators at $p-$Lebesgue points of a function in L_p are given.

2. PRELIMINARIES

The family of integral operators

$$T_\lambda (g, x) = \int_{\mathbb{R}} K_\lambda (x, t, g(t))\, dt \tag{2.1}$$

is a general form of the nonlinear Urysohn integral operator with kernel $K_\lambda (x, t, g)$, where $\mathbb{R}$ represents the real axis [16]. Here, λ is a positive parameter varying in the set of real numbers. In addition, the kernel function $K_\lambda (x, t, g(t))$ is an entire analytic function with respect to the variable g.

If the equation

$$K_\lambda (x, t, g(t)) = H_\lambda (x, t)\, E(t, g(t))$$

is satisfied, the operator (2.1) turns into a nonlinear family of Hammerstein integrals. Gadjiev [11] showed that this family of nonlinear integrals converges to the original function.

When the equation

$$K_\lambda (x, t, g(t)) = g(t)\, W_\lambda (x, t) \tag{2.2}$$

is satisfied, the operators defined in (2.1) turn into a family of linear integral operators that have many applications in approximation theory [9]. Also, the kernel of linear integral operators $W_\lambda (x, t)$ in approximation theory satisfies the equation

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} W_\lambda (x, t)\, dt = 1$$

for all values of x [3]. Therefore, if any constant value $g(x_0)$ is taken instead of $g(t)$ in (2.2), the equation

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} K_\lambda (x, t, g(x_0))\, dt = \lim_{\lambda\to\infty} \int_{\mathbb{R}} g(x_0)\, W_\lambda (x, t)\, dt = g(x_0)$$

is obtained.

In this study, the pointwise convergence problem of the Urysohn-type nonlinear integral operator including the linear operator will be examined in accordance with the following explanations and notations. When g^* is independent of t, we have

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}} K_\lambda\left(x,t,g^*\right)dt=g^*$$

[21]. These formulas contain the kernel $K_\lambda\left(x,t,g\right)$ and its partial derivatives at the value $g=g\left(x_0\right)$. This will be represented by the notation $g_0=g\left(x_0\right)$. Here, K_λ will also be the notation for the kernel.

If the kernel $K_\lambda\left(x,t,g\right)$ is an entire analytic function of the variable g, then the Taylor expansion of this kernel function in powers of $\left[g\left(t\right)-g\left(x_0\right)\right]$ is

$$K_\lambda\left(x,t,g\left(t\right)\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial g^n}K_\lambda\left(x,t,g\right)|_{g=g(x_0)}\left[g\left(t\right)-g\left(x_0\right)\right]^n.$$

From now on, the notation

$$K_{\lambda,g}^{(n)}\left(x,t,g_0\right)=\frac{\partial^n K_\lambda\left(x,t,g\right)}{\partial g^n}|_{g=g(x_0)}$$

will be used. According to this notation, the kernel K_λ can be written as

$$K_\lambda\left(x,t,g\left(t\right)\right)=\sum_{n=0}^{\infty}\frac{1}{n!}K_{\lambda,g}^{(n)}\left(x,t,g_0\right)\left[g\left(t\right)-g\left(x_0\right)\right]^n. \tag{2.3}$$

3. THEOREMS ON POINTWISE CONVERGENCE

In this part of the study, firstly, the properties related to the kernel of the family of Urysohn-type nonlinear integral operators will be given. Afterwards, by considering some basic concepts of the analysis, the assumptions to be used in the theorems regarding the convergence of these operators in L_p will be expressed. Finally, convergence theorems and proofs of the family of Urysohn-type nonlinear integral operators will be given.

We let the following conditions be satisfied for the K_λ kernel (see [3]):

(a) The kernel $K_\lambda\left(x,t,g\right)$ is an entire analytic function of the variable g for each constant $x,t\in\mathbb{R}$.

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}} K_\lambda\left(x,t,g_0\right)dt=g_0$$

is provided, where $\lambda>0$ is a parameter and g_0 is a value independent of t.

(b) For any constant $x = x_0$, the expression $K_{\lambda,g}^{(n)}(x_0, t, g_0)$ is monotonically increasing when $t < x_0$ and decreasing monotonically when $t > x_0$ with respect to t.

(c) $K_{\lambda,g}^{(n)}(x, y, g_0)$ is a non-negative function and for any numbers x, y and $n = 1, 2, \ldots$ the inequality

$$K_{\lambda,g}^{(n)}(x, y, g_0) \leq a(\lambda)$$

is satisfied, where $a(\lambda) \to 0$ as $\lambda \to \infty$.

(d) For any $n = 1, 2, \ldots$ and $y \neq x_0$,

$$K_{\lambda,g}^{(n)}(x_0, y, g_0) \leq K_{\lambda,g}^{'}(x_0, y, g_0).$$

(e)

$$\int_{\mathbb{R}} K_{\lambda,g}^{(n)}(x_0, t, g_0)\, dt = a_n, \quad n = 1, 2, \ldots$$

where a_n is independent of λ, but can depend on x_0.

(f) For any $\delta > 0$,

$$\lim_{\lambda \to \infty} \int_{|t - x_0| \geq \delta} K_{\lambda,g}^{'}(x_0, t, g_0)\, dt = 0.$$

(g) For any $y \neq x_0$,

$$\lim_{\lambda \to \infty} K_{\lambda,g}^{'}(x_0, y, g_0) = 0.$$

Remark 3.1. *Condition (d) indicates that the sequence (a_n) in (e) is bounded.*

In order to see that the conditions (a)-(g) above are met, the example in Almali's [3] study can be examined.

Condition (c) shows that for a bounded function $g(x)$, the Taylor series of the kernel K_λ is absolutely uniformly convergent. Therefore, the remainder term of the Taylor series tends to 0. This allows to change the order of the integral and the sum. If both sides of equation (2.3) are integrated, the operator

$$T_\lambda(g, x) = \int_{\mathbb{R}} K_\lambda(x, t, g(t))\, dt = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} K_{\lambda,g}^{(n)}(x, t, g_0)\, [g(t) - g(x_0)]^n\, dt \tag{3.1}$$

is obtained. This operator depends on the Taylor series expansion under suitable conditions and will be at the center of the study.

Let us now give the theorems on the convergence of the family of nonlinear integral operators of Urysohn-type at $p-$Lebesgue point of $g \in L_p(\mathbb{R})$.

Theorem 3.1. *Let $g \in L_p(\mathbb{R})$ be a bounded function on $\mathbb{R}$ and the kernel $K_\lambda(x,t,g)$ satisfy the conditions (a)-(g). Then, at each $p-$Lebesgue point $x_0 \in \mathbb{R}$ of the function g, one has*

$$\lim_{\lambda\to\infty} T_\lambda(g,x_0) = g(x_0).$$

Proof. The equation in (3.1) can be written as

$$T_\lambda(g,x_0) = \int_{\mathbb{R}} K_\lambda(x_0,t,g_0)\,dt + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\mathbb{R}}[g(t)-g(x_0)]^n K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt.$$

If $g(x_0)$ is subtracted from both sides and the triangle inequality is applied, we have

$$\begin{aligned}|T_\lambda(g,x_0)-g(x_0)| &\leq \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\mathbb{R}}|g(t)-g(x_0)|^n K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt\\ &+ \left|\int_{\mathbb{R}} K_\lambda(x_0,t,g_0)\,dt - g(x_0)\right|.\end{aligned}$$

According to Hölder's integral inequality (see [20]), this inequality can be written as

$$\begin{aligned}|T_\lambda(g,x_0)-g(x_0)| \leq \sum_{n=1}^{\infty}&\left\{\begin{array}{c}\frac{1}{n!}\left(\int_{\mathbb{R}}|g(t)-g(x_0)|^{np}K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt\right)^{\frac{1}{p}}\\ \times\left(\int_{\mathbb{R}}K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt\right)^{\frac{1}{q}}\end{array}\right\}\\ &+\left|\int_{\mathbb{R}}K_\lambda(x_0,t,g_0)\,dt-g(x_0)\right|\end{aligned}$$

with $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Here, if

$$(a+b)^p \leq 2^p(a^p+b^p) \tag{3.2}$$

(see [19]) and Hölder's infinite series inequality (see [20]) are respectively applied, then

$$|T_\lambda(g,x_0)-g(x_0)|^p \leq 2^p\sum_{n=1}^{\infty}\left(\frac{1}{n!}\right)^p\int_{\mathbb{R}}|g(t)-g(x_0)|^{np}K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt$$

$$\times\left(\sum_{n=1}^{\infty}\int\limits_{\mathbb{R}}K_{\lambda,g}^{(n)}\left(x_0,t,g_0\right)dt\right)^{\frac{p}{q}}+2^p\left|\int\limits_{\mathbb{R}}K_{\lambda}\left(x_0,t,g_0\right)dt-g\left(x_0\right)\right|^p \tag{3.3}$$

is obtained. At this stage of the proof, it will be discussed that bounded function $g\in L_p\left(\mathbb{R}\right)$ has a $p-$Lebesgue point.

If the function g belongs to $L_p\left(\mathbb{R}\right)$ and is bounded on $\mathbb{R}$, there is a positive number $A\in\mathbb{R}$ for each $t\in\mathbb{R}$ such that $\left|g\left(t\right)\right|\leq A$. Accordingly, the inequalities

$$\begin{aligned}\left|g\left(t\right)-g\left(x_0\right)\right| &\leq 2A\\ \left|g\left(t\right)-g\left(x_0\right)\right|^n &\leq \left(2A\right)^{n-1}\left|g\left(t\right)-g\left(x_0\right)\right|,\ \ n=1,2,\ldots\\ \left|g\left(t\right)-g\left(x_0\right)\right|^{np} &\leq \left(2A\right)^{(n-1)p}\left|g\left(t\right)-g\left(x_0\right)\right|^p,\ \ 1<p<\infty \qquad (3.4)\end{aligned}$$

can be written.

If the point x_0 is a $p-$Lebesgue point of the function $g\in L_p\left(\mathbb{R}\right)$ for $1<p<\infty$, for every $\varepsilon>0,$ there is at least one number $\delta>0$ such that $0<h\leq\delta$, we have

$$\int\limits_{x_0-h}^{x_0}\left|g\left(t\right)-g\left(x_0\right)\right|^p dt<\varepsilon^p h \tag{3.5}$$

and

$$\int\limits_{x_0}^{x_0+h}\left|g\left(t\right)-g\left(x_0\right)\right|^p dt<\varepsilon^p h. \tag{3.6}$$

Therefore, according to the inequalities (3.4) and (3.5), the inequality

$$\int\limits_{x_0-h}^{x_0}\left|g\left(t\right)-g\left(x_0\right)\right|^{np} dt\leq\left(2A\right)^{(n-1)p}\varepsilon^p h \tag{3.7}$$

is satisfied. According to the inequalities (3.4) and (3.6), the inequality

$$\int\limits_{x_0}^{x_0+h}\left|g\left(t\right)-g\left(x_0\right)\right|^{np} dt\leq\left(2A\right)^{(n-1)p}\varepsilon^p h \tag{3.8}$$

is satisfied.

Now, returning to the inequality given in (3.3), this inequality can be represented as

$$
\begin{aligned}
&\left|T_\lambda(g,x_0)-g(x_0)\right|^p \\
\leq\;& 2^p\sum_{n=1}^{\infty}\left(\frac{1}{n!}\right)^p\left\{\begin{array}{l}\displaystyle\int\limits_{|t-x_0|\leq\delta}\left|g(t)-g(x_0)\right|^{np}K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt\\ \displaystyle+\int\limits_{|t-x_0|\geq\delta}\left|g(t)-g(x_0)\right|^{np}K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt\end{array}\right\}\\
\times\;&\left(\sum_{n=1}^{\infty}\int\limits_{\mathbb{R}}K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt\right)^{\frac{p}{q}}+2^p\left|\int\limits_{\mathbb{R}}K_\lambda(x_0,t,g_0)\,dt-g(x_0)\right|^p
\end{aligned}
$$

$$
\begin{aligned}
&\left|T_\lambda(g,x_0)-g(x_0)\right|^p \\
\leq\;& 2^p\sum_{n=1}^{\infty}\left(\frac{1}{n!}\right)^p\{I_1+I_2\}\times\left(\sum_{n=1}^{\infty}\int\limits_{\mathbb{R}}K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt\right)^{\frac{p}{q}}\\
+\;&2^p\left|\int\limits_{\mathbb{R}}K_\lambda(x_0,t,g_0)\,dt-g(x_0)\right|^p
\end{aligned}
\tag{3.9}
$$

according to the determined number $\delta>0$. Here, the integral under the sum is bounded by condition (e). The absolute value of expression, on the other hand, tends to be 0 when $\lambda\to\infty$ by condition (a). Therefore, to complete the proof, it remains to examine the integrals I_1 and I_2 in (3.9).

First, we consider the integral I_2. When inequality (3.4) and condition (d) are applied respectively, this integral can be written as

$$
I_2\leq\int\limits_{|t-x_0|\geq\delta}(2A)^{(n-1)p}\left|g(t)-g(x_0)\right|^p K_{\lambda,g}^{'}(x_0,t,g_0)\,dt.
$$

If this inequality is rearranged using the triangle inequality first and then using the inequality (3.2), we obtain

$$
\begin{aligned}
I_2\leq\;&\frac{(2A)^{np}}{A^p}\int\limits_{|t-x_0|\geq\delta}\left|g(t)\right|^p K_{\lambda,g}^{'}(x_0,t,g_0)\,dt\\
&+(2A)^{np}\int\limits_{|t-x_0|\geq\delta}K_{\lambda,g}^{'}(x_0,t,g_0)\,dt.
\end{aligned}
\tag{3.10}
$$

According to condition (b), $K_{\lambda,g}^{'}(x_0,t,g_0)$ is a monotonically increasing function in the interval $(-\infty, x_0-\delta]$ and monotonically decreasing in the interval $[x_0+\delta,\infty)$. If the definition of the norm is also used, we can write

$$\int\limits_{|t-x_0|\geq\delta} |g(t)|^p K_{\lambda,g}^{'}(x_0,t,g_0)\,dt \leq \|g\|_{L_p(\mathbb{R})}^p \Big(K_{\lambda,g}^{'}(x_0,x_0-\delta,g_0)+K_{\lambda,g}^{'}(x_0,x_0+\delta,g_0)\Big)$$

for the first integral to the right of the inequality. Then,

$$\begin{aligned} I_2 &\leq \frac{(2A)^{np}}{A^p}\|g\|_{L_p(\mathbb{R})}^p \Big(K_{\lambda,g}^{'}(x_0,x_0-\delta,g_0)+K_{\lambda,g}^{'}(x_0,x_0+\delta,g_0)\Big) \\ &\quad + (2A)^{np} \int\limits_{|t-x_0|\geq\delta} K_{\lambda,g}^{'}(x_0,t,g_0)\,dt \end{aligned}$$

is obtained from the inequality (3.10). This means that $I_2 \to 0$ for $\lambda\to\infty$ considering conditions (f) and (g).

Finally, we consider the integral I_1. This integral can be written as

$$\begin{aligned} I_1 &= \int\limits_{x_0-\delta}^{x_0} |g(t)-g(x_0)|^{np} K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt \\ &\quad + \int\limits_{x_0}^{x_0+\delta} |g(t)-g(x_0)|^{np} K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt = I_{11}+I_{12}. \end{aligned} \tag{3.11}$$

In this case, it is necessary to show separately that the integrals I_{11} and I_{12} converge to 0.

First, we calculate the integral I_{11}. For this, we define a function $F(t)$ of the form

$$F(t)=\int\limits_{t}^{x_0} |g(s)-g(x_0)|^{np}\,ds$$

taking into account that inequality (3.7) is satisfied. Thus, for $F(t)$, when $x_0-t\leq\delta$ (that is, when $x_0-t=h$), the inequality

$$|F(t)| \leq (2A)^{(n-1)p}\varepsilon^p(x_0-t) \tag{3.12}$$

is satisfied. Also, the differential of the function $F(t)$ becomes

$$dF(t) = -|g(t)-g(x_0)|^{np}\,dt.$$

The integral I_{11} can be written as

$$I_{11} = \int\limits_{x_0-\delta}^{x_0} |g(t) - g(x_0)|^{np} K_{\lambda,g}^{(n)}(x_0, t, g_0)\, dt = -\int\limits_{x_0-\delta}^{x_0} K_{\lambda,g}^{(n)}(x_0, t, g_0)\, dF(t)$$

according to the theorem expressing the conversion of the Lebesgue integral to the Stieltjes integral (see [15]). Here, if the method of integration by parts, triangle inequality and (3.12) inequality are applied to the Stieltjes integral, respectively,

$$\begin{aligned} |I_{11}| &\leq (2A)^{(n-1)p} \varepsilon^p \delta K_{\lambda,g}^{(n)}(x_0, x_0-\delta, g_0) \\ &\quad + (2A)^{(n-1)p} \varepsilon^p \int\limits_{x_0-\delta}^{x_0} (x_0 - t)\, d_t \left[K_{\lambda,g}^{(n)}(x_0, t, g_0)\right] \end{aligned}$$

is obtained. If the method of integration by parts is applied here again, the result

$$\begin{aligned} |I_{11}| &\leq (2A)^{(n-1)p} \varepsilon^p \int\limits_{x_0-\delta}^{x_0} K_{\lambda,g}^{(n)}(x_0, t, g_0)\, dt \\ &\leq (2A)^{(n-1)p} \varepsilon^p \int\limits_{\mathbb{R}} K'_{\lambda,g}(x_0, t, g_0)\, dt \\ &= (2A)^{(n-1)p} \varepsilon^p a_1 \end{aligned} \tag{3.13}$$

is reached.

Now, we calculate the integral I_{12}. For this aim, we define a $G(t)$ function in the form

$$G(t) = \int\limits_{x_0}^{t} |g(z) - g(x_0)|^{np}\, dz,$$

taking into account the (3.8) inequality. Thus, for $G(t)$, when $t - x_0 \leq \delta$ (that is, when $t - x_0 = h$), the inequality

$$|G(t)| \leq (2A)^{(n-1)p} \varepsilon^p (t - x_0) \tag{3.14}$$

is satisfied. Also, the differential of the $G(t)$ function is

$$dG(t) = |g(t) - g(x_0)|^{np}\, dt.$$

Therefore, the integral I_{12} in the sense of Lebesgue can be converted to the Stieltjes integral as

$$I_{12} = \int_{x_0}^{x_0+\delta} |g(t) - g(x_0)|^{np} K_{\lambda,g}^{(n)}(x_0, t, g_0)\, dt = \int_{x_0}^{x_0+\delta} K_{\lambda,g}^{(n)}(x_0, t, g_0)\, dG(t).$$

From here,

$$\begin{aligned} I_{12} &= K_{\lambda,g}^{(n)}(x_0, x_0+\delta, g_0)\, G(x_0+\delta) - K_{\lambda,g}^{(n)}(x_0, x_0, g_0)\, G(x_0) \\ &\quad + \int_{x_0}^{x_0+\delta} G(t)\, d_t\left[-K_{\lambda,g}^{(n)}(x_0, t, g_0)\right] \end{aligned}$$

is obtained by using the method of integration by parts. Since the expression $-K_{\lambda,g}^{(n)}(x_0, t, g_0)$ in the equation will be an increasing function according to the condition (b), the differential $d_t\left[-K_{\lambda,g}^{(n)}(x_0, t, g_0)\right]$ becomes positive. Accordingly, if the triangle inequality and the inequality at (3.14) are applied to the equation, respectively,

$$\begin{aligned} |I_{12}| &\le (2A)^{(n-1)p}\, \varepsilon^p \delta K_{\lambda,g}^{(n)}(x_0, x_0+\delta, g_0) \\ &\quad + (2A)^{(n-1)p}\, \varepsilon^p \int_{x_0}^{x_0+\delta} (t - x_0)\, d_t\left[-K_{\lambda,g}^{(n)}(x_0, t, g_0)\right] \end{aligned}$$

is obtained. If the method of integration by parts is applied here again, the result

$$|I_{12}| \le (2A)^{(n-1)p}\, \varepsilon^p a_1 \tag{3.15}$$

is reached.

Therefore, if we substitute the results (3.13) and (3.15) into the equation in (3.11), we get

$$|I_1| \le 2\,(2A)^{(n-1)p}\, \varepsilon^p a_1.$$

This indicates that for a sufficiently small ε, the integral I_1 converges to 0 when $\lambda \to \infty$. Hence the proof is completed. □

The case that the kernel does not have the monotony property is expressed and proved by the following theorem.

Theorem 3.2. *Let the kernel $K_\lambda(x_0,t,g)$ satisfy conditions (a), (c), and (e). Suppose $D_{\lambda,n}(t,x_0)$ is a majorant function satisfying the condition*

$$K_{\lambda,g}^{(n)}(x_0,t,g_0) \leq D_{\lambda,n}(t,x_0) \tag{3.16}$$

for any $n = 1,2,\ldots$, and $D_{\lambda,n}$ satisfies the following conditions:

(b*) *The function $D_{\lambda,n}(t,x_0)$ is monotonically increasing for $t < x_0$ and monotonically decreasing for $t > x_0$.*

(d*)

$$\int_{\mathbb{R}} D_{\lambda,n}(t,x_0)\,dt \leq b < \infty,$$

where b is independent of λ.

(f*) *For any constant $\delta > 0$,*

$$\lim_{\lambda\to\infty} \int_{|t-x_0|\geq\delta} D_{\lambda,n}(t,x_0)\,dt = 0.$$

(g*) *For any $y \neq x_0$,*

$$\lim_{\lambda\to\infty} D_{\lambda,n}(y,x_0) = 0.$$

Then, at each p–Lebesgue point $x_0 \in \mathbb{R}$ of the globally bounded function $g \in L_p(\mathbb{R})$, one has

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} K_\lambda(x_0,t,g(t)) = g(x_0).$$

Proof. In the proof of this theorem, a method similar to the proof of Theorem 3.1 will be followed. To begin the proof, the equation in (3.1) can be written as

$$\int_{\mathbb{R}} K_\lambda(x_0,t,g(t))\,dt = \int_{\mathbb{R}} K_\lambda(x_0,t,g_0)\,dt + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} [g(t)-g(x_0)]^n K_{\lambda,g}^{(n)}(x_0,t,g_0)\,dt.$$

First, $g(x_0)$ is subtracted from both sides of this equation, then the triangle inequality is applied, and then the inequality

$$\begin{aligned}\left|\int_{\mathbb{R}} K_\lambda(x_0,t,g(t))\,dt - g(x_0)\right| &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} |g(t)-g(x_0)|^n D_{\lambda,n}(t,x_0)\,dt \\ &+ \left|\int_{\mathbb{R}} K_\lambda(x_0,t,g_0)\,dt - g_0\right|\end{aligned}$$

is obtained if condition (3.16) is taken into account. As in the proof of Theorem 3.1, if this inequality is rearranged considering Hölder inequality and the inequality in (3.2),

$$\begin{aligned} &\left| \int_{\mathbb{R}} K_{\lambda}\left(x_0, t, g\left(t\right)\right) dt - g\left(x_0\right) \right|^p \\ \leq\ & 2^p \sum_{n=1}^{\infty} \left(\frac{1}{n!}\right)^p \int_{\mathbb{R}} \left|g\left(t\right) - g\left(x_0\right)\right|^{np} D_{\lambda,n}\left(t, x_0\right) dt \\ \times\ & \left(\sum_{n=1}^{\infty} \int_{\mathbb{R}} D_{\lambda,n}\left(t, x_0\right) dt\right)^{\frac{p}{q}} + 2^p \left| \int_{\mathbb{R}} K_{\lambda}\left(x_0, t, g_0\right) dt - g_0 \right|^p \end{aligned} \tag{3.17}$$

is obtained. Noting that we will return to this inequality at this stage of the proof, let us first consider the property that g is a bounded function, and then the property that x_0 is the $p-$Lebesgue point of g.

If g is a bounded function in $L_p\left(\mathbb{R}\right)$, then for every $t \in \mathbb{R}$ there is a positive number $A \in \mathbb{R}$ such that $\left|g\left(t\right)\right| \leq A$. Accordingly, the inequality

$$\left|g\left(t\right) - g\left(x_0\right)\right|^{np} \leq \left(2A\right)^{(n-1)p} \left|g\left(t\right) - g\left(x_0\right)\right|^p \tag{3.18}$$

can be written, where $1 < p < \infty$.

If the point x_0 is the p–Lebesgue point of the function $g \in L_p\left(\mathbb{R}\right)$ for $1 < p < \infty$, there is at least one number $\delta > 0$ for every $\varepsilon > 0$ such that when $0 < h \leq \delta$ the inequalities

$$\int_{x_0-h}^{x_0} \left|g\left(t\right) - g\left(x_0\right)\right|^{np} dt \leq \left(2A\right)^{(n-1)p} \varepsilon^p h$$

and

$$\int_{x_0}^{x_0+h} \left|g\left(t\right) - g\left(x_0\right)\right|^{np} dt \leq \left(2A\right)^{(n-1)p} \varepsilon^p h$$

are provided.

Turning now to inequality (3.17), this inequality can be represented as

$$\left|\int_{\mathbb{R}} K_\lambda\left(x_0, t, g\left(t\right)\right) dt - g\left(x_0\right)\right|^p$$

$$\leq \quad 2^p \sum_{n=1}^{\infty}\left(\frac{1}{n!}\right)^p \left\{ \begin{array}{c} \int\limits_{|t-x_0|\leq\delta} \left|g\left(t\right)-g\left(x_0\right)\right|^{np} D_{\lambda,n}\left(t,x_0\right) dt \\ + \int\limits_{|t-x_0|\geq\delta} \left|g\left(t\right)-g\left(x_0\right)\right|^{np} D_{\lambda,n}\left(t,x_0\right) dt \end{array} \right\}$$

$$\times \quad \left(\sum_{n=1}^{\infty}\int_{\mathbb{R}} D_{\lambda,n}\left(t,x_0\right) dt\right)^{\frac{p}{q}} + 2^p \left|\int_{\mathbb{R}} K_\lambda\left(x_0,t,g_0\right) dt - g_0\right|^p$$

$$\left|\int_{\mathbb{R}} K_\lambda\left(x_0, t, g\left(t\right)\right) dt - g\left(x_0\right)\right|^p \leq 2^p \sum_{n=1}^{\infty}\left(\frac{1}{n!}\right)^p \left\{J_1 + J_2\right\}$$

$$\times \quad \left(\sum_{n=1}^{\infty}\int_{\mathbb{R}} D_{\lambda,n}\left(t,x_0\right) dt\right)^{\frac{p}{q}} + 2^p \left|\int_{\mathbb{R}} K_\lambda\left(x_0,t,g_0\right) dt - g_0\right|^p \tag{3.19}$$

according to the determined number $\delta > 0$. Here, the integral under the sum is bounded by (d*) condition. The absolute value expression, on the other hand, tends to 0 for $\lambda \to \infty$ by condition (a). In this case, it is necessary to examine the integrals J_1 and J_2 in (3.19) to complete the proof.

First, we consider the J_2 integral. Considering the inequalities (3.18) and (3.2),

$$J_2 \leq \frac{\left(2A\right)^{np}}{A^p} \int\limits_{|t-x_0|\geq\delta} \left|g\left(t\right)\right|^p D_{\lambda,n}\left(t,x_0\right) dt + \left(2A\right)^{np} \int\limits_{|t-x_0|\geq\delta} D_{\lambda,n}\left(t,x_0\right) dt \tag{3.20}$$

can be written for this integral. According to the (b*) condition, $D_{\lambda,n}\left(t,x_0\right)$ majorant function is a monotonically increasing function in the interval $\left(-\infty, x_0-\delta\right]$ and monotonically decreasing in the interval $\left[x_0+\delta, \infty\right)$. If the definition of the norm is also used, the inequality

$$\int\limits_{|t-x_0|\geq\delta} \left|g\left(t\right)\right|^p D_{\lambda,n}\left(t,x_0\right) dt \leq \left\|g\right\|_{L_p(\mathbb{R})}^p \left(D_{\lambda,n}\left(x_0-\delta,x_0\right) + D_{\lambda,n}\left(x_0+\delta,x_0\right)\right)$$

can be written for the first integral to the right of the above inequality. Therefore, the result

$$J_2 \leq \frac{(2A)^{np}}{A^p} \|g\|^p_{L_p(\mathbb{R})} \left(D_{\lambda,n}(x_0-\delta,x_0)+D_{\lambda,n}(x_0+\delta,x_0)\right)$$
$$+ (2A)^{np} \int\limits_{|t-x_0|\geq\delta} D_{\lambda,n}(t,x_0)\,dt$$

is obtained from the inequality (3.20). This means $J_2 \to \infty$ for $\lambda \to \infty$ considering the conditions (f*) and (g*).

Finally, we consider the integral J_1. This integral can be expressed as

$$J_1 = \int\limits_{x_0-\delta}^{x_0} |g(t)-g(x_0)|^{np} D_{\lambda,n}(t,x_0)\,dt + \int\limits_{x_0}^{x_0+\delta} |g(t)-g(x_0)|^{np} D_{\lambda,n}(t,x_0)\,dt = J_{11}+J_{12}. \tag{3.21}$$

To show that the integral J_1 converges to 0, similar to the proof of Theorem 3.1, we consider the integrals J_{11} and J_{12}, separately. In this case, the results of the integrals J_{11} and J_{12} are calculated as

$$|J_{11}| \leq (2A)^{(n-1)p}\,\varepsilon^p b \tag{3.22}$$

and

$$|J_{12}| \leq (2A)^{(n-1)p}\,\varepsilon^p b, \tag{3.23}$$

respectively.

Therefore, if the results (3.22) and (3.23) are substituted into equation (3.21), the result

$$|J_1| \leq 2\,(2A)^{(n-1)p}\,\varepsilon^p b$$

is obtained for the integral J_1. This shows that for a sufficiently small $\varepsilon > 0$, $\lambda \to \infty$ while $J_1 \to \infty$. Thus, the proof of the theorem is completed. □

For some examples of these theorems that have been stated and proved, the study of Almali [3] can be seen.

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DEPARTMENT OF MATHEMATICS
SCIENCE AND ART FACULTY
KIRIKKALE UNIVERSITY
TURKEY
Email address: sevgi_esen@hotmail.com

DEPARTMENT OF MATHEMATICS
SCIENCE AND ART FACULTY
KIRIKKALE UNIVERSITY
TURKEY
Email address: davutkayabasi46@gmail.com