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SPECTRAL GALERKIN METHOD FOR STOCHASTIC SPACE-TIME FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

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ABSTRACT. This work is devoted to deal with a stochastic space-time fractional integro-differential equation in the Hilbert space $L^2(0,1)$, by studing its spatial approximation. Precisely, we use the spectral Galerkin method to prove that the spatial approximation converges strongly (i.e. in the space $L^p(\Omega, L^2(0,1))$), by imposing only a regularity condition on the initial value.

1. INTRODUCTION

Recently, a considerable interest in the theoretical study of the stochastic fractional integral or integro-differential equations (see [8, 11, 12, 14–16, 20] and the references therein), due to the fact that such class of equations have been used frequently as a mathematical models of many physical phenomena as the anomalous diffusions of memory processes with random effects. In general, it is not easy to solve these kind of equations analytically, for this the numerical study plays an important role by providing a numerical approximations of the analytic solutions with respect to time, to space or to both simultaneously. The main task of the numerical study for stochastic partial differential equations is to elaborate

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schemes, generally based on the deterministic numerical mathods, such as the spectral Galerkin mathod.

To the best of our knowledge, there is no work in the literatures until now is concerned with the numerical study of these kind of equations, although the importance of such study. Moreover, we can find a few new papers have dealt with the numerical approximations of the fractional stochastic partial differential equations, see e.g. [2, 4, 9, 10, 18, 19, 23].

From these facts, our contribution in the current paper will be the study of the spatial approximation of such class of equations, which is given in the following general form:

$$
\begin{aligned}
u(t) = u_0 &+ \frac{1}{\Gamma(\alpha)}\int_0^t \frac{A_\beta\, u(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{F(u(s))}{(t-s)^{1-\alpha}}ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t \frac{G}{(t-s)^{1-\alpha}}dW(s),
\end{aligned}
\tag{1.1}
$$

for any $t \in [0,T]$ with $T > 0$ be fixed, $\alpha \in (\frac{1}{2}, 1]$, where $A_\beta := (-\frac{\partial^2}{\partial^2 x})^{\frac{\beta}{2}} = A^{\frac{\beta}{2}}$, $\beta > 1$ is the fractional Laplacian and A is the minus Laplacian equiped with the Dirichlet boundary conditions, the initial condition $u_0 := u(0)$ is a $L^2(0,1)$-valued $\mathcal{F}_0$-measurable random variable, $F : L^2(0,1) \to L^2(0,1)$ and $G : L^2(0,1) \to L^2(0,1)$ are two operators, W is a $L^2(0,1)$-valued cylindrical Wiener process. The fractional integrals appear in Pr.(1.1) are considered in the Riemann-Liouville sense.

It is worth mentioning that in [7], Arab Z. and Tunc C. have studied and proved the wellposedness of Pr.(1.1) and its spatial and temporal regularity.

The paper is ordered by the following: we introduce in Section 2 some notations and preliminaries are concerned with the wellposedness of Pr.(1.1). In Section 3 we state and prove the spatial approximation of the mild solution via spectral Galerkin method. Finally, conclusion is presented in Section 4.

2. Preliminaries and notations

This section is devoted to give the wellposedness result of Pr.(1.1), that has been proved in [7]. In order to do this, we need first some notations.

Notations. $\mathbb{N}^* := \mathbb{N} - \{0\}$. For $\mathcal{O}$ an operator we mean by $D(\mathcal{O})$ its domain of definition, the Hilbert space $L^2(0,1)$, its norm and inner product are denoted respectively by H, $|.|_H$, $\langle ., . \rangle_H$, the space of linear bounded operators defined on H into it self and its norm are denoted respectively by $\mathcal{L}(H)$ and $\|.\|_{\mathcal{L}(H)}$. HS is the space of Hilbert-Schmidt operators defined from the Hilbert space H into it self, and we indicate its norm by $\|.\|_{HS}$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is a normal filtration and X be a Banach space, $L^p(\Omega, X)$, for $p \geq 2$ is the space of X-valued p-th integrable random variables on Ω, its norm is denoted by $\|.\|_{L^p(\Omega,X)}$, $\Lambda([0,T]; H) := \{v \in C([0,T]; L^p(\Omega, H)), v \text{ is } \mathbb{F} - adapted\}$ is a Banach space equiped with the norm $\|v\|_\Lambda := \sup_{t\in[0,T]} \|v(t)\|_{L^p(\Omega,H)}$. We use respectively the abbreviations RHS, Est, Pr and ONB for right hand side, estimate, problem and orthogonormal basis.

According to the spectral decomposition, we define the fractional Laplacian as follows (see [1, 3, 5, 6]).

Definition 2.1. *Let $\beta > 1$, and let $(e_n, \lambda_n)_{n=1}^{+\infty}$ be the eigenpairs of the operator A, such that $\lambda_n := (n\pi)^2$ and $e_n(.) := \sqrt{2}sin(\pi.)$. Then, for any $u \in D(A_\beta)$ where*

$$D(A_\beta) := \{v \in H, \text{ such that } \ |v|^2_{D(A_\beta)} := \sum_{n=1}^{+\infty} \lambda_n^\beta \langle v, e_n\rangle_H^2 < +\infty\},$$

we have

$$A_\beta u := \sum_{n=1}^{+\infty} \lambda_n^{\frac{\beta}{2}} \langle u, e_n\rangle_H e_n. \tag{2.1}$$

The system $(e_n)_{n\in\mathbb{N}^*}$ can be considered as an ONB of the space H. Then, from (2.1), we see that for any $n \in \mathbb{N}^*$,

$$A_\beta e_n = \sum_{k=1}^{+\infty} \lambda_k^{\frac{\beta}{2}} \langle e_n, e_k\rangle_{L^2(0,1)} e_k = \lambda_n^{\frac{\beta}{2}} e_n,$$

and so, $(e_n, \lambda_n^{\frac{\alpha}{2}})_{n\in\mathbb{N}^*}$ represents the eigenpairs of the fractional Laplacian A_β.

Lemma 2.1. *The operator A_β satisfies the following.*

(i) *Is symmetric.*

(ii) *Is the infinitesimal generator of an analytic semigroup $(S_\beta(t) := e^{-tA_\beta})_{t\geq 0}$ on H satisfies for all $v \in H$,*

$$S_\beta(t)v = \sum_{k\in\mathbb{N}^*} e^{-t\lambda_k^{\frac{\beta}{2}}} \langle v, e_k\rangle_H e_k. \tag{2.2}$$

(iii) *For all $\gamma \geq 0$ there exists a positive constant C_γ such that*

$$\|A^\gamma S_\beta(t)\|_{\mathcal{L}(H)} \leq C_\gamma t^{-\frac{2\gamma}{\alpha}}. \tag{2.3}$$

(iiii) *For all $\xi > \frac{1}{4}$, there exists $C_\xi > 0$ such that*

$$\|\mathcal{A}^{-\xi}\|_{HS} \leq C_\xi. \tag{2.4}$$

Proof. The proof of the symmetry is fulfilled directely from the definition of A_β. For the second and the third assertions see [3, 17] and for the last one see [3, 7]. □

Definition 2.2. *([13, 15, 23]) Let $u := (u(t))_{t\in[0,T]}$ be an H-valued stochastic process. u is said to be a **mild solution** of Pr.(1.1) if*

- *for all $t \in [0,T]$, $u(t)$ is $\mathcal{F}_t$-adapted,*
- *u satisfies the following equality in H, $\mathbb{P}$-a.s.,*

$$\begin{aligned} u(t) &= \int_0^\infty \xi_\alpha(\theta) S_\beta(t^\alpha\theta) u_0 d\theta \\ &+ \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \xi_\alpha(\theta) S_\beta((t-s)^\alpha\theta) F(u(s))\, d\theta\, ds \\ &+ \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \xi_\alpha(\theta) S_\beta((t-s)^\alpha\theta) G\, d\theta\, dW(s), \end{aligned} \tag{2.5}$$

for all $t \in [0,T]$, where ξ_α is a probability density function defined on $(0,\infty)$.

Arab Z. and Tunc C. in [7] have proved the wellposedness of Pr.(1.1) (Theorem 2.1 bellow), after imposing the following assumptions. For $p \geq 2$:

$\mathcal{H}_F$- The operator $F : H \to H$ (not necessarily linear) satisfies the global Lipschitz and the linear growth conditions, i.e.,

$$|F(u) - F(v)|_H \leq C_F |u - v|_H, \tag{2.6}$$

and

$$|F(u)|_H \leq C_F |u|_H, \tag{2.7}$$

for some positive constant C_F.

Assumption $\mathcal{H}_F$ can be reformulated in the random context as follows. For x and y be two H-valued random variables, it holds

$$\|F(x)-F(y)\|^p_{L^p(\Omega,H)} = \mathbb{E}|F(x)-F(y)|^p_H \leq C_F^p \mathbb{E}|x-y|^p_H = C_F^p \|x-y\|^p_{L^p(\Omega,H)}, \tag{2.8}$$

and

$$\|F(x)\|^p_{L^p(\Omega,H)} = \mathbb{E}|F(x)|^p_H \leq C^p_F \mathbb{E}|x|^p_H = C^p_F \|x\|^p_{L^p(\Omega,H)}. \tag{2.9}$$

$\mathcal{H}_G$ **-** The operator $G : H \to H$ is linear and bounded, i.e., $\|G\|_{\mathcal{L}(H)} \leq C_G$, for some positive constant C_G.

$\mathcal{H}_{u_0}$ **-** The initial condition u_0 is an $\mathcal{F}_0$-measurable random variable, satisfies $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, i.e. $\|u_0\|_{L^p(\Omega,H)} < \infty$.

Remark 2.1. *In the rest of this paper, when we need to use estimations in the random context as it has been proved above for Assumption $\mathcal{H}_F$, we will do it without proof in order to avoid the repetitions.*

Theorem 2.1. *([7]) Let $\alpha \in (\frac{1}{2}, 1)$, $\beta > \frac{2\alpha}{2\alpha-1}$ and $p \geq 2$. Under the Assumptions $\mathcal{H}_F$, $\mathcal{H}_G$ and $\mathcal{H}_{u_0}$, Pr.*(1.1) *admits a unique mild solution $u \in \Lambda([0,T]; H)$, provided that*

$$C_\gamma C_{\alpha,1} C_F T^\alpha < 1,$$

where $\gamma \in [0, 1 - \frac{1}{2\alpha})$ and $C_{\alpha,1} := \frac{\Gamma(2)}{\Gamma(1+\alpha)}$.

To make the proof of our main result more easier, we need the following useful lemmas.

Lemma 2.2. *([23]) Let $\alpha \in (0,1)$ and $\nu \in (-1, +\infty)$. It is true that*

$$\int_0^\infty \theta^\nu \xi_\alpha(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)} =: C_{\alpha,\nu},$$

where ξ_α is a probability density function defined on $(0,\infty)$ and Γ is Gamma function.

Lemma 2.3. *Let the continuous function $g : [0,T] \to [0, +\infty)$, for a fixed $T > 0$. If $\exists\ \varrho > 0$ such that*

$$g(t) \leq C_1 + C_2 \int_0^t (t-\tau)^{\varrho-1} g(\tau) d\tau,\ \forall\ t \in (0,T],$$

for some $C_1, C_2 > 0$. Then, $\exists\ C_{(C_2,T,\varrho)} > 0$, such that

$$g(t) \leq C_1 C_{(C_2,T,\varrho)}.$$

Lemma 2.4. *[22, Chapter 7; Est.(7.5) and Est.(7.6), p.112]. Let U be a Hilbert space and let $\mathcal{A}$ be a linear (not necessarily bounded), self-adjoint and positive definite operator defined on $D(U) \subseteq U$, which has eigenvalues $\{\mu_j\}_{j=1}^N$, for $1 < N \leq \infty$*

corresponding to a basis of orthogonormal eigenfunctions $\{\varphi_j\}_{j=1}^N$. *Then, for an arbitrary function* $\mathcal{G}$ *defined on the spectrum* $\sigma(\mathcal{A}) = \{\mu_j\}_{j=1}^N$ *of* $\mathcal{A}$*, it holds*

$$\|\mathcal{G}(\mathcal{A})\|_{\mathcal{L}(U)} = \sup_{1\leq j\leq N} |\mathcal{G}(\mu_j)|_U. \tag{2.10}$$

Lemma 2.5. *([4, Lemma A.8].)* $\forall \gamma > 0, \exists \quad C_\gamma := \gamma^\gamma e^{-\gamma} > 0$ *such that* $\forall x \geq 0, \;\; x^\gamma e^{-x} \leq C_\gamma$.

3. Spatial approximation of problem (1.1) by using Spectral Galerkin method

In this main section we study and prove the spatial approximation of the mild solution u by using the spectral Galerkin method. To do this, we fix $N \in \mathbb{N}^*$, let $h := \frac{1}{N}$, and let $(H_h)_{h\in(0,1]}$ be a sequence of finite dimensional subspaces of the Hilbert space H, such that

$$H_h := span\{e_1, \dots, e_N\}.$$

Let $\mathcal{P}_h : H \to H_h$ be the Galerkin projection onto H_h. Thus, for any $v \in H$ we have

$$\mathcal{P}_h v = \sum_{k=1}^{N} \langle v, e_k\rangle_H \; e_k.$$

We give the definition of the discrete version of A_β as follows.

Definition 3.1. *The discrete version of* A_β *is an operator* $A_{\beta,h} : H_h \to H_h$*, defined for any* $v_h \in H_h$ *by*

$$A_{\beta,h} v_h := \sum_{k=1}^{N} \langle v_h, e_k\rangle_H A_\beta e_k.$$

It is easy to see that, $A_{\beta,h} v_h = \sum_{k=1}^{N} \langle v_h, e_k\rangle_H \lambda_k^{\frac{\beta}{2}} e_k$, and so, for any $n \in \{1, \dots, N\}$,

$$A_{\beta,h} e_n = \sum_{k=1}^{N} \langle e_n, e_k\rangle_H A_\beta e_k = A_\beta e_n = \lambda_n^{\frac{\beta}{2}} e_n.$$

Then, $(e_k, \lambda_k^{\frac{\beta}{2}})_{k=1}^N$ is the set of eigenpairs of $A_{\beta,h}$.

Lemma 3.1. *The operator $-A_{\beta,h}$ is a generator of a semigroup of contraction $(S_{\beta,h}(t) := e^{-tA_{\beta,h}})_{t\in[0,T]}$ on H_h, acting on the spectrum as*

$$S_{\beta,h}(t)e_k = e^{-t\lambda_k^{\frac{\beta}{2}}} e_k, \quad \forall\, k \in \{1,\dots,N\}.$$

Moreover, for all $\gamma \geq 0$, there exists $C_\gamma > 0$ such that

$$\|A^{\gamma}_{\beta,h} S_{\beta,h}(t)\|_{\mathcal{L}(H)} \leq C_\gamma t^{-\gamma}, \text{ for all } t \in (0,T]. \tag{3.1}$$

Proof. The operator $A_{\beta,h}$ is self-adjoint and positive definite. Indeed, its symmetry is fulfilled directely from the symmetry of A_β and since $D(A_{\beta,h}) = H_h$, then $A_{\beta,h}$ is self-adjoint (see [3, Corolarry 1.32], [24]). About the second property, we have for any $u_h := \sum_{i=1}^N u_h^i e_i \in H_h$ where $u_h^i := \langle u_h, e_i\rangle_H$,

$$\begin{aligned}
\langle A_{\beta,h}u_h, u_h\rangle_H &= \langle \sum_{i=1}^N u_h^i A_\beta e_i, \sum_{j=1}^N u_h^j e_j\rangle_H = \sum_{i,j=1}^N u_h^i u_h^j \langle A_\beta e_i, e_j\rangle_H \\
&= \sum_{i,j=1}^N u_h^i u_h^j \langle \lambda_i^{\frac{\beta}{2}} e_i, e_j\rangle_H = \sum_{i=1}^N (u_h^i)^2 \lambda_i^{\frac{\beta}{2}} \geq 0.
\end{aligned}$$

Then, $A_{\beta,h}$ is positive definite. Hence, $-A_{\beta,h}$ is a generator of a C_0-semigroup $(e^{-tA_{\beta,h}})_{t\in[0,T]}$ on H_h (see [3, Proposition 1.58], [21, Proposition 9.4, p. 519]), let us denote it by $S_{\beta,h}(t)$.

About the smoothing property Est.(3.1), the use of Est.(2.10) in Lemma 2.4 and Lemma 2.5 gives

$$\|A^{\gamma}_{\beta,h} e^{-tA_{\beta,h}}\|_{\mathcal{L}(H)} = \sup_{1\leq i\leq N} \left(\lambda_i^{\frac{\beta\gamma}{2}} e^{-t\lambda_i^{\frac{\beta}{2}}} \right) \leq C_\gamma t^{-\gamma}.$$

□

Now, we are able to introduce the discrete version of Pr.(1.1) by using the spectral Galerkin method.

$$\begin{aligned}
u_h(t) = \mathcal{P}_h u_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{A_{\beta,h} u_h(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\mathcal{P}_h F(u_h(s))}{(t-s)^{1-\alpha}} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\mathcal{P}_h G}{(t-s)^{1-\alpha}} dW(s),
\end{aligned} \tag{3.2}$$

for all $t \in [0,T]$.

Theorem 2.1 is ensured the existence and the uniquness of a mild solution $u_h \in \Lambda([0,T];H_h)$, that satisfies the following equality in H_h, $\mathbb{P}-$ a.s.

$$\begin{aligned} u_h(t) &= \int_0^\infty \xi_\alpha(\theta) S_{\beta,h}(t^\alpha\theta)\mathcal{P}_h u_0 d\theta \\ &+ \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S_{\beta,h}((t-s)^\alpha\theta)\mathcal{P}_h F(u_h(s))\, d\theta\, ds \\ &+ \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S_{\beta,h}((t-s)^\alpha\theta)\mathcal{P}_h G\, d\theta\, dW(s), \quad \forall\, t\in[0,T]. \end{aligned} \tag{3.3}$$

Our main result in this work is the following.

Theorem 3.1. *For $\alpha \in (\frac{1}{2},1)$, $\beta > \frac{2\alpha}{2\alpha-1}$ and $p \geq 2$, let $u := (u(t))_{t\in(0,T]}$ be the mild solution of Pr.(1.1) with initial condition u_0 satisfies $\|A^\sigma u_0\|_{L^p(\Omega,L^2(0,1))} < \infty$, for some $\sigma > 0$, and let $u_h := (u_h(t))_{t\in(0,T]}$ be the mild solution of its discrete version Pr.(3.2). Then, u_h converges strongly to u with order of convergence $\delta := \min\{\sigma, \zeta, \frac{\beta\acute{\zeta}}{2}\}$, i.e.*

$$\|u(t)-u(t)_h\|_{L^p(\Omega,H)} \leq Ch^\delta,\ for\ all\ t\in(0,T], \tag{3.4}$$

for some positive constant C independent of h, where $\zeta < \beta$ and $\acute{\zeta} \in (\frac{1}{\beta}, 1-\frac{1}{2\alpha})$.

The proof of our main result needs also the following useful Lemma, that is concerned the family of operators $(E_{\beta,h}(t))_{t\in[0,T]}$ such that

$$\forall\, t\in[0,T],\ E_{\beta,h}(t) := S_\beta(t) - S_{\beta,h}(t)\mathcal{P}_h.$$

It is easy to see that $S_{\beta,h}(t)\mathcal{P}_h = \mathcal{P}_h S_\beta(t)$, and so $E_{\beta,h}(t) = (I-\mathcal{P}_h)S_\alpha(t)$.

Lemma 3.2. *([3, Lemma 6.13]) Let $\beta > 1$ and $t\in(0,T]$. We have*

(i) *For all $\zeta \geq 0$ and all $\eta\in\mathbb{R}$ there exists $C_{\zeta,\eta} > 0$ such that*

$$|E_{\beta,h}(t)x|_H \leq C_{\zeta,\eta} h^\zeta\, t^{-\frac{(\zeta-\eta)}{\beta}}\, |A^{\frac{\eta}{2}}x|_H,\ \ \forall x\in D(A^{\frac{\eta}{2}}). \tag{3.5}$$

(ii) *For all $\zeta > \frac{1}{\beta}$ there exists $C_{\zeta,\beta} > 0$ such that*

$$\|E_{\beta,h}(t)\|^2_{HS} \leq C_{\zeta,\beta}\, h^{\beta\zeta}\, t^{-\zeta}. \tag{3.6}$$

3.1. **Proof of Theorem 3.1.** Let $\alpha\in(\frac{1}{2},1)$, $\beta > \frac{2\alpha}{2\alpha-1}$ and $p\geq 2$. From equations (2.5) and (3.3) we have,

$$\|u(t)-u_h(t)\|_{L^p(\Omega,H)} \ \leq\ R_1+R_2+R_3+R_4, \tag{3.7}$$

where

$$R_1 := \| \int_0^\infty \xi_\alpha(\theta) E_{\beta,h}(t^\alpha \theta) u_0 d\theta \|_{L^p(\Omega,H)},$$

$$R_2 := \alpha \| \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha \theta) F(u(s)) d\theta ds \|_{L^p(\Omega,H)},$$

$$R_3 := \alpha \| \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) S_{\beta,h}((t-s)^\alpha \theta) \mathcal{P}_h \left(F(u(s)) - F(u_h(s)) \right) d\theta ds \|_{L^p(\Omega,H)},$$

$$R_4 := \alpha \| \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha \theta) G d\theta dW(s) \|_{L^p(\Omega,H)}.$$

To estimate R_1, let $\sigma > 0$. By using Est.(3.5) (with $\zeta = \eta = \sigma$) and Lemma 2.2 (with $\nu = 0$), we end up with

$$\begin{aligned} R_1 &:= \| \int_0^\infty \xi_\alpha(\theta) E_{\beta,h}(t^\alpha \theta) u_0 d\theta \|_{L^p(\Omega,H)} \leq \int_0^\infty \xi_\alpha(\theta) \| E_{\beta,h}(t^\alpha \theta) u_0 \|_{L^p(\Omega,H)} d\theta \\ &\leq C_\sigma h^\sigma \, \| A^{\frac{\eta}{2}} u_0 \|_{L^p(\Omega,H)} \int_0^\infty \xi_\alpha(\theta) d\theta = C_\sigma h^\sigma \, \| A^{\frac{\eta}{2}} u_0 \|_{L^p(\Omega,H)} C_{\alpha,0}. \end{aligned} \tag{3.8}$$

For the second estimate R_2, we use Est.(3.5) (with $\zeta < \beta, \eta = 0$), Assumption $\mathcal{H}_F$ and Lemma 2.2 (with $\nu = 1 - \frac{\zeta}{\beta}$), to get

$$\begin{aligned} R_2 &:= \alpha \| \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha \theta) F(u(s)) d\theta ds \|_{L^p(\Omega,H)} \\ &\leq \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) \| E_{\beta,h}((t-s)^\alpha \theta) F(u(s)) \|_{L^p(\Omega,H)} d\theta ds \\ &\leq \alpha C_{\zeta,0} h^\zeta \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) ((t-s)^\alpha \theta)^{-\frac{\zeta}{\beta}} \| F(u(s)) \|_{L^p(\Omega,H)} d\theta ds \\ &\leq \alpha C_{\zeta,0} h^\zeta C_F \int_0^t (\int_0^\infty \theta^{1-\frac{\zeta}{\beta}} \xi_\alpha(\theta) d\theta) (t-s)^{\alpha(1-\frac{\zeta}{\beta})-1} \| u(s) \|_{L^p(\Omega,H)} ds \\ &\leq \alpha C_{\zeta,0} h^\zeta C_F C_{\alpha,1-\frac{\zeta}{\beta}} \int_0^t (t-s)^{\alpha(1-\frac{\zeta}{\beta})-1} \| u(s) \|_{L^p(\Omega,H)} ds \\ &\leq \alpha C_{\zeta,0} h^\zeta C_F C_{\alpha,1-\frac{\zeta}{\beta}} \| u \|_\Lambda \int_0^t (t-s)^{\alpha(1-\frac{\zeta}{\beta})-1} ds \\ &\leq \alpha C_{\zeta,0} h^\zeta C_F C_{\alpha,1-\frac{\zeta}{\beta}} \| u \|_\Lambda \frac{T^{\alpha(1-\frac{\zeta}{\beta})}}{\alpha(1-\frac{\zeta}{\beta})}. \end{aligned} \tag{3.9}$$

Thanks to the facts that $S_{\beta,h}((t-s)^\alpha\theta)\mathcal{P}_h = \mathcal{P}_h S_\beta((t-s)^\alpha\theta)$ and $\|\mathcal{P}_h\|_{\mathcal{L}(H)} \leq 1$, we have

$$R_3 :=$$

$$:= \alpha\| \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) S_{\beta,h}((t-s)^\alpha\theta)\mathcal{P}_h \left(F(u(s)) - F(u_h(s))\right) d\theta ds\|_{L^p(\Omega,H)}$$

$$= \alpha\| \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \mathcal{P}_h S_\beta((t-s)^\alpha\theta) \left(F(u(s)) - F(u_h(s))\right) d\theta ds\|_{L^p(\Omega,H)}$$

$$\leq \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|\mathcal{P}_h S_\beta((t-s)^\alpha\theta) \left(F(u(s)) - F(u_h(s))\right)\|_{L^p(\Omega,H)} d\theta ds$$

$$\leq \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|\mathcal{P}_h S_\beta((t-s)^\alpha\theta)\|_{\mathcal{L}(H)} \|F(u(s)) - F(u_h(s))\|_{L^p(\Omega,H)} d\theta ds$$

$$\leq \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|\mathcal{P}_h\|_{\mathcal{L}(H)} \|S_\beta((t-s)^\alpha\theta)\|_{\mathcal{L}(H)}$$
$$\|F(u(s)) - F(u_h(s))\|_{L^p(\Omega,H)} d\theta ds$$

$$\leq \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|S_\beta((t-s)^\alpha\theta)\|_{\mathcal{L}(H)} \|F(u(s)) - F(u_h(s))\|_{L^p(\Omega,H)} d\theta ds.$$

The use of the semigroup property (2.3) (with $\gamma = 0$), Assumption $\mathcal{H}_F$ and Lemma 2.2 (with $\nu = 1$) help us to estimate R_3 as follows

$$\begin{aligned}
R_3 &\leq \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|S_\beta((t-s)^\alpha\theta)\|_{\mathcal{L}(H)} \\
&\quad \|F(u(s)) - F(u_h(s))\|_{L^p(\Omega,H)} d\theta ds \\
&\leq \alpha C_0 \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|F(u(s)) - F(u_h(s))\|_{L^p(\Omega,H)} d\theta ds \\
&\leq \alpha C_0 C_F \int_0^t (\int_0^\infty \theta\xi_\alpha(\theta) d\theta)(t-s)^{\alpha-1} \|u(s) - u_h(s)\|_{L^p(\Omega,H)} ds \\
&\leq \alpha C_0 C_F C_{\alpha,1} \int_0^t (t-s)^{\alpha-1} \|u(s) - u_h(s)\|_{L^p(\Omega,H)} ds. \qquad (3.10)
\end{aligned}$$

To estimate R_4, we use Burkholder-Davis-Gundy inequality as follows

$$\begin{aligned} R_4 &:= \alpha\| \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha\theta) G d\theta dW(s)\|_{L^p(\Omega,H)} \\ &\leq \alpha C_p \left(\mathbb{E}\left(\int_0^t \| \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha\theta) G d\theta\|_{HS}^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &= \alpha C_p \| \int_0^t \| \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha\theta) G d\theta\|_{HS}^2 ds \|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}^{\frac{1}{2}}, \end{aligned} \tag{3.11}$$

where $C_p := (\frac{p}{2}(p-1))^{\frac{1}{2}}(\frac{p}{p-1})^{\frac{p}{2}-1}$.

We need first to estimate $\| \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha\theta) G d\theta\|_{HS}^2$. To do this, we use the fact that $\|AB\|_{HS} \leq \|A\|_{HS}\|B\|_{\mathcal{L}(H)}$, for any $A \in HS$ and any $B \in \mathcal{L}(H)$, the Est.(3.6) (with $\zeta = \acute{\zeta} \in (\frac{1}{\beta}, 1-\frac{1}{2\alpha})$, which is possible thanks to $\beta > \frac{2\alpha}{2\alpha-1}$), Assumption $\mathcal{H}_G$ and Lemma 2.2 (with $\nu = 1-\frac{\acute{\zeta}}{2}$) as follows

$$\begin{aligned} &\| \int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) E_{\beta,h}((t-s)^\alpha\theta) G d\theta\|_{HS}^2 \\ &\leq \left(\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|E_{\beta,h}((t-s)^\alpha\theta) G\|_{HS} d\theta \right)^2 \\ &\leq \left(\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) \|E_{\beta,h}((t-s)^\alpha\theta)\|_{HS} \|G\|_{\mathcal{L}(H)} d\theta \right)^2 \\ &\leq C_{\acute{\zeta},\beta} h^{\beta\acute{\zeta}} (t-s)^{2\alpha(1-\frac{\acute{\zeta}}{2})-2} \|G\|_{\mathcal{L}(H)}^2 \left(\int_0^\infty \theta^{1-\frac{\acute{\zeta}}{2}} \xi_\alpha(\theta) d\theta \right)^2 \\ &\leq C_{\acute{\zeta},\beta} h^{\beta\acute{\zeta}} (t-s)^{2\alpha(1-\frac{\acute{\zeta}}{2})-2} \|G\|_{\mathcal{L}(H)}^2 (C_{\alpha,1-\frac{\acute{\zeta}}{2}})^2. \end{aligned} \tag{3.12}$$

From Est.(3.11) and Est.(3.12), we arrive at

$$\begin{aligned} R_4 &\leq \alpha C_p C_{\acute{\zeta},\beta}^{\frac{1}{2}} h^{\frac{\beta\acute{\zeta}}{2}} \|G\|_{\mathcal{L}(H)} C_{\alpha,1-\frac{\acute{\zeta}}{2}} \| \int_0^t (t-s)^{2\alpha(1-\frac{\acute{\zeta}}{2})-2} ds \|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}^{\frac{1}{2}} \\ &\leq \alpha C_p C_{\acute{\zeta},\beta}^{\frac{1}{2}} h^{\frac{\beta\acute{\zeta}}{2}} \|G\|_{\mathcal{L}(H)} C_{\alpha,1-\frac{\acute{\zeta}}{2}} \left(\int_0^t (t-s)^{2\alpha(1-\frac{\acute{\zeta}}{2})-2} ds \right)^{\frac{1}{2}} \\ &\leq \alpha C_p C_{\acute{\zeta},\beta}^{\frac{1}{2}} h^{\frac{\beta\acute{\zeta}}{2}} \|G\|_{\mathcal{L}(H)} C_{\alpha,1-\frac{\acute{\zeta}}{2}} \frac{T^{\alpha(1-\frac{\acute{\zeta}}{2})-\frac{1}{2}}}{(2\alpha(1-\frac{\acute{\zeta}}{2})-1)^{\frac{1}{2}}}. \end{aligned} \tag{3.13}$$

Coming back to Est.(3.7), by relpacing Est.(3.8), Est.(3.9), Est.(3.10) and Est.(3.13) in it, we end up with

$$\|u(t)-u_h(t)\|_{L^p(\Omega,H)} \leq \mathcal{C}_1 h^{\delta} + \mathcal{C}_2 \int_0^t (t-s)^{\alpha-1} \|u(s)-u_h(s)\|_{L^p(\Omega,H)} ds,$$

where $\delta := \min\{\sigma, \zeta, \frac{\beta\acute{\zeta}}{2}\}$,

$$\begin{aligned}\mathcal{C}_1 &:= C_\sigma \|A^{\frac{\eta}{2}} u_0\|_{L^p(\Omega,H)} C_{\alpha,0} \\ &+ \alpha C_{\zeta,0} C_F C_{\alpha,1-\frac{\zeta}{\beta}} \|u\|_\Lambda \frac{T^{\alpha(1-\frac{\zeta}{\beta})}}{\alpha(1-\frac{\zeta}{\beta})} \\ &+ \alpha C_p C_{\acute{\zeta},\beta}^{\frac{1}{2}} \|G\|_{\mathcal{L}(H)} C_{\alpha,1-\frac{\acute{\zeta}}{2}} \frac{T^{\alpha(1-\frac{\acute{\zeta}}{2})-\frac{1}{2}}}{(2\alpha(1-\frac{\acute{\zeta}}{2})-1)^{\frac{1}{2}}},\end{aligned}$$

and $\mathcal{C}_2 := \alpha C_0 C_F C_{\alpha,1}$. An application of Gronwall Lemma 2.3 yields

$$\|u(t)-u_h(t)\|_{L^p(\Omega,H)} \leq \mathcal{C}_1 C_{\mathcal{C}_2,T,\alpha}\, h^{\delta}.$$

By this the desired result is obtained.

4. Conclusion

Stochastic fractional integro-differential equations have been used as a mathematical models of many physical phenomena in applied sciences. In this paper, we have considered the stochastic space-time fractional integro-differential equation in the Hilbert space $L^2(0,1)$. By using the spectral Galerkin method, we have proved that the approximate solution u_h, for $h \in (0,1]$ converges strongly (i.e. in the space $L^p(\Omega, L^2(0,1))$, for $p \geq 2$) to the mild solution u, by imposing only a regularity condition on the initial value, i.e. $\|A^\sigma u_0\|_{L^p(\Omega,L^2(0,1))} < \infty$, for some $\sigma > 0$.

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