ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.4, 383–413
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.4.7

AMPLITUDE ADJUSTMENT WITH FIWASVJ MODEL

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ABSTRACT. Andrianantenainarinoro [2] remarked that the price amplitudes of financial models may not correspond to the reality and we propose here a model in continuous time Fractionally Integrated WASC Stochastic Volatility Jump. To do this, we introduce a fractal index in the WASC Stochastic Volatility Jump model and we have two others characteristics: amplitude adjustment and memory of process. We present also several theories in stochastic calculus, algebraic, differential geometry, numerical method and estimating method which can use to financial such us: sense of a fractional integral, relationship between trace and determinant operator, Euler's approximation for an unresolved differential equation and convergence speed.

1. INTRODUCTION

Andrianantenainarinoro [2] showed that some price amplitudes of financial models may be abnormal. Hence, we must regularize the amplitude of asset to adjust it to the reality and he proposed a technical by using the Matérn process. In this article, we propose a model in continuous time Fractionally Integrated WASC Stochastic Volatility Jump noted FIWASVJ(d,a):

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2020 *Mathematics Subject Classification.* 60G22, 91G30.
Key words and phrases. fractional Brownian motion, asset pricing models.
Submitted: 30.03.2022; *Accepted:* 15.04.2022; *Published:* 28.04.2022.

$$
\begin{cases}
d\log S_t = (r - \frac{1}{2}vec[tr(e_{ii}\Gamma_t)])dt + \sqrt{\Gamma_t}dZ_t + d\psi_t\varphi \\
d\Gamma_t = \frac{\nu}{(2a+1)G(a+1)^2}Q'Q(dt)^{2a+1} + (\Phi\Gamma_t + \Gamma_t\Phi')dt \\
\qquad +\sqrt{\Gamma_t}d\tilde{B}_{a,t}Q + Q'(d\tilde{B}_{a,t})'\sqrt{\Gamma_t} + d\psi_t \\
dZ_t = \sqrt{1-\rho'\rho}dW_{d,t} + d\tilde{B}_{d,t}\rho \\
d\psi_t = \sqrt{\Gamma_t}d\tilde{P}_t + (d\tilde{P}_t)'\sqrt{\Gamma_t} + (d\tilde{P}_t)(d\tilde{P}_t)' \quad \text{where:}
\end{cases}, \tag{1.1}
$$

(i) ν is a positive integer nonzero;

(ii) Q, Φ are $n \times n$ dimensional real matrices;

(iii) e_{ii} is the $n \times n$ dimensional matrix defined by $e_{ii} = (\delta_{ijk})_{j,k=1,\ldots,n}$ where $\delta_{ijk} = \begin{cases} 1 \text{ if } (j,k) = (i,i) \\ 0 \text{ otherwise} \end{cases}$;

(iv) If $a_1, \ldots, a_n \in \mathbb{R}$, we define $vec(a_i) = (a_1, \ldots, a_n)'$ which is a vector in $\mathbb{R}^n$;

(v) φ and r are vectors in $\mathbb{R}^n$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_n)'$ where $\rho_i \in [-1, 1]$;

(vi) ψ_t is the jump process defined in WASVJ model;

(vii) $dZ_t = \sqrt{1-\rho'\rho}dW_{d,t} + d\tilde{B}_{d,t}\rho$ defines the stochastic correlation noise between the yield $\log S_t$ and its volatility Γ_t on the continuous part of the trajectory where $d \in \left]-\frac{1}{2}, \frac{1}{2}\right[$;

(viii) $\tilde{B}_{a,t}$ is a $n \times n$ dimensional stochastic matrix whose components are the fractional Brownian motion (fBm) order a defined in Mandelbrot [13] by

$$
B_{a,t} = \int_0^t \frac{(t-s)^a}{G(a+1)}dW_s \tag{1.2}
$$

where $a \in \left]-\frac{1}{2}, \frac{1}{2}\right[$, G is the Gamma function, $G(\alpha) = \int_0^{+\infty} u^{\alpha-1}e^{-u}du$, $\alpha > 0$ and W_t is a standard Brownian motion (sBm);

(ix) $W_{d,t}$ is a $n \times n$-dimensional stochastic matrix whose components are the fBm order d;

(x) $\tilde{P}_t$ is a $n \times n$ dimensional stochastic matrix whose components are the compounded Poisson processes (cPp);

(xi) H' is the transpose of the matrix H.

(xii) $tr(H)$ is the trace of the matrix H

(xiii) y' is the transpose of the vector y.

The model is obtained by changing the sBm in the WASC Stochastic Volatility Jump model (WASVJ) of Andrianantenainarinoro [1] by the fBm. The fractal index on the asset of model adjusts its course to the reality and the fractal in the volatility is to obtain its memory. The volatility Γ_t is a new process called Jump and Fractionally Integrated Wishart Autoregressive noted JAFIWAR(a).

The purpose of this article is therefore to build a financial model with FIWASVJ (d,a) in the market without friction. To do this, it is therefore necessary to set up a modeling with the uncertainty linked to the future evolution of the financial market. But before that, we must study the positiveness of volatility Γ_t and its law. Next, we discuss the sense of the integral $\int_0^T \sqrt{\Gamma_s} dZ_s$ and the law of yield $\log S_t$. Its law is related to an unresolved differential equation and we approximate the solution by Euler's approximation. In the practical part, we will show how to estimate the parameter of model and we look the impact of its adjustment on the option pricing. We find in this paper severals theories in stochastic calculus, algebraic, differential geometric, numerical method and estimating method which can be essential for the financial market.

2. The model

In this study, we work in the probability space ($\mathbb{R}^n$, $\mathbb{P}$) where $\mathbb{P}$ is the "risk–neutral"probability such that the price of any option is a conditional expectation of its payoff. Consider a market of a basket carrying n underlying assets such that S_t is the value of this basket at time t, $\log S_t$ is its return.

2.1. **Positive definite of volatility.** Let x_t a process in $\mathbb{R}^n$ defined by:

$$dx_t = \Phi x_t dt + \sqrt{Q'Q} dB_{a,t} + dP_t, \tag{2.1}$$

where Φ and Q are $n \times n$ dimensional real matrices; $B_{a,t}$ is a n-dimensional stochastic vector whose components are the fBm order $a \in \left]-\frac{1}{2}, \frac{1}{2}\right[$ and P_t is a n-dimensional vector of cPp.

Let z_t a process of the form:

$$z_t = \sum_{i=1}^{\nu} x_{i,t}(x_{i,t})', \tag{2.2}$$

where ν is a positive integer nonzero and $(x_{i,t})_t$ $i = 1, \ldots, \nu$ are the n-dimensional vector process defined by (2.1).

Proposition 2.1. *z_t is a positive definite matrix if and only if $\nu \geq n \geq 1$.*

Proof. "$\Rightarrow$"Firstly, if $\nu = 1$, then we work in $\mathbb{R}$ and the process z_t is the sum of the real processes squared.

Let us now consider for $\nu \geq 2$. We use absurd reasoning. Suppose that $n > \nu$ and z_t is a positive definite matrix. Build a $n \times \nu$ dimensional process X_t:

$$d(X_t)' = \Phi(X_t)'dt + \sqrt{Q'Q}d\check{B}_{a,t} + (d\check{P}_t)', \tag{2.3}$$

where $(X_t)' = (x_{1,t}, \ldots, x_{\nu,t})$ is a $n \times \nu$ dimensional stochastic matrix; $\check{B}_{a,t} = (B_{1,a,t}, B_{2,a,t}, \ldots, B_{\nu,a,t})$ is the $n \times \nu$ dimensional matrix where the $B_{a,i,t}$ are the fBm vectors of $x_{i,t}$, $i = 1, \ldots, \nu$ and $(\check{P}_t)' = (P_{1t}, P_{2,t}, \ldots, P_{\nu,t})$ is the $n \times \nu$ dimensional matrix where the $P_{i,t}$ are n-dimensional vectors of cPp of $x_{i,t}$, $i = 1, \ldots, \nu$.

We have $z_t = (X_t)'X_t$. Thus $rank(z_t) \leq min(n, \nu)$. Since $\nu < n$, then we have $rank(z_t) < n$ where z_t is a $n \times n$ dimensional matrix. Thus z_t is singular and therefore it is not positive definite matrix. A contradiction with z_t is positive definite matrix.

"$\Leftarrow$"Let $y \in \mathbb{R}^p$, $y = (y_1, y_2, \ldots, y_p)'$. Let suppose also that $x_{i,t} = (x_1^i, \ldots, x_n^i)'$. Developing z_t of the form (2.2), we get

$$z_t = \sum_{i=1}^{\nu} \begin{bmatrix} (x_1^i)^2 & x_1^i x_2^i & \cdots & x_1^i x_n^i \\ x_2^i x_1^i & (x_2^i)^2 & \cdots & x_2^i x_n^i \\ \vdots & \vdots & \cdots & \vdots \\ x_n^i x_1^i & x_n^i x_2^i & \cdots & (x_n^i)^2 \end{bmatrix}.$$

After the calculation,

$$y' z_t y = \sum_{i=1}^{\nu} \left(\sum_{j=1}^{n} (y_j x_j^i) \right)^2 \geq 0.$$

Thus, if the latter is zero, then we obtain ν equations with n unknowns which are:

$$\begin{cases} \sum_{j=1}^{n} (y_j x_j^1) = 0 \\ \sum_{j=1}^{n} (y_j x_j^2) = 0 \\ \vdots \\ \sum_{j=1}^{n} (y_j x_j^{\nu}) = 0. \end{cases}$$

Since $\nu \geq n$, we have $y_j = 0$ for all j. Thus, the later is strictly positive for all $0 \neq y \in \mathbb{R}^n$ and it follows that Γ_t is positive definite matrix. $\square$

2.2. **Marginal dynamic of the model.**

Proposition 2.2. *Let us $dP_{i,t} = Y_i dN_t$ where $P_{i,t}$ is the cPp of $x_{i,t}$, $i = 1, \ldots, \nu$, the Y_i are n-dimensional vectors of i.i.d (independent and identically distributed) random variables and N_t is a Poisson process of intensity $\lambda > 0$. If $\nu \geq n$ and $a \in \left[0, \frac{1}{2}\right[$, then the process z_t satisfies the SDE (Stochastic Differential Equation) of type:*

$$\begin{aligned} dz_t = {} & \frac{\nu}{(2a+1)G(a+1)^2} Q'Q(dt)^{2a+1} + (\Phi z_t + z_t \Phi')dt \\ & + \sqrt{z_t}(d\tilde{B}_{a,t})'\sqrt{Q'Q} + \sqrt{Q'Q}d\tilde{B}_{a,t}\sqrt{z_t} \\ & + \sqrt{z_t}(d\tilde{P}_t)' + d\tilde{P}_t\sqrt{z_t} + d\tilde{P}_t(d\tilde{P}_t)' \end{aligned} \tag{2.4}$$

with Q and Φ are the above matrices; $\tilde{B}_{a,t}$ is a $n \times n$ dimensional stochastic matrix whose components are independent fBm order a defined by $d\tilde{B}_{a,t} = d\check{B}_{a,t}X_t(\sqrt{z_t})^{-1}$; $(\tilde{P}_t)$ is a $n \times n$ dimensional stochastic matrix whose components are the cPp such that $d\tilde{P}_t = (d\check{P}_t)'X_t(\sqrt{z_t})^{-1}$ where $\check{B}_{a,t}$, $\check{P}_t$ and X_t are defined in (2.3).

Proof. Applying Ito's formula with respect to fBm on the process $f(x_t) = \sum\limits_{i=1}^{\nu} x_{i,t}(x_{i,t})'$, we obtain

$$\begin{aligned} dz_t = {} & \sum_{i=1}^{\nu} dx_{i,t}^c(x_{i,t}^c)' + \sum_{i=1}^{\nu} x_{i,t}(dx_{i,t}^c)' + \frac{\nu(dt)^{2a+1}}{(2a+1)G(a+1)^2}Q'Q \\ & + \left[\sum_{i=1}^{\nu}(x_{i,t}+Y_i)(x_{i,t}+Y_i)' - x_{i,t}(x_{i,t})'\right] dN_t, \end{aligned}$$

where X_t^c is the continuous part of X_t (cf. reference [1] but fBm instead of sBm), and so

$$\begin{aligned} dz_t = {} & \sum_{i=1}^{\nu}(\Phi x_{i,t}dt + \sqrt{Q'Q}dB_{i,a,t})(x_{i,t})' + \sum_{i=1}^{\nu} x_{i,t}(\Phi x_{i,t}dt + \sqrt{Q'Q}dB_{i,a,t})' \\ & + \frac{\nu(dt)^{2a+1}}{(2a+1)G(a+1)^2}Q'Q + [x_{i,t}(x_{i,t})' + x_{i,t}(Y_i)' + Y_i(x_{i,t})' \\ & + Y_i(Y_i)' - x_{i,t}(x_{i,t})']dN_t \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&= \frac{\nu(dt)^{2a+1}}{(2a+1)G(a+1)^2}Q'Q + (\Phi z_t + z_t\Phi')dt + \sum_{i=1}^{\nu}\sqrt{Q'Q}dB_{i,a,t}(x_{i,t})' \\
&+ x_{i,t}(dB_{i,a,t})'\sqrt{Q'Q} + \sum_{i=1}^{\nu}[x_{i,t}(Y_i)' + Y_i(x_{i,t})' + Y_i(Y_i)']dN_t \\
&= \frac{\nu(dt)^{2a+1}}{(2a+1)G(a+1)^2}Q'Q + (\Phi z_t + z_t\Phi')dt + \sqrt{Q'Q}d\check{B}_{a,t}X_t \\
&+ (X_t)'(d\check{B}_{a,t})'\sqrt{Q'Q} + (X_t)'d\check{P}_t + (d\check{P}_t)'X_t + (d\check{P}_t)'d\check{P}_t.
\end{aligned}
$$

Since $d\tilde{P}_t = (d\check{P}_t)'X_t(\sqrt{z_t})^{-1}$ and $d\tilde{B}_{a,t} = d\check{B}_{a,t}X_t(\sqrt{z_t})^{-1}$. Then, we have $d\tilde{P}_t(d\tilde{P}_t)' = (d\check{P}_t)'d\check{P}_t$. Thus

$$
\begin{aligned}
(2.5)\quad &= \frac{\nu}{(2a+1)G(a+1)^2}Q'Q(dt)^{2a+1} + (\Phi z_t + z_t\Phi')dt + \sqrt{z_t}(d\tilde{B}_{a,t})'\sqrt{Q'Q} \\
&+ \sqrt{Q'Q}d\tilde{B}_{a,t}\sqrt{z_t} + \sqrt{z_t}(d\tilde{P}_t)' + d\tilde{P}_t\sqrt{z_t} + d\tilde{P}_t(d\tilde{P}_t)'.
\end{aligned}
$$

The first term of the right-hand side of the equality later vanishes as a consequence of application of the Ito formula with respect fBm. □

Let Γ_t a process solution of the SDE defined by (2.4). We call the former approach by JAFIWAR(a) process. We remark that if jump does not exist then the process reduces to a FIWAR2 (see [3]).

2.3. **The law of Γ_t.** In this section, we try to give the explicit expressions of Laplace transform of volatility Γ_t.

Proposition 2.3. *: If $A \in \mathcal{M}_n(\mathbb{R})$ such that $\|A\| < 1$ then $I_n - A$ and $I_n + A$ are the definite positives matrices and they are true for any norms.*

Proof. Let be $0 \neq y \in \mathbb{R}^n$ and $A \in \mathcal{M}_n(\mathbb{R})$, $\|A\| < 1$. We have $| y'Ay | \leq \| y' \| \| A \| \| y \| = \| A \| \| y \|^2 < \| y \|^2$. Thus $- \| y \|^2 < -y'Ay < \| y \|^2$. Then $0 < \| y \|^2 - y'Ay = y'(I_n - A)y < 2 \| y \|^2$.

Same reasoning for the other. □

Let $F : t \in \mathbb{R} \longmapsto F(t) = F_t \in GL_n(\mathbb{R})$ be a differentiable function. Thus, we obtain the following differential (cf. Le Stum [11]):

$$
d\det(F_t) = \det(F_t)tr(F_t^{-1}dF_t). \tag{2.6}
$$

Hence, if $\log F_t$ is defined, then by assuming $F_0 = I_n$ and integrating between 0 to t member to member equality $[\det(F_s)]^{-1} d \det(F_s) = tr(F_s^{-1} dF_s)$, we find $\log(\det(F_t)) = tr(logF_t)$ and

$$\det(F_t) = \exp tr(logF_t). \tag{2.7}$$

This relationship is useful when we want have the characteristic function from the Laplace transform.

Let us $d\tilde{P}_t = JdN_t$ with $J = (J_{lk})_{1 \leq l,k \leq n}$ where J_{lk} are the i.i.d normal random variables with $J_{lk} \rightsquigarrow N(m, \sigma^2)$. Let Λ be a $n \times n$ dimensional symmetric matrix. The Laplace transform of Γ_{t+h} given Γ_t is defined by:

$$\Psi_{\Gamma_t}(\Lambda, h) = \mathbb{E}\left\{e^{tr(\Lambda \Gamma_{t+h})} / \Gamma_t\right\} \text{ where } t, h \geq 0. \tag{2.8}$$

Since Γ_t is an affine function, we have

$$\Psi_{\Gamma}(\Lambda, h) = e^{tr(B(h)\Gamma_t) + c(h)} \tag{2.9}$$

with $B(h)$ and $c(h)$ are deterministic functions expressed by, in the trace operator

Proposition 2.4. *If* $\| 2\sigma^2 B(h) \| < 1$ *and* $B(h)$ *) is a symmetric matrix for all* $h \geq 0$*, then*

$$\begin{aligned} B(h) &= \left[I_n - \frac{2}{G(a+1)^2} \Lambda Q' Q \int_0^h s^{2a} e^{(\Phi+\Phi')s} ds\right]^{-1} \Lambda e^{(\Phi+\Phi')h}, \\ c(h) &= tr\left[-\frac{\nu}{2} \log\left(I_n - \frac{2}{G(a+1)^2} \Lambda Q' Q \int_0^h s^{2a} e^{(\Phi+\Phi')s} ds\right)\right] \\ &\quad + \lambda \int_0^h e^{tr\left[B(u)(I_n - 2\sigma^2 B(u))^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t-}}(m\tilde{1}) + 2\Gamma_{t-} B(u)\sigma^2\right) - \frac{n}{2}\log\Delta(u)\right]} - 1 du \end{aligned}$$

with $\Delta(u) = I_n - 2\sigma^2 B(u)$ *and* $\tilde{1}$ *is a* $n \times n$ *dimensional matrix whose components are equal to* 1*.*

Proof. Let be $t, h \geq 0$. By using the Feynmann–Kac argument on the SDE of Γ_t, we get

$$\frac{\partial \Psi_{\Gamma_{t-}}(\Lambda, h)}{\partial h} = \mathcal{L}_\Gamma \Psi_{\Gamma_{t-}}(\Lambda, h) \tag{2.10}$$

with

$$\mathcal{L}_\Gamma \Psi_{\Gamma_t} = \mathcal{L}_{(\Gamma)^c} \Psi_{\Gamma_{t-}} + \mathcal{L}_{jumps} \tag{2.11}$$

where

(i) $\mathcal{L}_{(\Gamma)^c}$ is the infinitesimal generator of $(\Gamma)^c$;

(ii) Using the work of Bru [4], we have

$$\mathcal{L}_{(\Gamma)^c} = tr\left[\left(\frac{\nu h^{2a}}{G(a+1)^2}Q'Q + \Phi\Gamma_{t^-} + \Gamma_{t^-}\Phi\right)D + \frac{2h^{2a}}{G(a+1)^2}\Gamma_{t^-}DQ'QD\right]; \tag{2.12}$$

(iii) $\mathcal{L}_{jumps}$ is the infinitesimal generator of the jumps such as:

$$\mathcal{L}_{jumps} = \lambda\Psi_{\Gamma_{t^-}}\mathbb{E}\left\{e^{tr\left(B(h)(2\sqrt{\Gamma_{t^-}}J+JJ'\right)} - 1/\Gamma_t\right\}; \tag{2.13}$$

(iv) $D = (D_{ij})_{ij}$ such that $D_{ij} = \frac{\partial}{\partial\Gamma_{ij,t}}$.

We have also

$$\frac{\partial\Psi_{\log S_{t^-}}(\gamma,h)}{\partial h} = \left[tr\left(\frac{\partial B(h)}{\partial h}\Gamma_{t^-}\right) + \frac{\partial c(h)}{\partial h}\right]\Psi_{\Gamma_{t^-}}(\Lambda,h).$$

So, from the expression (2.10), we have

$$\begin{aligned} & tr\left(\frac{\partial B(h)}{\partial h}\Gamma_{t^-}\right) + \frac{\partial c(h)}{\partial h} \\ =& tr\left[\left(\frac{\nu h^{2a}}{G(a+1)^2}Q'Q + \Phi\Gamma_{t^-} + \Gamma_{t^-}\Phi'\right)B(h)\right] \\ &+ tr\left[\frac{2h^{2a}}{G(a+1)^2}\Gamma_{t^-}B(h)Q'QB(h)\right] \\ &+ \lambda\mathbb{E}\left\{e^{tr\left[B(h)\left(2\sqrt{\Gamma_{t^-}}J+JJ'\right)\right]} - 1/\Gamma_t\right\} \text{ for all } h>0. \end{aligned} \tag{2.14}$$

We have also $B(0) = \Lambda$ and $c(0) = 0$ and from the work of Andrianantenainarinoro [1, page 13],

$$\begin{aligned} &\mathbb{E}\left[e^{tr\left[B(h)\left(2\sqrt{\Gamma_{t^-}}J+JJ'\right)\right]}/\Gamma_t\right] \\ =& \prod_{k=1}^{n} e^{(m\check{1})'B(h)(m\check{1})+2vec(\sigma_k)'B(h)(m\check{1})} \int \frac{e^{-\varepsilon'\frac{1}{2}\Delta(h)\varepsilon+\left(2(m\check{1})'B(h)\sigma+2vec(\sigma_k)'B(h)\sigma\right)\varepsilon}}{\sqrt{2}^n\sqrt{\pi}^n}d\varepsilon \end{aligned}$$

where $\check{1}$ is a n-dimensional vector whose components are equal to 1, and further equal to

$$\begin{aligned} &\prod_{k=1}^{n} e^{(m\check{1})'B(h)m\check{1}+2(vec(\sigma_k))'B(h)(m\check{1})-\frac{1}{2}tr(\log\Delta(h))} \\ &e^{\left((m\check{1})'B(h)\sigma+(vec(\sigma_k))'B(h)\sigma\right)\left(2\Delta(h)^{-1}\right)\left(\sigma B(h)(m\check{1})+\sigma B(h)vec(\sigma_k)\right)} \end{aligned}$$

through Gourrieroux [9, Lemma, page 213], the Proposition 2.3 and the equality (2.7), and also to

$$e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]}.$$

Thus, (2.14)) is equal to

$$tr\left[\left(\frac{\nu h^{2a}}{G(a+1)^2}QQ'+\Phi\Gamma_{t^-}+\Gamma_{t^-}\Phi'\right)B(h)+\frac{2h^{2a}}{G(a+1)^2}\Gamma_{t^-}B(h)Q'QB(h)\right]$$
$$+\lambda\left[e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]}-1\right].$$

Identifying the coefficient of Γ_{t^-}, we get

$$\frac{\partial B(h)}{\partial h}=\Phi B(h)+B(h)\Phi'+\frac{2h^{2a}}{G(a+1)^2}B(h)Q'QB(h).$$

In the trace operator, $B(h)$ is the solution of following SDE:

$$\frac{\partial z}{\partial h}=(\Phi+\Phi')z+\frac{2h^{2a}}{G(a+1)^2}Q'Qz^2. \tag{2.15}$$

Let us $y=z^{-1}$. The SDE (2.15) is equivalent to SDE:

$$\frac{\partial y}{\partial h}+(\Phi+\Phi')y=-\frac{2h^{2a}}{G(a+1)^2}Q'Q. \tag{2.16}$$

$f(h)=-\frac{2}{G(a+1)^2}Q'Qe^{-(\Phi+\Phi')h}\int_0^h s^{2a}e^{(\Phi+\Phi')s}ds+Ke^{-(\Phi+\Phi')h}$ is the solution of SDE (2.16) where K is a constant.

Thus, $g(h)=\left[I_n-\frac{2}{G(a+1)^2}K_1Q'Q\int_0^h s^{2a}e^{(\Phi+\Phi')s}ds\right]^{-1}K_1e^{(\Phi+\Phi')h}$ is the solution of SDE (2.15) where K_1 is a constant. We get the result by taking $B(h)=g(h)$ where $K_1=\Lambda$.

Finally, by identification we have

$$\begin{aligned}
\frac{\partial c(h)}{\partial h} &= tr\left[\frac{\nu h^{2a}}{G(a+1)^2}QQ'B(h)\right]\\
&+ \lambda\left[e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]}-1\right]\\
&= tr\left[\frac{-\nu}{2}B(h)^{-1}\frac{\partial B(h)}{\partial h}-\frac{\nu}{2}(\Phi+\Phi')\right]\\
&+ \lambda\left[e^{tr\left[B(h)\Delta(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}B(h)\sigma^2\right)-\frac{n}{2}\log\Delta(h)\right]}-1\right]
\end{aligned}$$

with $c(0) = 0$. Thus,

$$\begin{aligned} c(h) &= tr\left[\frac{-\nu}{2}(\log B(h) - \log B(0)) - \frac{\nu h}{2}(\Phi + \Phi')\right] \\ &+ \lambda \int_0^h e^{tr\left[B(u)\Delta(u)^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t-}}(m\tilde{1}) + 2\Gamma_{t-}B(u)\sigma^2\right) - \frac{n}{2}\log\Delta(u)\right]} - 1du \\ &= tr\left[-\frac{\nu}{2}\log\left(I_n - \frac{2}{G(a+1)^2}\Lambda Q'Q\int_0^h s^{2a}e^{(\Phi+\Phi')s}ds\right)\right] \\ &+ \lambda \int_0^h e^{tr\left[B(u)\Delta(u)^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_{t-}}(m\tilde{1}) + 2\Gamma_{t-}B(u)\sigma^2\right) - \frac{n}{2}\log\Delta(u)\right]} - 1du \end{aligned}$$

where $\Delta(u) = I_n - 2\sigma^2 B(u)$ for all $u \in [0; h]$. □

Remark 2.1. *When $a = d = 0$, Andrianantenainarinoro [1] studied the stationarity of Γ_t.*

2.4. **The law of asset returns.** Let be a market of the form (1.1). What meaning can give to $\int_0^T \sqrt{\Gamma_s}dZ_s$ or $\int_0^T \sigma_{kl,s}dB_s$? where $\sqrt{\Gamma_t} = (\sigma_{ij,t})_{1\le i,j\le n}$.

Let's consider

$$\sum_{t_i \in \Delta} \sigma_{kl,t_i}(B_{t_{i+1}} - B_{t_i}) \tag{2.17}$$

with $\Delta = \{0 = t_0 < t_1 < \ldots < t_p = T\}$ and watch what's going when $\Delta \longrightarrow 0$.

If $d = 0$, then B_t is sBm which is martingale. Since $\sigma_{kl,t}$ is adapted continue, the sum (2.17) converges to $\int_0^T \sigma_{kl,s}dB_s$ through [14, Proposition 122].

Now, suppose that $\frac{1}{2} > d > 0$. Let's admit that for all $t, t' \in [0, T]$ closer, there exist $M > 0$, $\| \sigma_{kl,t'} - \sigma_{kl,t} \| \le M \mid t' - t \mid^{\frac{1}{2}}$ and put $T = 1$ (to simplify) and $\Delta = \Delta_p = \{k2^{-p}, k = 0, \ldots, 2^{p-1}\}$. If $t \in \Delta_p$, we denote $t' = t + 2^{-p}$ and $\tau = \frac{t+t'}{2}$. Let us $u_p = \sum_{t\in\Delta_p} \sigma_{kl,t}(B_{t'} - B_t)$. So, we have

$$\begin{aligned} u_{p+1} - u_p &= \sum_{t\in\Delta_p} \sigma_{kl,t}(B_\tau - B_t) + \sigma_{kl,\tau}(B_{t'} - B_\tau) - \sum_{t\in\Delta_p} \sigma_{kl,t}(B_{t'} - B_\tau) \\ &+ \sigma_{kl,t}(B_\tau - B_t) \\ &= \sum_{t\in\Delta_p} (\sigma_{kl,\tau} - \sigma_{kl,t})(B_{t'} - B_\tau). \end{aligned}$$

Hence

$$\begin{aligned}
\| u_{p+1} - u_p \| &\leq \sum_{t \in \Delta_p} \| \sigma_{kl,\tau} - \sigma_{kl,t} \| \| B_{t'} - B_\tau \| \\
&\leq \sum_{t \in \Delta_p} M \mid \tau - t \mid^{\frac{1}{2}} \left(\mathbb{E}(B_{t'} - B_\tau)^2 \right)^{\frac{1}{2}} \\
&= \sum_{t \in \Delta_p} \frac{M 2^{-(p+1)\frac{1}{2}} 2^{-(p+1)(d+\frac{1}{2})}}{\sqrt{2d+1} G(d+1)} \\
&= \frac{M 2^{-(p+1)d}}{2\sqrt{2d+1} G(d+1)} \longrightarrow 0 \text{ if } 0 < d < \frac{1}{2} \text{ when } p \longrightarrow \infty.
\end{aligned}$$

Hence u_p converges to the Young integral [19] $\int_0^T \sigma_{kl,s} dB_s$ for $0 < d < \frac{1}{2}$.

Now, demonstrate that for all $t, t' \in [0, T]$ closer, there exist $M > 0$, $\| \sigma_{kl,t'} - \sigma_{kl,t} \| \leq M \mid t' - t \mid^{\frac{1}{2}}$.

Let $U = \left\{ (x_{11,t}, \ldots, x_{1n,t}, x_{22,t}, \ldots, x_{2n,t}, , , , x_{nn,t}) \in \mathbb{R}^{\frac{n(n+1)}{2}} \right\}$, $x_{ii} > 0$ and the main miners of the symmetric matrix $(x_{kl})_{k,l=1\ldots n}$ are positives. Put $F : U \longrightarrow U$, $F(\sigma_{11,t}, \ldots, \sigma_{1n,t}, \sigma_{22,t}, \ldots, \sigma_{2n,t}, , , , \sigma_{nn,t}) = (\Gamma_{11,t}, \ldots, \Gamma_{1n,t}, \Gamma_{22,t}, \ldots, \Gamma_{2n,t}, , , , \Gamma_{nn,t})$. Admit that F is a global dimorphism. Thus, there exist a function $g_{ki} : U \longrightarrow \mathbb{R}$ of $\mathcal{C}^1$ class such that $g_{ki}(\Gamma_t) = \sigma_{ki,t}$ and

$$d\sigma_{ki,t} = \sum_{s,r=1}^{n} \frac{\partial g_{ki}(\Gamma_t)}{\partial \Gamma_{sr,t}} d\Gamma_{sr,t}.$$

Thus, $\| \sigma_{kl,t'} - \sigma_{kl,t} \| \leq M \mid t' - t \mid^{\frac{1}{2}}$ through to SDE of Γ_t where

$$\begin{aligned}
M &= A \sum_{k,l=1}^{n} N_{kl} \\
A &= \sup_{\substack{t \in [0,T] \\ s,r,l,q=1,\ldots,n}} \left(\left| \frac{\partial g_{sr}(\Gamma_t)}{\partial \Gamma_{lq,t}} \right| ; |\Gamma_{sr,t}| ; |g_{sr}(\Gamma_t)| \right) < \infty
\end{aligned}$$

because the trajectory of Γ_t is right continuous with a left limit and the g_{sr} are $\mathcal{C}^1(U)$ classes, and further,

$$\begin{aligned}
N_{kl} = &\left| \nu \sum_{j=1}^{n} Q_{kj} Q_{jl} \right| + A \sum_{i=1}^{n} |\Phi_{il}| + |\Phi_{ki}| + \frac{A}{G(a+1)} \sum_{i,j=1}^{n} |Q_{jl}| + |Q_{ki}| + \\
&\lambda \left(2nA\sqrt{\sigma^2 + m^2} + n(\sigma^2 + m^2) \right),
\end{aligned}$$

where $\Phi = (\Phi_{kl})_{kl}$, $\sqrt{Q'Q} = (Q_{kl})_{kl}$.

Prove now that for $n = 1$ and $n = 2$, F is a global dimorphism.

For $n = 1$, we have $U =]0, +\infty[$, and $F(\sigma) = \sigma^2$. Thus $DF = 2\sigma > 0$ on U (DF is the derivative of F). The result follows using the global inversion theorem (see in the reference [6]).

For $n = 2$, we have $U = \{(x_{11}, x_{12}, x_{22}) \in \mathbb{R}^3 : x_{11}, x_{22} > 0; x_{11}x_{22} > (x_{12})^2\}$ and $F(\sigma_{11}, \sigma_{12}, \sigma_{22}) = ((\sigma_{11})^2 + (\sigma_{12})^2, (\sigma_{11} + \sigma_{22})\sigma_{12}, (\sigma_{22})^2 + (\sigma_{12})^2)$. Thus, we have $\det(DF) = \det \begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & 0 \\ \sigma_{12} & \sigma_{11} + \sigma_{22} & \sigma_{12} \\ 0 & 2\sigma_{12} & 2\sigma_{22} \end{bmatrix} = 4(\sigma_{11} + \sigma_{22})(\sigma_{11}\sigma_{22} - (\sigma_{12})^2) > 0$ for all $(\sigma_{11}, \sigma_{12}, \sigma_{22}) \in U$. The result follows using the global inversion theorem.

We can develop on $-\frac{1}{2} < d < 0$. To do this, to give a sense to $\int_0^T \sqrt{\Gamma_s} dZ_s$, we can use the rough paths theory of Lyon [12]. To converge the Riemann sum (2.17), we add a fix term built from the Levy's areas (see [15]). But for now, let's stay in $d \in \left[0, \frac{1}{2}\right[$.

Let γ be a vector in $\mathbb{R}^n$. The Laplace transform of $\log S_{t+h}$ given $\log S_t$ and Γ_t is defined by:

$$\Psi_{\log S_t}(\gamma, h) = \mathbb{E}\{e^{\gamma' \log S_{t+h}} / logS_t, \Gamma_t\} \text{ where } t, h \geq 0. \tag{2.18}$$

As the yield $\log S_t$ is affine, we have

$$\Psi_{\log S_t}(\gamma, h) = e^{tr(A(h)\Gamma_t) + B(h) \log S_t + C(h)} \tag{2.19}$$

with $A(h)$, $B(h)$ and $C(h)$ are the functions. Using the Feynmann–Kac argument to the model, we have

$$\frac{\partial \Psi_{\log S_{t-}}(\gamma, h)}{\partial h} = \mathcal{L}_{\log S, \Gamma} \Psi_{\log S_{t-}}(\gamma, h) \tag{2.20}$$

where $t, h \geq 0$; $\mathcal{L}_{\log S, \Gamma}$ is the infinitesimal generator of the joint $(\log S_t, \Gamma_t)$ defined by:

Proposition 2.5.

$$\begin{aligned} \mathcal{L}_{\log S,\Gamma} &= tr\left[\left(\frac{\nu h^{2a}}{G(a+1)^2} Q'Q + \Phi\Gamma_{t^-} + \Gamma_{t^-}\Phi\right) D + \frac{2h^{2a}}{G(a+1)^2} \Gamma_{t^-} D Q' Q D\right] \\ &+ \nabla_Y \left(r - \frac{1}{2} vec[tr(e_{ii}\Gamma_{t^-})]\right) + \frac{h^{2d}}{2G(d+1)^2} \nabla_Y \Gamma_{t^-} \nabla'_Y \end{aligned}$$

$$+ \quad \frac{h^{d+a}}{G(d+1)G(a+1)} tr\left(D\sqrt{Q'Q}\rho\nabla_Y\Gamma_{t^-} + \Gamma_{t^-}\nabla'_Y\rho'\sqrt{Q'Q}D\right)$$

$$+ \quad \lambda\Psi_{\log S_{t^-}}\mathbb{E}\left\{e^{(\gamma'+A(u))(2\sqrt{\Gamma_{t^-}}J\varphi+JJ'\varphi)} - 1/logS_t, \Gamma_t\right\}, \tag{2.21}$$

with

(i) $D = (D_{ij})_{1\leq i,j\leq n}$ *where* $D_{ij} = \frac{\partial}{\partial\Gamma_{ij,t}}$ *and* $\Gamma_{ij,t}$*,* $1 \leq i,j \leq n$ *are the components of the volatility matrix* Γ_t*;*

(ii) $\nabla_Y = \left(\frac{\partial}{\partial Y_1}, \cdots, \frac{\partial}{\partial Y_n}\right)'$ *where* $Y_i = \log S_{i,t}$ *is the yield of the i-th underlying in the basket,* $i = 1, \ldots, n$*.*

Proof. Let $t, h \geq 0$. The operator $\mathcal{L}_{\log S,\Gamma}$ can be broken down into the following four components:

$$\mathcal{L}_{\log S,\Gamma}\Psi_{\log S_{t^-}} = \mathcal{L}_{(\log S)^c}\Psi_{\log S_{t^-}} + \mathcal{L}_{(\Gamma)^c}\Psi_{\log S_{t^-}} + \mathcal{L}_{<(\log S)^c,(\Gamma)^c>}\Psi_{\log S_{t^-}} + \mathcal{L}_{jumps} \tag{2.22}$$

with

(i) Applying the same reasoning of Da Fonseca [5, equations (11)–(12)–(13)], we can obtain the infinitesimal generators $\mathcal{L}_{(\Gamma)^c}$, $\mathcal{L}_{(\log S)^c}$ and $\mathcal{L}_{<(\log S)^c,(\Gamma)^c>}$ defined by:

$$\mathcal{L}_{(\log S)^c} = \nabla_Y\left(r - \frac{1}{2}vec[tr(e_{ii}\Gamma_{t^-})]\right) + \frac{h^{2d}}{2G(d+1)^2}\nabla_Y\Gamma_{t^-}\nabla'_Y; \tag{2.23}$$

$$\mathcal{L}_{(\Gamma)^c} = tr\left[\left(\frac{\nu h^{2a}}{G(a+1)^2}Q'Q + \Phi\Gamma_{t^-} + \Gamma_{t^-}\Phi\right)D + \frac{2h^{2a}}{G(a+1)^2}\Gamma_{t^-}DQ'QD\right]; \tag{2.24}$$

$$\mathcal{L}_{<(\log S)^c,(\Gamma)^c>} = \frac{h^{d+a}}{G(d+1)G(a+1)}tr(D\sqrt{Q'Q}\rho\nabla_Y\Gamma_{t^-} + \Gamma_{t^-}\nabla'_Y\rho'\sqrt{Q'Q}D); \tag{2.25}$$

(ii) $\mathcal{L}_{jumps}$ is the infinitesimal generator of the jumps defined by:

$$\mathcal{L}_{jumps} = \lambda\mathbb{E}\left\{\Psi((\log S_{t+h} + H), \Gamma_{t+h} + G) - \Psi(\log S_{t+h}, \Gamma_{t+h})/logS_t, \Gamma_t\right\}$$

$$= \lambda\Psi_{\log S_{t^-},\Gamma_{t^-}} \times \mathbb{E}\left\{e^{\gamma' H + tr(A(h)G)} - 1/logS_t, \Gamma_t\right\} \tag{2.26}$$

where $H = 2\sqrt{\Gamma_{t^-}}J\varphi + JJ'\varphi$ and $G = 2\sqrt{\Gamma_{t^-}}J + JJ'$.

□

Proposition 2.6. *If* $\| 2\sigma^2\omega(h) \| < 1$*, then*

$$\begin{aligned} A(h) &= \int_0^h \Upsilon_3(u)A(u)^2 + \Upsilon_2(u)A(u) + \Upsilon_1(u)du, \\ B(h) &= \gamma', \\ C(h) &= tr\left[r\gamma' h + \frac{\nu}{G(a+1)^2}Q'Q\int_0^h u^{2a}A(u)du\right] \\ &+ \lambda\int_0^h \left[e^{tr\left[\omega(u)\mu(u)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_t}(m\tilde{1})+2\sigma^2\Gamma_t\omega(u)\right)-\frac{n}{2}\log\mu(u)\right]} - 1\right] du. \end{aligned}$$

where $\tilde{1}$ *is a* $n \times n$ *dimensional matrix whose components are equal to* 1 *and*

$$\begin{aligned} \omega(h) &= \frac{\varphi\gamma' + \gamma\varphi}{2} + A(h); \\ \mu(h) &= I_n - 2\sigma^2\omega(h); \\ \Upsilon_1(h) &= -\frac{1}{2}\sum_{i=1}^n \gamma_i e_{ii} + \frac{h^{2d}}{2G(d+1)^2}\gamma\gamma' \\ \Upsilon_2(h) &= (\Phi + \Phi') + \frac{h^{d+a}}{G(d+1)G(a+1)}(\gamma\rho'\sqrt{Q'Q} + \sqrt{Q'Q}\rho\gamma'), \\ \Upsilon_3(h) &= \frac{2h^{2a}}{G(a+1)^2}Q'Q. \end{aligned}$$

Proof. Let $t, h \geq 0$. We have

$$\frac{\partial\Psi_{\log S_{t^-}}(\gamma,h)}{\partial h} = \left[tr(\frac{\partial A(h)}{\partial h}\Gamma_{t^-}) + \frac{\partial B(h)}{\partial h}\log S_{t^-} + \frac{\partial C(h)}{\partial h}\right]\Psi_{\log S_{t^-}}(\gamma,h).$$

Then, from the expression (2.20), we deduce

$$\begin{aligned} & tr\left[\frac{\partial A(h)}{\partial h}\Gamma_{t^-}\right] + \frac{\partial B(h)}{\partial h}\log S_{t^-} + \frac{\partial C(h)}{\partial h} \\ &= B(h)(r - \frac{1}{2}vec[tr(e_{ii}\Gamma_{t^-})]) + \frac{h^{2d}}{2G(d+1)^2}B(h)\Gamma_{t^-}B(h)' \\ &+ tr\left[\left(\frac{\nu h^{2a}Q'Q}{G(a+1)^2} + \Phi\Gamma_{t^-} + \Gamma_{t^-}\Phi'\right)A(h) + \frac{2h^{2a}\Gamma_{t^-}A(h)Q'QA(h)}{G(a+1)^2}\right] \\ &+ \frac{h^{d+a}tr\left[A(h)\sqrt{Q'Q}\rho B(h)\Gamma_{t^-} + \Gamma_{t^-}B(h)'\rho'\sqrt{Q'Q}A(h)\right]}{G(d+1)G(a+1)} \end{aligned}$$

$$+\lambda\mathbb{E}\left\{e^{tr[(\varphi\gamma'+A(h))H]}-1/logS_t,\Gamma_t\right\} \tag{2.27}$$

with the initial conditions $A(0)=0$, $B(0)=\gamma'$ and $C(0)=0$.

Applying the some computations in (2.14) but $\omega(h)=\frac{\varphi\gamma'+\gamma\varphi}{2}+A(h)$ to the place of $B(h)$, we have
$\mathbb{E}\left\{e^{tr[\omega(h)(2\sqrt{\Gamma_{t^-}}J+JJ')]}/logS_t,\Gamma_t\right\}=e^{tr\left[\omega(h)\mu(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_{t^-}}(m\tilde{1})+2\Gamma_{t^-}\omega(h)\sigma^2\right)-\frac{n}{2}\log\mu(h)\right]}$
where $\mu(h)=I_n-2\sigma^2\omega(h)$. So, (2.27) is equal to

$$\begin{aligned}
&B(h)\left(r-\frac{1}{2}vec[tr(e_{ii}\Gamma_{t^-})]\right)+\frac{h^{2d}}{2G(d+1)^2}B(h)\Gamma_{t^-}B(h)'\\
&+tr\left[\left(\frac{\nu h^{2a}}{G(a+1)^2}Q'Q+\Phi\Gamma_{t^-}+\Gamma_{t^-}\Phi'\right)A(h)+\frac{2h^{2a}}{G(a+1)^2}\Gamma_{t^-}A(h)Q'QA(h)\right]\\
&+\frac{h^{d+a}}{G(d+1)G(a+1)}tr\left[A(h)\sqrt{Q'Q}\rho B(h)\Gamma_{t^-}+\Gamma_{t^-}B(h)'\rho'\sqrt{Q'Q}A(h)\right]\\
&+\lambda\left[e^{tr\left[\omega(h)\mu(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_t}(m\tilde{1})+2\sigma^2\Gamma_t\omega(h)\right)\right]-\frac{n}{2}tr\log\mu(h)}-1\right].
\end{aligned}$$

By identifying the coefficient of $\log S_{t^-}$, we have $\frac{\partial B(h)}{\partial h}=0$ which follows that $B(h)=B(0)=\gamma'$ for all $h\geq 0$.

Identifying the coefficient of Γ_{t^-}, we have

$$\frac{\partial A(h)}{\partial h} = -\frac{1}{2}\sum_{i=1}^{n}\gamma_i e_{ii}+\frac{h^{2d}}{2G(d+1)^2}\gamma\gamma' \tag{2.28}$$

$$+\ \Phi A(h)+A(h)\Phi'+\frac{2h^{2a}}{G(a+1)^2}A(h)Q'QA(h) \tag{2.29}$$

$$+\ \frac{h^{d+a}}{G(d+1)G(a+1)}(A(h)\sqrt{Q'Q}\rho\gamma'+\gamma\rho'\sqrt{Q'Q}A(h)), \tag{2.30}$$

which is integrable in $[0,1]$ (see the Euler approximation above doing tend n to $+\infty$). In the trace operator, $A(h)$ is the solution of Riccati SDE:

$$\frac{\partial z}{\partial h}=\Upsilon_3(h)z^2+\Upsilon_2(h)z+\Upsilon_1(h) \tag{2.31}$$

where

$$\Upsilon_1(h) = -\frac{1}{2}\sum_{i=1}^{n}\gamma_i e_{ii}+\frac{h^{2d}}{2G(d+1)^2}\gamma\gamma'$$

$$\begin{aligned}\Upsilon_2(h) &= (\Phi + \Phi') + \frac{h^{d+a}}{G(d+1)G(a+1)}(\gamma\rho'\sqrt{Q'Q} + \sqrt{Q'Q}\rho\gamma')\\ \Upsilon_3(h) &= \frac{2h^{2a}}{G(a+1)^2}Q'Q.\end{aligned}$$

Thus

$$A(h) = \int_0^h \Upsilon_3(u)A(u)^2 + \Upsilon_2(u)A(u) + \Upsilon_1(u)du. \tag{2.32}$$

And finally, by identification

$$\begin{aligned}\frac{\partial C(h)}{\partial h} &= tr\left[r\gamma' + \frac{\nu h^{2a}}{G(a+1)^2}Q'QA(h)\right]\\ &+ \lambda\left[e^{tr\left[\omega(h)\mu(h)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_t}(m\tilde{1})+2\sigma^2\Gamma_t\omega(h)\right)-\frac{n}{2}\log\mu(h)\right]} - 1\right]\end{aligned}$$

where $C(0) = 0$. And thus

$$\begin{aligned}C(h) &= tr\left[r\gamma' h + \frac{\nu}{G(a+1)^2}Q'Q\int_0^h u^{2a}A(u)du\right]\\ &+ \lambda\int_0^h \left[e^{tr\left[\omega(u)\mu(u)^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_t}(m\tilde{1})+2\sigma^2\Gamma_t\omega(u)\right)-\frac{n}{2}\log\mu(u)\right]} - 1\right]du. \end{aligned}\tag{2.33}$$

□

Remark 2.2. *$A(h)$ is a solution of Riccati SDE and it exists. Indeed, we can bring the problem back to that of Cauchy (cf. Raymond [17]):*

$$\begin{cases}\frac{dvec(z)}{dh} = g(h, vec(z)) = vec(f(h,z)) \in \mathbb{R}^{n^2}\\ vec(z(0)) = 0 \in \mathbb{R}^{n^2}.\end{cases}\tag{2.34}$$

We can easily see that for fixed h, the function $vec(z) \in \mathbb{R}^{n^2} \to g(h, vec(z)) \in \mathbb{R}^{n^2}$ is continue and locally Lipschitz in $vec(z)$. The result follows through the theorem of Cauchy–Lipschitz (cf. Raymond [17]). However, we don't have the explicit expression of $A(h)$ because the SDE is a Riccati type with variables coefficients and the expression of general solution is not determined. Hence the use of the approximation.

Proposition 2.7. *Let $n = 2$. Let ι be the interest rate. If $a = d = 0$ and*

$$r = \iota\check{1} - \lambda\begin{bmatrix} e^{tr\left[\omega_1\mu_1^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_0}(m\tilde{1})+2\sigma^2\Gamma_0\omega_1\right)-\log\mu_1\right]} - 1\\ e^{tr\left[\omega_2\mu_2^{-1}\left((m\tilde{1})^2+2\sqrt{\Gamma_0}(m\tilde{1})+2\sigma^2\Gamma_0\omega_2\right)-\log\mu_2\right]} - 1\end{bmatrix}$$

where $\omega_j = \frac{\varphi\gamma_j' + \gamma_j\varphi'}{2}$*;* $\mu_j = I_n - 2\sigma^2\omega_j$ *and* $\gamma_j = \begin{cases} (1,0)' \text{ if } j = 1 \\ (0,1)' \text{ if } j = 2 \end{cases}$*, then the market is without friction.*

Proof. Let be $a = d = 0$. In the market without friction, the hoped value of $j = 1; 2$ asset under the risk–neutral probability $\mathbb{P}$ is:

$$\begin{aligned} S_{j,0}e^{\iota t} &= \mathbb{E}[S_{j,t}|S_{j,0}, \Gamma_0] \\ &= \mathbb{E}[exp(\log S_{j,t})|\log S_{j,0}, \Gamma_0] \\ &= \Psi_{\log S_0}(\gamma_j, t) \\ &= e^{tr[A(t)\Gamma_0] + \log S_{j,0} + C(t)} \\ &= S_{j,0}e^{tr[A(t)\Gamma_0] + C(t)} \end{aligned} \tag{2.35}$$

where $\Psi_{\log S_0}(\gamma_j, t)$ is the Laplace transform above by taking $h = t$; $t = 0$ and $\gamma = \gamma_j$. For $a = d = 0$, the SDE (2.28) is resolved and we find $A(t)$ by imposing $A(t) = F(t)^{-1}G(t)$ with $F(t) \in GL_n(\mathbb{R})$ and $G(t) \in \mathcal{M}_n(\mathbb{R})$. We have $0 = A(0) = F(0)^{-1}G(0)$ and thus take $G(0) = 0$ and $F(0) = I_n$. Thus, we get

$$A(t) = A_{22}(t)^{-1}A_{21}(t) \tag{2.36}$$

where

$$\begin{aligned} \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} &= \exp\left(t\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & -M_{11} \end{bmatrix}\right) \\ M_{11} &= \frac{\Phi + \Phi'}{2} + \frac{\gamma_j\rho'\sqrt{Q'Q} + (\gamma_j\rho'\sqrt{Q'Q})'}{2} \\ M_{12} &= -2Q'Q \\ M_{21} &= \frac{1}{2}\gamma_j\gamma_j' - \frac{1}{2}\sum_{l=1}^{n}\gamma_{j,l}e_{ll} = 0 \end{aligned}$$

We have

$$\begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} = \begin{bmatrix} e^{M_{11}t} & -\frac{1}{2}M_{11}^{-1}(e^{M_{11}t} - e^{-M_{11}t})M_{12} \\ 0 & e^{-M_{11}t} \end{bmatrix} \tag{2.37}$$

$$A(t) = 0 \tag{2.38}$$

$$C(t) = r_j t + \lambda t\left[e^{tr\left[\omega_j\mu_j^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_0}(m\tilde{1}) + 2\sigma^2\Gamma_0\omega_j\right) - \log\mu_j\right]} - 1\right]. \tag{2.39}$$

where r_j is the j-th component of vector r. Indeed, let be $T = \begin{bmatrix} M_{11} & M_{12} \\ 0 & -M_{11} \end{bmatrix}$, $T^s = (T_{ij}^{(s)})_{ij}$, $s \in \mathbb{N}$. In the trace operator, we have $T_{ij}^{(0)} = I_n$ if $i = j$ and 0 otherwise; $T_{11}^{(1)} = M_{11}$; $T_{12}^{(1)} = M_{12}$; $T_{21}^{(1)} = 0$; $T_{22}^{(1)} = -M_{11}$; $T_{11}^{(2)} = (M_{11})^2$; $T_{12}^{(2)} = M_{11}M_{12} - M_{21}M_{11} = 0$; $T_{21}^{(2)} = 0$; $T_{22}^{(2)} = (M_{11})^2$. Now, let us consider $p \geq 1$, in the trace operator, reasoning by recurrence, we have $T_{11}^{2(p+1)} = T_{11}^{(2)}T_{11}^{(2p)} + T_{12}^{(2)}T_{21}^{(2p)} = (M_{11})^{2(p+1)}$; $T_{12}^{2(p+1)} = T_{11}^{(2)}T_{12}^{(2p)} + T_{12}^{(2)}T_{22}^{(2p)} = 0$; $T_{21}^{2(p+1)} = T_{21}^{(2)}T_{11}^{(2p)} + T_{22}^{(2)}T_{21}^{(2p)} = 0$; $T_{22}^{2(p+1)} = T_{21}^{(2)}T_{12}^{(2p)} + T_{22}^{(2)}T_{22}^{(2p)} = (M_{11})^{2(p+1)}$. Then using the values $T_{ij}^{(1)}$ and $T_{ij}^{(2p)}$ above, we have, for all $p \geq 1$ $T_{11}^{2p+1} = T_{11}^{(1)}T_{11}^{(2p)} + T_{12}^{(1)}T_{21}^{(2p)} = (M_{11})^{2p+1}$; $T_{12}^{2p+1} = T_{11}^{(1)}T_{12}^{(2p)} + T_{12}^{(1)}T_{22}^{(2p)} = (M_{11})^{2p}M_{12}$; $T_{21}^{2p+1} = T_{21}^{(1)}T_{11}^{(2p)} + T_{22}^{(1)}T_{21}^{(2p)} = 0$; $T_{22}^{2p+1} = T_{21}^{(1)}T_{12}^{(2p)} + T_{22}^{(1)}T_{22}^{(2p)} = -(M_{11})^{2p+1}$. Well, we have

$$\begin{aligned} \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} &= e^{tT} = \sum_{s=0}^{+\infty} \frac{(tT)^s}{s!} \\ &= \begin{bmatrix} e^{M_{11}t} & -\frac{1}{2}M_{11}^{-1}(e^{M_{11}t} - e^{-M_{11}t})M_{12} \\ 0 & e^{-M_{11}t} \end{bmatrix}. \end{aligned}$$

Thus, the value of $A(t)$ is obtained through the expression (2.36) and the $A_{ij}(t)$ above. Hence

$$(2.35) = S_{j,0}e^{r_j t + \lambda t\left[e^{tr\left[\omega_j \mu_j^{-1}\left((m\bar{1})^2 + 2\sqrt{\Gamma_0}(m\bar{1}) + 2\sigma^2\Gamma_0\omega_j\right) - \log \mu_j\right]} - 1\right]}$$

and the result follows by using identification method. □

Remark 2.3. *1) r is the rate interest if the model is without jump process. 2) For the fractals models, Rogers [18] and Guasoni [8] showed that $\mathbb{E}\left(\int \sigma(s, B_{a,s})dB_{a,s}\right) \neq 0$ products an arbitrage and so any transaction is payed with a rate $\epsilon > 0$.*

2.5. **Approximation of function $A(h)$.** We have not the closed form expression of the particular solution of SDE (2.31). Then, we approximate $A(h)$ by a value approached. An example is Euler's approximation.

Euler's method

Let be a differential equation $\frac{\partial y}{\partial x} = f(x, y)$ on the interval $[a, b]$. We propose a solution approached of differential equation on the interval $[a, b]$ using the Euler's method. We divide the interval $[a, b]$ using the regular subdivision of n order: $x_i = a + \frac{i(b-a)}{n}$, $\forall i \in \{0, 1 \ldots, n\}$. The step is $s = \frac{b-a}{n}$. In this case, we find $\forall i \in$

$\{0, 1 \ldots, n\}$ a approached value y_i of $y^*(x_i)$ where y^* is the solution of equation. And thus, we continue by the following principle: we suppose the initial condition $u_0 = (x_0, y_0)$, and:

- on $[a, x_1]$, we replace the function y^* by its tangent at u_0. We obtain $y_1 = y_0 + s\frac{\partial y^*}{\partial x}$. But since y^* is the solution of equation, we have that $\frac{\partial y^*(x_0)}{\partial x} = f(x_0, y_0)$. So, we have $y_1 = y_0 + sf(x_0, y_0)$ which gives $u_1 = (x_1, y_1)$.
- on $[x_1, x_2]$, we operate in the some style and suppose $y_2 = y_1 + sf(x_1, y_1)$ to obtain $u_2 = (x_2, y_2)$.
- We operate of this way $\forall i \in \{0, 1 \ldots, n\}$ and we have $y_i = y_{i-1} + sf(x_{i-1}, y_{i-1})$ which is a approximation of $y^*(x_i)$.

Thus, Euler's approximation on $[0, T]$ of $A(h)$ is:

$$A_n(h) = \begin{cases} y_0 = 0 \text{ if } h = 0 \\ y_1 = y_0 + sf(0, 0) \text{ if } h \in [0, x_1], \\ \vdots \\ y_i = y_{i-1} + sf(x_{i-1}, y_{i-1}) \text{ if } h \in [x_{i-1}, x_i], i = 1, \ldots, n \end{cases} \tag{2.40}$$

where $f(x, y) = \Upsilon_3(x)y^2 + \Upsilon_2(x)y + \Upsilon_1(x)$ such that $f(0, 0) = \Upsilon_1(0)$.

Convergence speed of approximation of $A(h)$

Let A_0 be the true value solution of SDE (2.31), N be a large enough integer and $A^{(N)}_{Euler}$ be an Euler's approximation with iteration N. The convergence speed of Euler's method is slow: the rest of $\| A^{(N)}_{Euler} - A_0 \|$ is $\frac{1}{N^2}$ order (cf. Raymond [17]).

2.6. **Dependence between yield and its volatility.** Let be a market FIWASVJ(d,a) where d, a are the fractals indexes such that $a, d \in \left[0, \frac{1}{2}\right[$. We assume that each component of the vector $\tilde{B}_{d,t}$ is independent with the one matrix $\tilde{W}_{d,t}$ (see the third equation in (1.1)).

Theorem 2.1. *The covariance between each component of vector yield noise $d \log S_t$ and the one volatility noise matrix $d\Gamma_t$ is given by for all $i, j, h = 1, \ldots, n$,*

$$cov(d(\log S_{h,t})^c, d(\Gamma_{ij,t})^c) = \frac{\left(\Gamma_{hi,t} \sum_{l=1}^{n} Q_{lj}\rho_l + \Gamma_{hj,t} \sum_{l=1}^{n} Q_{li}\rho_l\right)(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)}, \tag{2.41}$$ *with*

(i) $\log S_{.,t}$ *is the component of the yield vector* $\log S_t$,

(ii) $\rho_{.}$ *is the component of vector* ρ,

(iii) $\Gamma_{..,t}$ *is the component of the volatility matrix* Γ_t *and*

(iv) $Q_{..}$ *is the component of the matrix* $Q'Q$.

Proof. From the expressions $\sqrt{\Gamma_t} = (\sigma_{ij,t})_{1\le i,j\le n}$ which is symmetrical and $\Gamma_t = (\Gamma_{ij,t})_{i,j=1,\dots,n}$, we get

$$\Gamma_{ij,t} = \sum_{l=1}^{n} \sigma_{il,t}\sigma_{jl,t}. \tag{2.42}$$

Now, let be $i, j, h \in \{1, \dots, n\}$. $d(\log S_{h,t})^c$ is the yield noise of $\log S_{h,t}$ in the continuous part which is the h-th line of $d\log S_t$ defined in equation (1.1) by $d(\log S_{h,t})^c = \left(r_h - \frac{\Gamma_{hh,t}}{2}\right) dt + \sum_{k=1}^{n} \sigma_{hk,t} dZ_{k,t}$. And $d(\Gamma_{ij,t})^c$ is the component i-th row and j-th column of $d(\Gamma_t)^c$ with

$$\begin{aligned} d(\Gamma_{ij,t})^c &= \frac{\nu(dt)^{2a+1}}{(2a+1)G(a+1)^2}\sum_{l=1}^{n} Q_{il}Q_{jl} + \left(\sum_{l=1}^{n}\Phi_{il}\Gamma_{lj,t} + \sum_{l=1}^{n}\Gamma_{il,t}\Phi_{jl}\right) dt \\ &+ \sum_{m,l=1}^{n}(\sigma_{im,t}dW_{a,ml,t}Q_{lj} + \sigma_{jm,t}dW_{a,ml,t}Q_{li}). \end{aligned} \tag{2.43}$$

So,

$$\begin{aligned} &cov(d(\log S_{h,t})^c, d(\Gamma_{ij,t})^c) \\ &= cov\left(\sum_{k=1}^{n}\sigma_{hk,t}dZ_{k,t}, \sum_{m,l=1}^{n}(\sigma_{im,t}dW_{a,ml,t}Q_{lj} + \sigma_{jm,t}dW_{a,ml,t}Q_{li})\right) \end{aligned} \tag{2.44}$$

where $dZ_{k,t} = \sqrt{1-\rho'\rho}\, dB_{d,k,t} + \sum_{p=1}^{n} dW_{d,kp,t}\rho_p$.

Since each component of the vector B_t is independent with the one matrix $\tilde{W}_t$, we have $cov(dB_{d,k,t}, dW_{a,sm,t}) = 0\ \forall\ k$, s and m, then using

$$cov\left(dW_{d,kp,t}, dW_{a,ml,t}\right) = \begin{cases} 0 \text{ si } (k,p) \neq (m,l) \\ \frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)} \text{ otherwise} \end{cases}, \tag{2.45}$$

we have

$$(2.44) = \frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)}\sum_{k,l=1}^{n}\sigma_{hk,t}\sigma_{ik,t}Q_{lj}\rho_l +$$

$$\frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)}\sum_{k,l=1}^{n}\sigma_{hk,t}\sigma_{jk,t}Q_{li}\rho_l$$
$$= \frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)}\left(\Gamma_{hi,t}\sum_{l=1}^{n}Q_{lj}\rho_l+\Gamma_{hj,t}\sum_{l=1}^{n}Q_{li}\rho_l\right),$$

through (2.42). □

Let $\zeta_{pq,t}$ be the correlation between $\Gamma_{pp,t}$ and $\Gamma_{qq,t}$ defined by

$$\zeta_{pq,t}=\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}. \tag{2.46}$$

2.7. **Dependence between yield and its correlations.**

Theorem 2.2. *The expressions of covariances between each yield noise of the basket* $\log S_{p,t}$ *and the correlations noises* $\zeta_{pq,t}$*,* $p,q=1,\dots,n$ *and* $p\neq q$ *are given by:*

$$cov(d(\log S_{p,t})^c, d(\zeta_{pq,t})^c)=\frac{(dt)^{d+a+1}(1-\zeta_{pq,t}^2)}{(d+a+1)G(d+1)G(a+1)}\sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}}\sum_{l=1}^{n}Q_{lq}\rho_l \tag{2.47}$$

with $(\zeta_{pq,t})^c$ *is the continuous part of* $\zeta_{pq,t}$*.*

Proof. Let be $p,q\in\{1,\dots,n\}$, $p\neq q$. Applying Ito's formula on process $f(\Gamma_t)=\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}$, we get

$$\begin{aligned} & d(\zeta_{pq,t})^c \\ = & \frac{d(\Gamma_{pq,t})^c}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}-\frac{1}{2}\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}\left(\frac{d(\Gamma_{pp,t})^c}{\Gamma_{pp,t}}+\frac{d(\Gamma_{qq,t})^c}{\Gamma_{qq,t}}\right)+\frac{(dt)^{2a+1}}{2(2a+1)G(a+1)^2} \\ & \sum_{i,j,k,l=1}^{n}\frac{\partial^2 f(\Gamma_t)}{\partial\Gamma_{ij,t}\partial\Gamma_{kl,t}}\left[\sum_{r=1}^{n}\Gamma_{ik,t}Q_{rj}Q_{rl}+\Gamma_{il,t}Q_{rj}Q_{rk}+\Gamma_{jk,t}Q_{ri}Q_{rl}+\Gamma_{jl,t}Q_{ri}Q_{rk}\right]. \end{aligned} \tag{2.48}$$

Hence

$$\begin{aligned} & cov(d(\log S_{i,t})^c, d(\zeta_{pq,t})^c) \\ = & \frac{cov(d(\log S_{i,t})^c, d(\Gamma_{pq,t})^c)}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}-\frac{1}{2}\frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}} \\ & \left(\frac{cov(d(\log S_{i,t})^c, d(\Gamma_{pp,t})^c)}{\Gamma_{pp,t}}+\frac{cov(d(\log S_{i,t})^c, d(\Gamma_{qq,t})^c)}{\Gamma_{qq,t}}\right) \end{aligned} \tag{2.49}$$

$$
\begin{aligned}
&= \frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)} \frac{(\Gamma_{qq,t}\Gamma_{ip,t} - \Gamma_{pq,t}\Gamma_{iq,t})\sum_{l=1}^{n} Q_{lq}\rho_l}{\Gamma_{qq,t}\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}} \\
&+ \frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)} \frac{(\Gamma_{pp,t}\Gamma_{iq,t} - \Gamma_{pq,t}\Gamma_{ip,t})\sum_{l=1}^{n} Q_{lp}\rho_l}{\Gamma_{pp,t}\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}} \qquad (2.50)
\end{aligned}
$$

through (2.41). Assuming $i = p$, we have that (2.49) is equal to

$$
\begin{aligned}
&\frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)} \sum_{l=1}^{n} Q_{lq}\rho_l \left(\sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}} - \frac{\Gamma_{pq,t}}{\sqrt{\Gamma_{pp,t}\Gamma_{qq,t}}}\frac{\Gamma_{pq,t}}{\Gamma_{qq,t}} \right) \\
= &\frac{(dt)^{d+a+1}}{(d+a+1)G(d+1)G(a+1)} \sum_{l=1}^{n} Q_{lq}\rho_l \sqrt{\frac{\Gamma_{pp,t}}{\Gamma_{qq,t}}}(1 - \zeta_{pq,t}^2) \text{ through (2.46).}
\end{aligned}
$$

□

3. Estimating of the parameter of model

Consider a market with an interest rate ι by paying the transaction with rate $\epsilon > 0$:

$$
\begin{cases}
d\log S_t = \left(\iota\check{1} - \lambda \begin{bmatrix} e^{tr\left[\omega_1\mu_1^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_0}(m\tilde{1}) + 2\sigma^2\Gamma_0\omega_1\right) - \log\mu_1\right]} - 1 \\ e^{tr\left[\omega_2\mu_2^{-1}\left((m\tilde{1})^2 + 2\sqrt{\Gamma_0}(m\tilde{1}) + 2\sigma^2\Gamma_0\omega_2\right) - \log\mu_2\right]} - 1 \end{bmatrix} \right. \\
\qquad \left. -\frac{1}{2}vec[tr(e_{ii}\Gamma_t)] \right) dt + \sqrt{\Gamma_t}dZ_t + d\psi_t\varphi \\
d\Gamma_t = \frac{\nu}{(2a+1)G(a+1)^2}Q'Q(dt)^{2a+1} + (\Phi\Gamma_t + \Gamma_t\Phi')dt \\
\qquad +\sqrt{\Gamma_t}d\tilde{B}_{a,t}Q + Q'(d\tilde{B}_{a,t})'\sqrt{\Gamma_t} + d\psi_t \\
dZ_t = \sqrt{1-\rho'\rho}dW_{d,t} + d\tilde{B}_{d,t}\rho \\
d\psi_t = \sqrt{\Gamma_t}d\tilde{P}_t + (d\tilde{P}_t)'\sqrt{\Gamma_t} + (d\tilde{P}_t)(d\tilde{P}_t)'
\end{cases} \qquad (3.1)
$$

We pass by following two-steps to estimate the parameters of the model:

i) We estimate the order of the fBm a via the local Whittle estimator of L Kristoufek [10].

ii) We estimate the parameters Φ, φ, ρ, ν, m, σ, λ, $\sqrt{Q'Q}$ and the fractal index d using the estimating method presented in Andrianantenainarinoro [2].

4. Application

In this article, we propose to estimate the current price of CAC40 and S&P500. We used the daily CAC40 and S&P500 indexes. For each stock, the time series start the January 29, 2020 and end the February 19, 2020 which are presented by the following Figure 1.

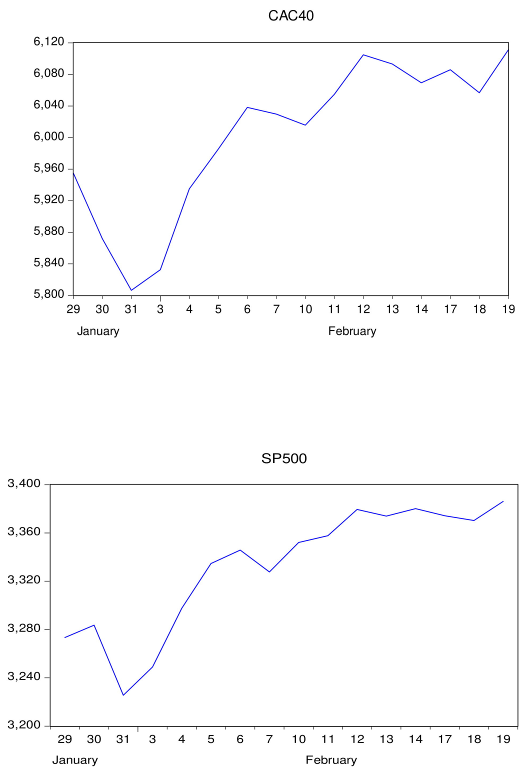

FIGURE 1. Historical Volume of CAC40 and S&P500 Indexes

We can see from the graphs that prices have rebounded recently and according to an economic analysis the covid-19 is the cause. The investors therefore rescue to buy the calls and the seller must feel a greater probability of losing and seeks an arbitrary model. Now is the time to find out if our model can bring the market back without friction. To do this, we church how the price evolves and what is the price of option.

4.1. **Results of estimation.**

Step 1: Estimating of fractal index a

We get $\hat{a} = 0$ with standard deviation error 0.1767767. Hence, the series have short memory.

Step 2: Estimating of C-GMM estimators Φ, φ, ρ, ν, m, σ, λ and $\sqrt{Q'Q}$

4.1.1. *Monte Carlo study.* The initial parameters used in the simulation are:

$$\Gamma_0 = \begin{bmatrix} 0.0225 & -0.0054 \\ -0.0054 & 0.0144 \end{bmatrix}, \Phi = \begin{bmatrix} -5 & -0.5 \\ -0.5 & -5 \end{bmatrix};$$

$$\varphi = (-1, -1);\ \rho = (-0.3, -0.4);\ \nu = 15;$$

$$m = 0;\ \sigma = 0.01;\ \lambda = 0.5;\ \alpha = 0.00225;$$

$$\sqrt{Q'Q} = \begin{bmatrix} 0.1204 & -0.01097 \\ -0.01097 & 0.09549 \end{bmatrix}.$$

The matrix $\sqrt{Q'Q}$ is obtained by using the long–term relationship:

$$\Gamma_\infty \Phi' + \Phi \Gamma_\infty + \lambda(m\tilde{1})\sqrt{\Gamma_\infty} + \lambda\sqrt{\Gamma_\infty}(m\tilde{1}) = -\nu Q'Q - \lambda n \sigma^2 I_n - n\lambda(m\tilde{1})^2$$

which is a necessary condition of WASC Stochastic Volatility Jump model for that process Γ_t is stationary (see Andrianantenainarinoro [1, equation (4.3)]). The annual interest rate ι is taken in the range $[0.015; 0.0175]$ which is a daily interest rate $0,000045$.

The table 1 shows the descriptives statistics of the data used. The figure 2 displays the C-GMM method criterion.

The table 4.1.2 presents the C-GMM estimator $\hat{\theta}_1$.

The results of the estimates of $\hat{\theta}$ with its standard deviations of errors are presented in the table 3.

The figure 3 gives the stylized facts captured by the model.

We use the uniform density on $[0, 1] \times [0, 1]$ for the density π.

The figure 4 shows the forecast of two courses CAC40 and SP500 using FIWASVJ model with $d = 0$ and the pricing option is presented in the tables 4 and 5. While, the figure 5 is for FIWASVJ model with $d = 0,49$ and the pricing option is presented in the tables 6 and 7.

4.1.2. *Empirical results.*

We present the results studying the data by statistics descriptives analysis.

TABLE 1. Analysis by descriptives statistics

Indice	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
log (CAC40)	8,867	8,691	8,705	8,700	8,712	8,718
log (SP500)	8,079	8,100	8,116	8,111	8,124	8,127

The two underlying are no dispersed with compared to average. The yield of CAC40 can be adjusted by the Gaussian distribution $N(8.7, 0.000265036)$ and the yield SP500 by $N(8.111, 0.000236996)$.

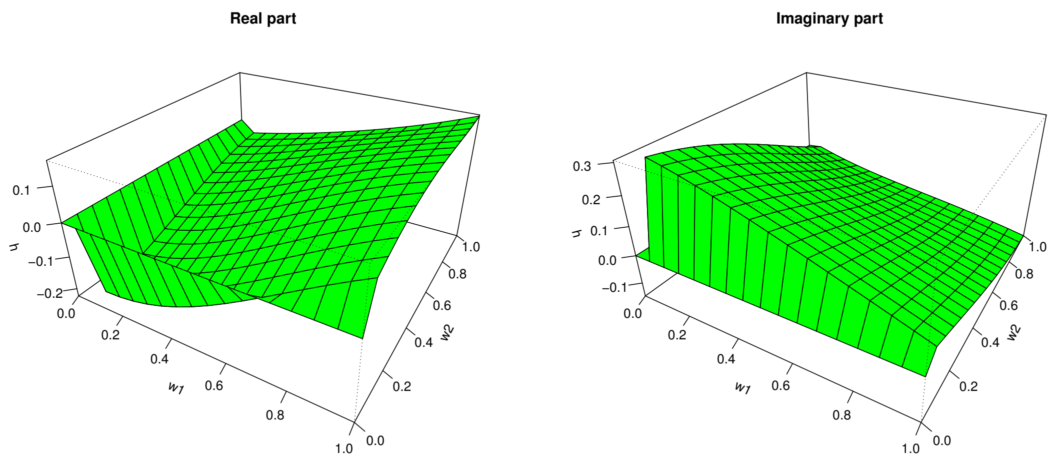

FIGURE 2. C-GMM estimation criterion

The figure 2 show us the values taken by real and imaginary part of the empirical moment of continuum of C-GMM method. It show us that we can minimize its function.

TABLE 2. C-GMM estimator $\hat{\theta}_1$.

parameter	ρ_1	ρ_2	Q_{11}	$Q_{12} = Q_{21}$	Q_{22}	Φ_{11}	Φ_{12}
estimator	-0.7	-0.7	0.1	0.0007876	0.1	-20.401852	-0.09999907

Φ_{21}	Φ_{22}	ν	λ	φ_1	φ_2	m	σ
-0.016926	-20.421835	2	0.003093	0.588434	-0.987034	0.00032	0.346291

with objective $2,280892.10^{-6}$.

TABLE 3. C-GMM estimator $\hat{\theta}$

parameter	estimator	standard deviation
ρ_1	-0.7	0.5564173
ρ_2	-0.7	0.5265019
Q_{11}	0.100000000	0.08094443
$Q_{12} = Q_{21}$	0.09374005	1.309801
Q_{22}	0.100000000	1.305106
Φ_{11}	-22.01260	2.588469
Φ_{12}	-0.1	1.721465
Φ_{21}	-0.1	0.8070361
Φ_{22}	-21.95342	1.288708
ν	2	0.5529048
λ	0.001703251	0.5522816
φ_1	0.1518992	$2,786253.10^{-8}$
φ_2	-0.9993954	$1,270706.10^{-8}$
m	0.000031086	0.004261457
σ	0.000038515	25.50306

with objective $1,630633.10^{-7}$.

4.1.3. *Study of prices CAC40 and SP500.*

It is better to recognize the asset noise if it is logic or not before using for the hedging or pricing option. Let be the fractal index $d \in \left[0, \frac{1}{2}\right[$. We works in $[0, 0.0001]$ and we use the time–step $= 0.0001 \times \frac{1}{300}$ (to facility the discretization of model) and the fractal index $d = 0$; $d = 0.05$ and $d = 0.1$.

The classified figures by fractal index are similar appearance but the scale makes the difference. We look in below a value of fractal index d which adjusts the course of model to the reality. We also acknowledge that on a period of strong volatility the asset have weak values and on the decrease period of volatility the asset increases. Likewise, the underlying prices studied are characterized by an asymmetric correlation between the correlation and asset. We perceive also the return to average. The model does not capture a jump.

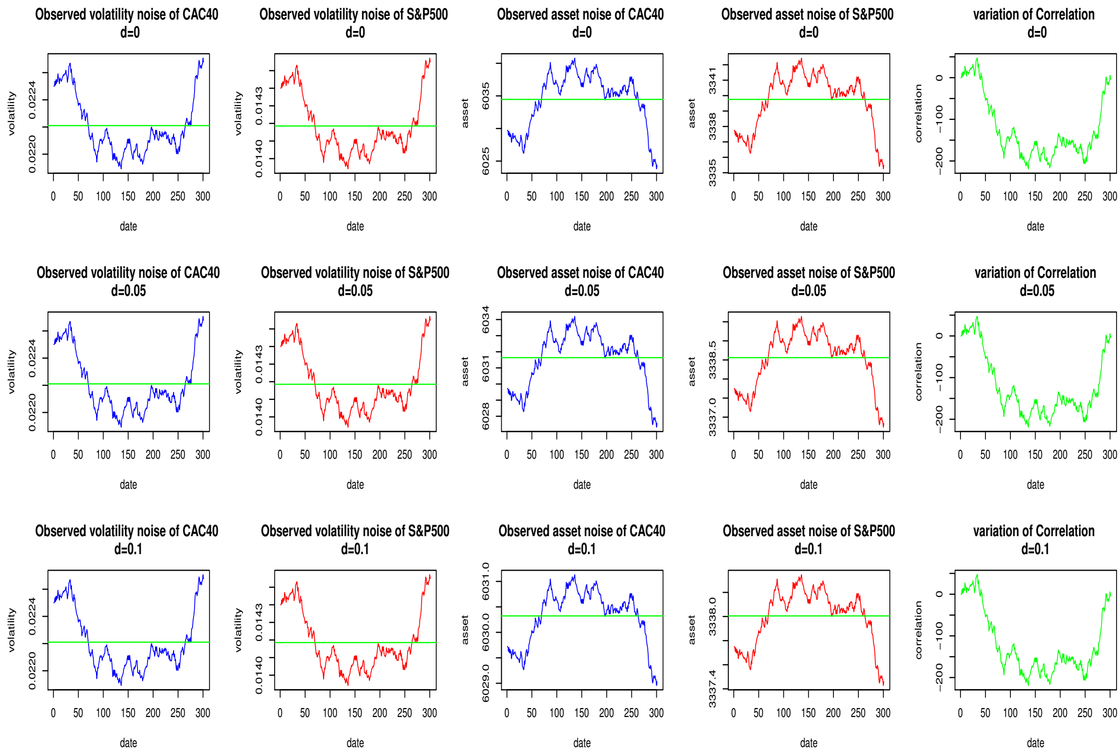

FIGURE 3. Stylized facts of model

Step 3: Estimate the fractal index d

Using the real data (see in a site of trading www.boursorama.com or m.fr.investing.com), the strong value of CAC40 (resp. S&P500) of daily later is 6111.41 (resp. 3393.52); the weak value of CAC40 (resp. S&P500) is 6072.66 (resp. 3378.83); the frequency data is 15 second; the market opens at 9:00 AM and closes at 5:35 PM.

We remark that the scale of CAC40 or S&P500 using WASVJ model is not logic according the figure 4.

The CAC40 (resp. S&P500) in this time can take a big value greater than 6111.41 (resp. 3393.52). We can correct that and we found, for $d = 0.49$, the course evolution is given in figure 5.

4.2. European call option of the basket CAC40 and SP500.

Let be a European call of the basket of indexes (CAC40, S&P500) and note by (K_1, K_2) the strike of index quoted by points. The maturity is half a day and one

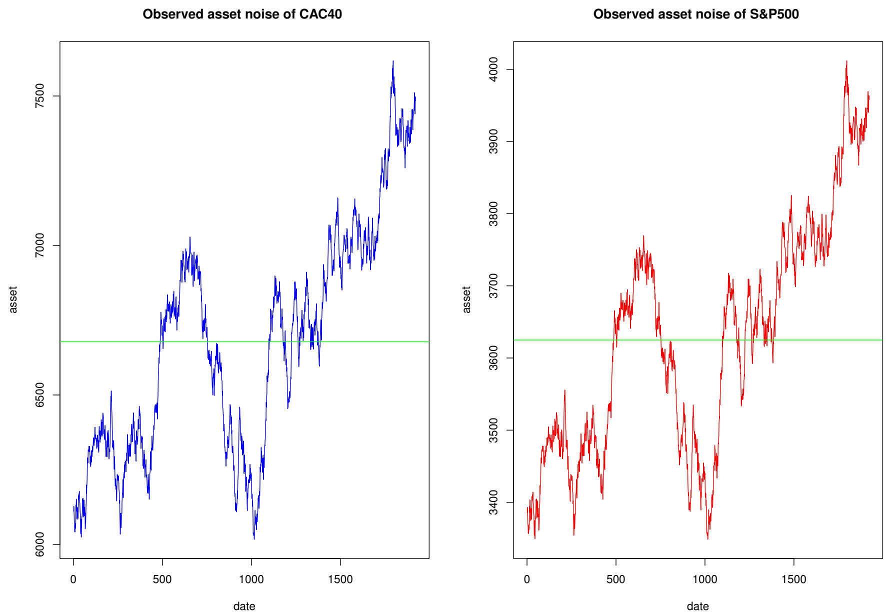

FIGURE 4. The course evolution in the one daily using WASVJ model

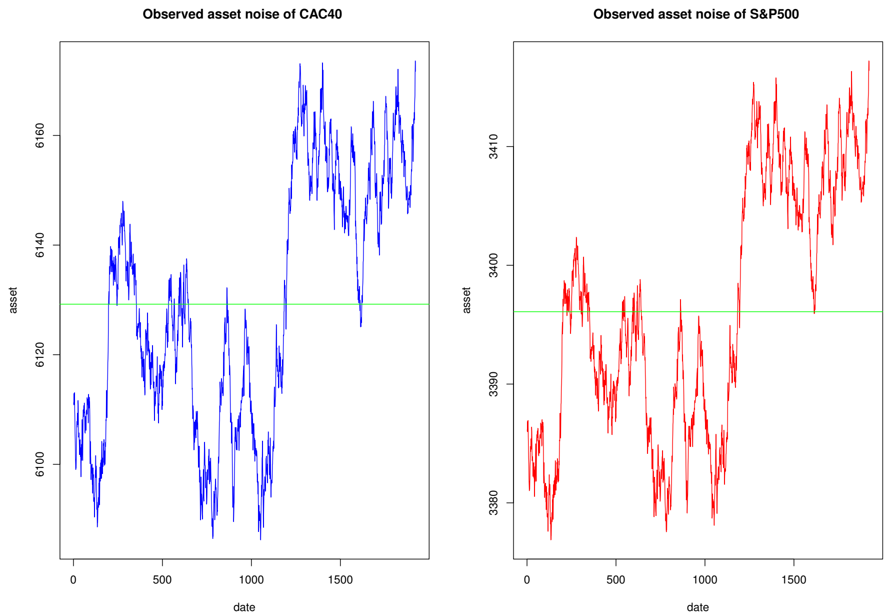

FIGURE 5. The course evolution in the one daily with $d = 0,49$

day. The rate interest is $\iota = 0$ for the date expired in half daily and $\iota = 0.000045$ for daily rate. The annual interest rate ι is taken in a fork $[0.015; 0.0175]$ which is a the daily rate $0,000045$. We use the correlation and spread options in the reference [7].

Take $N = 622$; $\epsilon_1 = -3$; $\epsilon_2 = 1$ and $(\bar{u}, K) = (47.80884, 2566.262)$ or $(\bar{u}, K) = (42.49675, 2673.852)$.

TABLE 4. Spread option with WASC and WASVJ models

model		WASC		WASVJ	
expiration T		0.5	1	0.5	1
K	2566.262	150.1792	150.1766	150.1792	150.1766
	2673.852	32.27406	32.28091	32.27406	32.28091

For correlation option, we take $\bar{u} = 40$; $N = 2^9$; $\alpha_1 = 0.3$ and $\alpha_2 = 0.4$.

TABLE 5. Correlation Option with WASC and WASVJ models

model		WASC		WASVJ	
expiration		0.5	1	0.5	1
(K_1, K_2)	(5649.82,3260.38)	32588.27	36788.34	32588.27	36788.34
	(6111.45,3260.38)	3155.289	4727.712	3155.289	4727.712

The price options using WASC and WASVJ are any equal.

Let see now the option when we regularize the amplitude.

TABLE 6. Spread option with FIWASVJ(0.49,0) model

T		0.5	1
K	2566.262	150.0673	150.0968
	2673.852	32.08414	32.18518

TABLE 7. Correlation option with FIWASVJ(0.49,0) model

expiration		0.5	1
(K_1, K_2)	(5649.82,3260.38)	28423.57	30776.66
	(6111.45,3260.38)	1552.37	2486.081

We observe that the price option changes for $d = 0.49$ and the difference is not significant for spride option but not for correlation option.

5. Discussion on the results and conclusion

The memory of volatility is short.

The jump does not exist because its intensity is almost zero, there explicates the equality between the option price of WASC and WASVJ models. The volatilities of volatility stochastic are smalls, then there is not an anomaly.

The scale regularization of assets is important because the scale of course is rectified. Moreover, the price option changes. For example, on February 20, 2020 (CAC40, S&P500)= (6062.3, 3373.23). With WASC model, the market sells an option correlation of 4727.712 points after one day with strike (6111.45, 3260.38). But, when we adjust the course (d=0.49), with FIWASVJ(0.49,0), the option is 2486.081 and the reduction is significant. Thus there exist a risk created by the strong amplitude on the option pricing and the fractal index can solve its risk.

In practical, we use the characteristic function inside the Laplace transform, then $A(h)$ can resolve directly as in work of Asai [3].

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