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CYCLOTOMIC AND INVERSE CYCLOTOMIC POLYNOMIAL

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ABSTRACT. In this article, the companion matrix of the cyclotomic and inverse cyclotomic polynomial is found. To determine the flat, gap, jump of the $\Phi_n(x)$ and $\Psi_n(x)$ for the binary, ternary, quaternary, quinary cyclotomic polynomial. To find some properties of the cyclotomic and inverse cyclotomic polynomial.

1. INTRODUCTION

The term cyclotomic means "circle-dividing" which comes from the nth roots of unity in C divide a circle into n arcs of equal length. The nth roots of unity lies on the unit circle as the vertices of a regular n-gon. A primitive nth root of unity is a complex number z satisfying $z^n = 1$, but not $z^d = 1$ for any $d < n$. Let ξ_n denote any primitive nth root of unity. $\xi_n = \{\zeta_n^j, 1 \leq j \leq n, (j,n) = 1, \zeta_n = e^{2\pi i/n}\}$ For any positive integer n, the nth cyclotomic polynomial is a divisor of $x^n - 1$ and is not a divisor of $x^k - 1$ for any $k < n$,

$$\Phi_n(x) = \prod_{1 \leq k \leq n, (k,n)=1} (x - e^{2i\pi k/n}).$$

The degree of $\Phi_n(x)$ is $\varphi(n)$, where $\varphi(n)$ denotes Euler's totient function. The index of $\Phi_n(x)$ is n. $\Phi_n(x)$ is monic with integer coefficients and is irreducible over

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Q and all of its coefficients are integral. The roots of the $\Phi_n(x)$ lies in the unit circle. The minimal polynomial of any nth root of unity over the rationals is a cyclotomic polynomial. $\Phi_n(x)$ is defined in the whole complex plane and for any integer n. $\Phi_n(0) = 1, n \neq 1$,

$$x^n - 1 = \prod_{d/n} \Phi_d(x). \tag{1.1}$$

The nine cyclotomic polynomials are (Refer Gary [11])

$$\Phi_1(x) = x - 1, \Phi_2(x) = x + 1, \Phi_3(x) = x^2 + x + 1, \Phi_4(x) = x^2 + 1,$$
$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1, \Phi_6(x) = x^2 - x + 1,$$
$$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \Phi_8(x) = x^4 + 1, \Phi_9(x) = x^6 + x^3 + 1.$$

The Mobius function is defined as follows

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n = p_1, \ldots, p_r \\ 0 & \text{if n is divisible by a square.} \end{cases}$$

Another form of the cyclotomic polynomial $\Phi_n(x) = \prod_{d|n, 1 \leq d < n} (x^d - 1)^{\mu(n/d)}$.

Cyclotomic concept is important concept in number theory,Galois theory, combinotorics, algebra and their applications, which encouraged the research on its structure, height, gap and jump. Thangadurai [15] proved important properties of $\Phi_n(x)$. Andrew [2] found the algorithms for calculating $\Phi_n(x)$.Camburu et al. [9] found the new techniques for gaps and jump. Bin [8] gave an infinite of ternary cyclotomic polynomials with height 3. Bzdega et al. [4] found the formula for the $\Phi_n(x)$ at the other roots for unity using Fourier analysis. Andrica et al. [1] found the integral formula for the cyclotomic polynomial.

2. Preliminaries

Notations

$\Phi_n(x)$ = Cyclotomic polynomial

$\Psi_n(x)$ =Inverse cyclotomic polynomial

$C(\Phi_n(x))$ = Companion matrix of $\Phi_n(x)$

$Det(C(\Phi_n(x)))$ = Determinant of $\Phi_n(x)$

$\rho(C(\Phi_n(x)))$ = Rank of $\Phi_n(x)$

$\|C(\Phi_n(x))\|$ = Norm of $\Phi_n(x)$
$tr(C(\Phi_n(x)))$ = Trace of $\Phi_n(x)$
λ is the Eigen value of $\Phi_n(x)$
$Det(C(\Psi_n(x)))$ = Determinant of $\Psi_n(x)$
$\rho(C(\Psi_n(x)))$ = Rank of $\Psi_n(x)$
$\|C(\Psi_n(x))\|$ = Norm of $\Psi_n(x)$
$tr(C(\Psi_n(x)))$ = Trace of $\Psi_n(x)$
$A_n = max_m |a_n(k)|$
$A_+ = max_m(a_n(k))$
$A_- = min_m(a_n(k))$
$S_n = \sum_m |a_n(k)|$
$Q_n = \sum_m (a_n(k))^2$
$C_n = max_m |c_n(k)|$
$D_A = A_+ - A_-, \qquad D_C = C_+ - C_-$

Lemma 2.1. *(1)* $\Phi_{pn}(x) = \Phi_n(x^p)$ *if* p *divides n.*
(2) $\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x)$ *if* p *does not divide n.*
(3) $\Phi_n(x) = x^{\varphi(n)}\Phi_n(1/x)$ *for* $n > 1$.
(4) If $n \in Nodd$*, then* $\Phi_{2n}(x)) = \Phi_n(-x)$.

Theorem 2.1. *Let p be a prime and m be a positive integer. If p does not divide m, then* $\Phi_{pm}(x)\Phi_m(x) = \Phi_m(x^p)$.

Proof. Let $d|pm$ and p does not divide d. Since p is prime, $gcd(d,p) = 1$. By Euclid Lemma, $d|m$. Suppose that d divides m. It follows that $d|pm$ and d does not divide m iff $d|m$. Since $m|pm$, $d|pm$. Therefore $d|pm$ and d does not divide p iff $d|m$. Since p is prime and is not a divisor of m, $gcd(p,m) = 1$. If d is not a divisor of m, then $gcd(m/d,p) = 1$. Since μ is a multiplicative function, $\mu(mp/d) = \mu(m/d)\mu(p) = -\mu(m/d)$. Assume that $p|d$. Therefore $d = pn$ for some integer n, where $n|m$. Suppose that $n|m$, then $pn|pm$, and if we let $d = pn$, then $p|d$ and $d|pm$. Thus $d|pm$ and $p|d$ iff $n|m$, where $n = d/p$.

$$\Phi_{pm}(x)\Phi_m(x) = \prod_{d|pm}(x^d-1)^{\mu(pm/d)}\Phi_m(x)$$

$$= \prod_{d|pm, p|d} (x^d - 1)^{\mu(pm/d)} \prod_{d|pm, p\nmid d} (x^d - 1)^{\mu(pm/d)} \Phi_m(x)$$
$$= \prod_{n|m} (x^{pn} - 1)^{\mu(pm/pn)} \prod_{d|m} (x^d - 1)^{\mu(pm/d)} \Phi_m(x)$$
$$= \Phi_m(x^p) \prod_{d|m} (x^d - 1)^{-\mu(m/d)} \Phi_m(x)$$
$$= \Phi_m(x^p)(\Phi_m(x))^{-1} \Phi_m(x) = \Phi_m(x^p).$$

□

Properties 2.1.

(1) If p is a prime, then $\partial(\Phi_n(x))/\partial x = \sum_{k=1}^{p-1} kx^{k-1}$,

$$\partial^\alpha(\Phi_p(x))/\partial x^\alpha = \sum_{k=0}^{p-1} (k! x^{k-\alpha}/\Gamma(k-\alpha+1)) \& \int \Phi_n(x)dx = \sum_{k=0}^{p-1} (x^{k+1}/(k+1)).$$

(2) If an explicit equation for $\Phi_n(x)$ for square-free n,

$$\Phi_n(x) = \sum_{j=0}^{\varphi(n)} a_n(j) x^{\varphi(n)-j},$$

where $a_n(j) = -(\mu(n)/j) \sum_{m=0}^{j-1} a_n(m)\mu(gcd(n, j-m))\varphi(gcd(n, j-m))$, $a_n(0) = 1$, then

$$\partial \Phi_n(x)/\partial x = \sum_{j=0}^{\varphi(n)} a_n(j)(\varphi(n) - j) x^{\varphi - j - 1},$$

$$\int \Phi_n(x)dx \sum_{j=0}^{\varphi(n)} a_n(j)(x^{\varphi(n)-j+1}/\varphi(n) - j + 1)$$

and

$$\partial^\alpha(\Phi_n(x))/\partial x^\alpha = \sum_{j=0}^{\varphi(n)} (a_n(j)(\varphi(n) - j)! x^{\varphi - j - \alpha}/\Gamma(\varphi(n) - j - \alpha + 1)).$$

3. The Matrix Representation of $\Phi_n(x)$

Definition 3.1. *The companion matrix of the monic polynomial* $\Phi_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n$ *is the square matrix defined as*

$$C(\Phi_n(x)) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}.$$

Table 1.

Companion matrix	Eigenvalues	Norm	Rank	Trace
$C(\Phi_3(x))$	$(-1)^{2/3}$, $-(-1)^{1/3}$	$\sqrt{(3+\sqrt{5})/2}$	2	-1
$C(\Phi_4(x))$	$i, -i$	1	2	0
$C(\Phi_5(x))$	$(-1)^{4/5}$, $-(-1)^{2/5}$, $(-1)^{2/5}$, $-(-1)^{3/5}$	$\sqrt{(5+\sqrt{21})/2}$	4	-1

Properties 3.1. *Let* $1 \leq n \leq 30$*, the properties of Companion matrix of the nth cyclotomic polynomial are*

(1) $Det(C(\Phi_n(x))) = 1$.
(2) $C(\Phi_n(x)$ *is not a positive definite matrix.*
(3) $1 \leq \rho(C(\Phi_n(x)) \leq 28$.
(4) All the elements of the last column of the upper triangular matrix of $(C(\Phi_n(x)))$ *are -1 if n is prime and* $3 \leq n \leq 29$.
(5) $\|C(\Phi_n(x))\| = (a + b\sqrt{N})/2$, *where* $3 \leq a \leq 29, 1 \leq b \leq 5, 3 \leq N \leq 357$ *and* $n \neq 4, 8, 16$.
(6) $tr(C(\Phi_n(x))) = 0$ *or* 1.
(7) $\lambda = -(-1)^{1/n}, (-1)^{2/n}, \ldots (-1)^{-1/n}$*, n is prime and* $n \geq 3$.

4. Inverse cyclotomic polynomial

Definition 4.1. *(Moree [14]) The nth inverse cyclotomic polynomial* $\Psi_n(x)$ *are defined by*

$$\Psi_n(x) = \prod_{1 \leq k < n, (k,n) > 1} (x - e^{2i\pi k/n}),$$

$$\Psi_n(x) = (x^n - 1)/\Phi_n(x), deg(\Psi_n) = n - \varphi(n).$$

$\Psi_n(x)$ is the monic polynomial whose roots are the nth non-primitive roots of unity. For a square-free $n, deg(\Psi_n) = \omega(n)$, where $\omega(n) =$ number of prime factors of n. If p is prime then $\Psi_p(x) = -1 + x$, $\Psi_{pq}(x) = -1 - x - x^2 - \ldots - x^{p-1} + x^q + x^{q+1} + \ldots + x^{p+q-1}$.

The nth inverse cyclotomic polynomials are

$\Psi_1(x) = 1$, $\Psi_4(x) = -1 + x^2$, $\Psi_6(x) = -1 - x + x^3 + x^4$,
$\Psi_8(x) = -1 + x^4$, $\Psi_9(x) = -1 + x^3$, $\Psi_{10}(x) = -1 - x + x^5 + x^6$,
$\Psi_{12}(x) = -1 - x^2 + x^6 + x^8$, $\Psi_{14}(x) = -1 - x + x^7 + x^8$,
$\Psi_{15}(x) = -1 - x - x^2 + x^5 + x^6 + x^7$, $\Psi_{16}(x) = -1 + x^8$,
$\Psi_{18}(x) = -1 - x^3 + x^9 + x^{12}$, $\Psi_{20}(x) = -1 - x^2 + x^{10} + x^{12}$,
$\Psi_{21}(x) = -1 - x - x^2 + x^7 + x^8 + x^9$, $\Psi_{22}(x) = -1 - x + x^{11} + x^{12}$,
$\Psi_{24}(x) = -1 - x^4 + x^{12} + x^{16}$, $\Psi_{25}(x) = -1 + x^5$,
$\Psi_{26}(x) = -1 - x + x^{13} + x^{14}$, $\Psi_{28}(x) = -1 - x^2 + x^{14} + x^{16}$,
$\Psi_{30}(x) = -1 + x - x^2 - x^5 + x^6 - x^7 + x^{15} - x^{16} + x^{17} + x^{20} - x^{21} + x^{22}$

Table 2.

Companion matrix	Eigenvalues	Norm	Rank	Trace
$C(\Psi_4(x))$	-1, 1	1	2	0
$C(\Psi_6(x))$	$(-1)^{2/3}, -(-1)^{1/3}, -1, 1$	$\sqrt{2+\sqrt{3}}$	4	-1
$C(\Psi_8(x))$	$-1, i, -i, 1$	1	4	0

Properties 4.1. *Let* $1 \leq n \leq 30$*, the properties of Companion matrix of the nth inverse cyclotomic polynomial are*

(1) $Det(C(\Psi_n(x))) = 1$ *or* -1.
(2) $C(\Psi_n(x)$ *is not a positive definite matrix.*
(3) $1 \leq \rho(C(\Psi_n(x)) \leq 22$

(4)

$$\|C(\Phi_n(x))\| = \begin{cases} 1 & \text{if } n = 4,8,9,16,25,27 \\ \sqrt{2+\sqrt{3}} & \text{if n = 6,10,12,14,18,20,22,24,26,28} \\ \sqrt{3+2\sqrt{2}} & \text{if n = 15,21} \\ \sqrt{6+\sqrt{35}} & \text{if n = 30} \end{cases}$$

(5) $tr(C(\Psi_n(x))) = 0$ *or* 1 *or* -1

Theorem 4.1.

(1) $\Psi_{2n}(x) = (1-x^n)\Psi_n(-x)$ *if n is odd.*
(2) $\Psi_{pn}(x) = \Psi_n(x^p)$ *if* $p|n$.
(3) $\Psi_{pn}(x) = \Psi_n(x^p)\Phi_n(x)$ *if* p *does not divide* n.

5. Height and Length

Definition 5.1. *Let* $\Phi_n(x) = \sum_{k=1}^{\varphi(n)} a_n(k)x^k$ *be the nth cyclotomic polynomial.* A_n *is called the height of* Φ_n. S_n *is called the length of* Φ_n. *Assume that n is odd and square free. Then the order of* Φ_n *is the number* $\omega(n)$ *of distinct odd prime divisors of n. Refer Bzdega [5]*

Definition 5.2. *Let* $\Psi_n(x) = \sum_{k=1}^{n-\varphi(n)} c_n(k)x^k$ *be the nth inverse cyclotomic polynomial.* C_n *is called the height of the inverse cyclotomic polynomial.* C_+ *and* C_- *are maximum and minimum of the elements of inverse cyclotomic polynomial. If* Φ_n *is the nth cyclotomic polynomial of order 2, then* Φ_n *is called the binary cyclotomic polynomial.To find cyclotomic polynomials with large heights, we required bounds on* A_n.

Definition 5.3. *If* $n = pq$ *with* $p < q$ *primes, then* Φ_n *is a binary cyclotomic polynomial. It is denoted by* Φ_{pq}. *If* $n = pqr$ *with* $2 < p < q < r$ *primes, then* Φ_n *is a ternary cyclotomic polynomial. It is denoted by* Φ_{pqr}. *If* $n = pqrs$ *with* $2 < p < q < r < s$ *primes, then* Φ_n *is a quaternary cyclotomic polynomial. It is denoted by* Φ_{pqrs}. *If* $n = pqrst$ *with* $2 < p < q < r < s < t$ *primes, then* Φ_n *is a quinary cyclotomic polynomial. It is denoted by* Φ_{pqrst}.

Definition 5.4. *The polynomials of the form*

$$P_n(x) = (1 - x^n) \prod_{1\le i<j\le k} (1 - x^{n/p_ip_j}) / \prod_{i=1}^{k} (1 - x^{n/p_i}),$$

where $n = p_1p_2 \dots p_k$, is called the relatives of the cyclotomic polynomial.

Definition 5.5. *For $n > 1$, the coefficients of $\Phi_n(x)$ are palindromic, if $a_n(k) = a_n(\varphi(n) - k)$. For $n > 1$, the coefficients of $\Psi_n(x)$ are anti - palindromic, if $c_k(k) = -c_n(n - \varphi(n) - k)$. For an integer i, $\rho(i) = (-1)^i$ is the parity.*

Properties 5.1.

(1) *A_n is not bounded above by any polynomial in n.*
(2) *For a cyclotomic polynomial of order 1, $A_p = 1$ and $S_p = Q_p = p$.*
(3) *For $\Phi_{pq}(x), A_{pq} = 1, S_{pq} = Q_{pq} = 2p^*q^* - 1 < pq/2$, where $p^* \in 1, \dots, q-1$ is the inverse of p modulo q and $q^* \in 1, \dots, p-1$ is the inverse of q modulo p.*
(4) *For ternary cyclotomic polynomials, $A_{pqr} \le 2p/3, S_{pqr} \le 4p^2qr/9$.*
(5) *$n = 1181895$ is the smallest n such that $A_n > n$.*
(6) *For quaternary cyclotomic polynomial $A_{pqrs} < p(p-1)(pq-1)$.*
(7) *For quinary cyclotomic polynomial $A_{pqrst} < p^7q^3r$.*
(8) *For cyclotomic polynomial of order 6, $A_{pqrstu} < p^{15}q^7r^3s$.*
(9) *The coefficients of binary cyclotomic polynomials are 1, -1, 0.*
(10) *$C_p = 1, C_{pq} = 1, C_{pqr} \le p - 1$. Refer Bzdega [7]*
(11) *A ternary cyclotomic $\Phi_{pqr}(x)$ has at most $p + 1$ distinct coefficients.*
(12) *$deg(\Psi_{pqr}) = qr + rp + pq - p - q - r + 1$.*
(13) *The smallest values of n such that $\Phi_n(x)$ has one or more coefficients $\pm1, \pm2, \pm3, \dots$ are 0, 105, 385, 1365, 1785, 2805, 3135, 6545, 6545, 10465, 10465, 10465, 10465,10465, 11305, ...*
(14) *Let $M(p)$ be the maximum of the height of the ternary cyclotomic polynomial, where p is the smallest prime factor of n. $M(3) = 2$. For $p > 2, M(p) = (p + 1)/2$.*

TABLE 3.

n		Φ_n			Ψ_n		Φ_n			Ψ_n
		A_n	S_n	Q_n	C_n	n	A_n	S_n	Q_n	C_n
$n = p$	2	1	2	2	1	5	1	5	5	1
	3	1	3	3	1	7	1	7	7	1
$n = pq$	6=2.3	1	3	3	1	15=3.5	1	7	7	1
	10=2.5	1	5	5	1	21=3.7	1	9	9	1
	14=2.7	1	7	7	1	22=2.11	1	11	11	1
$n = pqr$	105=3.5.7	2	35	39	1	1001=7.11.13	2	326	372	2
	385=5.7.11	3	219	309	1	2431=11.13.17	4	1822	3006	2
$n = pqrs$	660=3.5.7.11	2	67	87	1	5005=5.7.11.13	5	2493	4144	7

6. FLAT CYCLOTOMIC POLYNOMIAL

Definition 6.1. *A cyclotomic polynomial $\Phi_n(z)$ is said to be flat if $A_n = 1$. We call that $\Phi_n(z)$ is flatter than $\Phi_m(z)$ if $A_m < A_n$. Flat cyclotomic polynomial introduced in Bachman [3]. No one yet to found a flat cyclotomic polynomial of order 5.*

Let p, q, r are the consecutive odd prime divisors

TABLE 4.

n		Φ_n				Ψ_n			
		A_+	A_-	D_A	$\varphi(n)$	C_+	C_-	D_C	$n - \varphi(n)$
$n = pqr$	105	1	-2	**3**	48	1	-1	2	57
	385	2	-3	**5**	240	1	-1	2	145
	1001	1	-2	3	720	2	-2	4	281
	2431	3	-4	7	1920	2	-2	4	511
	4199	4	-3	7	3456	3	-3	6	743
	7429	3	-3	6	6336	3	-3	6	1090
$n = pqrs$	660	2	-2	4	160	1	-1	2	500
	5005	5	-5	10	2880	7	-7	14	2125
	17017	10	-10	20	11520	10	-10	20	5497
$n = pqrst$	15015	23	-22	45	5760	11	-11	22	9255

Properties 6.1.

(1) *All cyclotomic polynomials of order 1 and 2 are flat.*

(2) *The first ternary cyclotomic polynomial $\Phi_{105}(z)$ is not flat, because $a_{105}(7) = -2$.*

(3) *If Φ_n is not flat, then n has at least three distinct odd prime factors.*

(4) *If p is a prime, Φ_p then is trivially flat. $A_{2qr} = 1$.*

(5) $A_{2qr} = 1$ *if* $r = \pm(\mod pq)$. *Refer Kaplan [13]*
(6) *If* $\Psi_n(x)$ *is of order* ≤ 2, *then* $\Psi_n(x)$ *is flat.*
(7) *Let* r *be any prime, then* $\Psi_{15r}(x)$ *and* $\Psi_{21r}(x)$ *are flat.*

Theorem 6.1. *The smallest* n *for which* $\Psi_n(x)$ *is non-flat is* $n = 561$.

Proof. By calculation $c_{561}(17) = -2$. By property 6, $\Psi_n(x)$ is flat for every odd square-free $n \leq 560$,

$$A = \{105, 165, 195, 231, 255, 273, 285, 345, 357, 399, 435, 465, 483, 555\}$$

and $B = \{385, 429, 455\}$, where A has all its elements divisible by 15 or 21. $\Psi_n(x)$ is flat for every $n \in A$. By direct calculation $\Psi_{385}(x)$, $\Psi_{429}(x)$, $\Psi_{455}(x)$ are flat. □

7. Coefficient optimal

Assume

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{deg(f)} c_k x^k,$$

$C(f) = \{c_k : 0 \leq k \leq deg(f)\}$ and $C_0(f) = \{c_k : k \geq 0\}$. Therefore $C_0(f) = C(f) \cup \{0\}$. Assume $C(n) = C(\Phi_n)$ and $C_0(n) = C_0(\Phi_n)$. Refer Gallot [10]

Definition 7.1. *If* $C_0(n) \subseteq \{-1, 0, 1\}$, *then* Φ_n *is said to be flat. If* $C_0(n)$ *consists of a range of consecutive integers, then* Φ_n *is coefficient convex. If* $C_0(n) = I_n \cap Z$ *for some interval* I_n *in the reals, then* Φ_n *is said to be coefficient convex. If* $C(n) = I_n \cap Z$ *for some interval* I_n *in the reals, then* Φ_n *is said to be strongly coefficient convex.*

Definition 7.2. *If the difference between the largest and the smallest coefficient is exactly* p, *then* $\Phi pq, \Phi pqr, \Phi pqrs, \Phi pqrst$ *are called coefficient optimal. Similarly we can define for* Ψ_n .

Properties 7.1.

(1) *Ternary* Φ_n *is coefficient convex.*
(2) *If* Φ_n *is flat, then* Φ_n *is coefficient convex.*
(3) *The difference between the largest and the smallest coefficients of* Φ_{pqr} *is at most* p.
(4) *When* p *is an odd prime,* Φ_{2p} *is coefficient convex but not strongly coefficient convex.*

(5) *Suppose that n has at most 3 distinct odd prime factors, then Φ_n is coefficient convex.*

(6) *Suppose that n has at most 3 distinct odd prime factors, then Ψ_n is coefficient convex.*

(7) *If n has four or more distinct odd prime factors, then Ψ_n need not be coefficient convex.*

(8) *Bachman found two infinite families of coefficient optimal ternary polynomials Φ_{pqr}, with $C(pqr) = [-((p-1)/2), ((p+1)/2)]$ for one family and $C(pqr) = [-((p+1)/2), ((p-1)/2)]$ for the other family.*

(9) *If n is ternary, then ϕ_{2n} does not have the jump one property.*

(10) *Let $p < q < r$ be odd primes. If $a, b \in C(pqr)$ and $b - a = p$, then $C(pqr) = -a, a+1, \ldots, b-1, b$*

Theorem 7.1. *If the ternary polynomial Φ_n, is not flat, then Φ_n does not have the jump one property.*

Proof. Assume that $a_k = m$ and $|m| > 1$. Then by property and Lemma 2.1(d),

$$|a_{2n}(k) - a_{2n}(k-1)| = |a_n(k) - a_n(k-1)| \geq 2\,|m| - 1 > 1.$$

□

TABLE 5.

n	Φ_n				Ψ_n	
	$C(n)$	$C_0(n)$	Coefficient optimal	Strongly Coefficient convex	$C(\Psi_n)$ and $C_0(\Psi_n)$	Coefficient optimal
6	$\{-1, 1\}$	$\{-1, 0, 1\}$	Yes	No	$\{-1, 0, 1\}$	No
10	$\{-1, 1\}$	$\{-1, 0, 1\}$	Yes	No	$\{-1, 0, 1\}$	No
14	$\{-1, 1\}$	$\{-1, 0, 1\}$	Yes	No	$\{-1, 0, 1\}$	No
15	$\{-1, 0, 1\}$		No	Yes	$\{-1, 0, 1\}$	No
21	$\{-1, 0, 1\}$		No	Yes	$\{-1, 0, 1\}$	No
105	$\{-2, -1, 0, 1\}$		Yes	Yes	$\{-1, 0, 1\}$	No
385	$\{-3, -2, -1, 0, 1, 2\}$		Yes	Yes	$\{-1, 0, 1\}$	No
1001	$\{-2, -1, 0, 1\}$		No	Yes	$\{-2, -1, 0, 1, 2\}$	No
2431	$\{-4, -3, -2, -1, 0, 1, 2, 3\}$		No	Yes	$\{-2, -1, 0, 1, 2\}$	No
4199	$\{-3, -2, -1, 0, 1, 2, 3, 4\}$		No	Yes	$\{-3, -2, -1, 0, 1, 2, 3\}$	No

8. Maximum gap

Definition 8.1. *Hong [12] Let* $f = a_0x^{e_1} + \ldots + a_rx^{e_r}$*, where* $a_0, \ldots, a_r \neq 0$ *and* $e_1 < \ldots < e_r$*. Then the maximum gap of* f *is defined by* $g(f) = max_{1\leq i\leq t}(e_{i+1} - e_i)$*, where* $g(f) = 0$ *when* $t = 1$*. The maximum gap,* $g(f)$ *which is the maximum of the differences(gaps) between two consecutive exponents occurring in* f*, where* $f = \Phi_n$ *or* $f = \Psi_n$*.* $g(\Phi_1) = 1, g(\Psi_1) = 0$*.*

If n is a prime, $\Phi_p(x) = 1 + x + ... + x^{p-1}, \Psi_p(x) = -1 + x.g(\Phi_p) = 1, g(\Psi_p) = 1,$ $g(\Phi_{pq}) = p - 1, g(\Psi_{pq}) = q - (p - 1), g(\Psi_{pqr}) = 2qr - \varphi(pqr).$

Table 6.

n		$g(\Phi_n)$	$g(\Psi_n)$	n	$g(\Phi_n)$	$g(\Psi_n)$
$n = pqr$	105	3	13	4199	173	12
	385	13	9	7429	265	16
	1001	47	7	12673	367	18
	2431	103	10	20677	585	22
$n = pqrs$	660	8	160	17017	6	30
	5005	8	3	46189	36	-
$n = pqrst$	15015	3	5	85085	3	-

9. Jump

Definition 9.1. *We say that* $f \in Z[x]$ *has the jump one property if neighbouring coefficients differ by at most one. Refer Bzdega [6]*

Definition 9.2. *The number of jumps of the cyclotomic coefficients defines as the* $J_n(\Phi_n) = \sum_k |a_n(k) - a_n(k-1)|$ *and The number of jumps of the inverse cyclotomic coefficients defines as the* $J_n(\Psi_n) = \sum_k |c_n(k) - c_n(k-1)|$*.* $J_n(\Phi_n) > n^{1/3}$ *for any ternary cyclotomic polynomial* Φ_n*.*

Theorem 9.1. *Camburu [9] For infinitely many* $n = pqr$ *with pairwise disjoint odd primes* p, q *and* r*,* $J_n << n^{(7/8)+O(1)}$*.*

TABLE 7.

n		$J_n(\Phi_n)$	$J_n(\Psi_n)$	n	$J_n(\Phi_n)$	$J_n(\Psi_n)$
$n = pq$	6	4	2	14	12	2
	10	8	2	15	12	2
$n = pqr$	105	16	40	1001	116	278
	385	80	90	2431	413	445
$n = pqrs$	660	114	66	5005	2988	-

TABLE 8.

f	$\int(1/f(x))dx$	$\int_{-\infty}^{\infty}(1/f(x))dx$
Φ_3	$(2tan^{-1}[(1+2x)/\sqrt{3}])/\sqrt{3}$	3.6276
Φ_4	$tan^{-1}x$	3.1416
Φ_5	$\sum_\omega log(x-\omega)/(4\omega^3+3\omega^2+2\omega+1)$, where $1+\omega+\omega^2+\omega^3+\omega^4=0$	2.39027
Φ_6	$(2tan^{-1}[(-1+2x)/\sqrt{3}])/\sqrt{3}$	3.6272
Φ_7	$\sum_\omega log(x-\omega)/(6\omega^5+5\omega^4+4\omega^3+3\omega^2+2\omega+1)$, where $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$	2.18245
Φ_8	$(-2tan^{-1}(1-\sqrt{2}x)+2tan^{-1}(1+\sqrt{2}x)-log(1-\sqrt{2}x+x^2)+log(1+\sqrt{2}x+x^2))/4\sqrt{2}$	2.2214

10. CYCLOTOMIC POLYNOMIAL $\Phi_n(x)$

Example 1. *For* $p=2, n=3, p$ *does not divide* n *then* $\Phi_6(x) = 1-x+x^2, \Phi_3(x^2) = 1+x^2+x^4, \Phi_3(x) = 1+x+x^2$, *therefore* $\Phi_6(x) = \Phi_3(x^2)/\Phi_3(x)$. *Use Lemma 2.1*

The eigen values of Φ_{29} in the complex plane is shown in figure 1.The graph of the values of the cyclotomic polynomials $\Phi_n(x)$, where $1 \le n \le 30, -1 \le x \le 1$ is shown in figure 2. The degree and number of terms of $\Phi_n(x)$, $1 \le n \le 1000$ are shown in figures 3 and 4.

The ternary inverse cyclotomic polynomial $\Psi_{385}(x)$ is $-1+x-x^5+x^6-x^7+x^8-x^{10}+x^{13}-x^{14}-x^{21}+x^{23}-x^{25}-x^{32}+x^{34}-x^{36}-x^{43}+x^{45}-x^{47}-x^{54}+x^{55}-x^{58}+x^{60}-x^{61}+x^{62}-x^{63}+x^{67}-x^{68}+x^{77}-x^{78}+x^{82}-x^{83}+x^{84}-x^{85}+x^{87}-x^{90}+x^{91}+x^{98}-x^{100}+x^{102}+x^{109}-x^{111}+x^{113}+x^{120}-x^{122}+x^{124}+x^{131}-x^{132}+x^{135}-x^{137}+x^{138}-x^{139}+x^{140}-x^{144}+x^{145}$

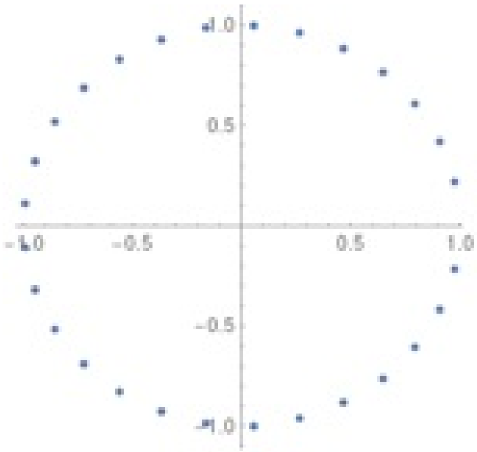

FIGURE 1.

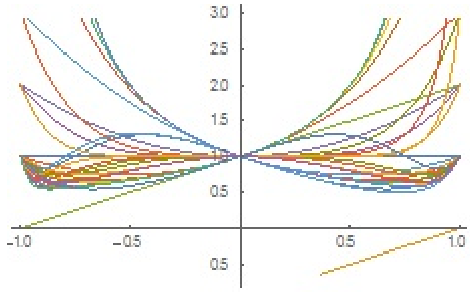

FIGURE 2.

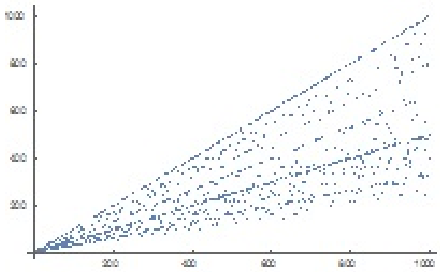

FIGURE 3.

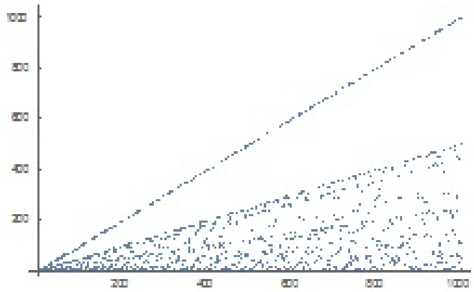

FIGURE 4.

TABLE 9.

k	The values of $\Phi_n(k), 1 \leq n \leq 30$
$k = 1$	0, 2, 3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19, 1, 1, 1, 23, 1, 5, 1, 3, 1, 29, 1
$k = 2$	1, 3, 7, 5, 31, 3, 127, 17, 73, 11, 2047, 13, 8191, 43, 151, 257, 131071, 57, 524287, 205, 2359, 683, 8388607, 241, 1082401, 2731, 262657, 3277, 536870911, 331
$k = 3$	2, 4, 13, 10, 121, 7, 1093, 82, 757, 61, 88573, 73, 797161, 547, 4561, 6562, 4570081, 703, 581130733, 5905, 368089, 44287, 47071589413, 6481, 501192601, 398581, 387440173, 478297, 34315188682441, 8401

11. SELF -RECIPROCAL POLYNOMIAL

Definition 11.1. *A polynomial f of degree d is said to be self-reciprocal if $f(x) = x^d f(1/x)$. If $f(x) = -x^d f(1/x)$, then is said to be anti-self reciprocal.*

By lemma 2.1, Φ_n is self reciprocal for $n \geq 2$. Φ_1 is anti-self-reciprocal. Ψ_n is anti- self reciprocal for $n \geq 2$. Ψ_1 is self-reciprocal. Let $a(n)$ be the number of polynomials f of degree n with $p(0)$ non zero. Let $b(n)$ be the number of such polynomials which are additionally self-reciprocal. Let $c(n)$ be the number of those which are self-reciprocal and where $p(-1)$ is the square of an integer.

TABLE 10.

n	1	2	3	4	5	6	7	8	9	10	11	12
$a(n)$	2	6	10	24	38	78	118	224	330	584	838	1420
$b(n)$	1	5	5	19	19	59	59	165	165	419	419	1001
$c(n)$	1	3	5	12	19	34	59	99	165	244	419	598

Theorem 11.1. *Let $f \in R[x]$ be a self-reciprocal polynomial. Then for $|z| = 1$, $f(z) = \pm |f(z)| z^{degf/2}$, Let $f \in R[x]$ be an anti- self-reciprocal polynomial, then for $|z| = 1$, $f(z) = \pm i |f(z)| z^{degf/2}$.*

Proof. Let $d = degf$. If f is self reciprocal and $|z| = 1, f(z) = z^d f(1/z) = z^d \overline{f(z)}$, $\Rightarrow f(z)f(z) = z^d \overline{f(z)} f(z) \Rightarrow f(z)^2 = z^d |f(z)|^2 \Rightarrow f(z) = \pm |f(z)| z^{degf/2}$.

If f is self reciprocal and $|z| = 1, f(z) = -z^d f(1/z) = -z^d \overline{f(z)} \Rightarrow f(z)f(z) = -z^d \overline{f(z)} f(z) \Rightarrow f(z)^2 = -z^d |f(z)|^2 \Rightarrow f(z) = \pm i |f(z)| z^{degf/2}$ □

Proposition 11.1. *Let f be a polynomial of degree $d \geq 1$. Suppose that f is self-reciprocal.*

(1) $f'(1) = df(1)/2$.
(2) If 2 does not divide d, then $f(-1) = 0$. If 2 divides d, then $f'(-1) = -f(-1)d/2$.

Suppose that is anti-self-reciprocal

(1) $f(1) = 0$.
(2) If 2 does not divide d, then $f'(-1) = -f(-1)d/2$. If 2 divides d, then $f(-1) = 0$.

Proof. Suppose that f is self reciprocal.

(1) $f(z) = z^d f(1/z) \Rightarrow f'(z) = -z^{d-2}f'(1/z) + f(z)dz^{d-1} \Rightarrow f'(1) = df(1)/2$.

(2) If $d = 2a$, then $d-1$ is an odd, $f'(-1) = -(-1)^d f'(-1) + f(-1)d(-1)^{d-1} \Rightarrow f'(-1) = -df(-1)/2$. If $d \neq 2a$, then $d-1$ is even, $f'(-1) = f'(-1) + f(-1)d \Rightarrow f(-1) = 0$.

Suppose that f is anti-self reciprocal.

(1) $f(z) = -z^d f(1/z) \Rightarrow f'(z) = -z^{d-2}f'(1/z) - f(z)dz^{d-1} \Rightarrow f(1) = 0$.

(2) If $d = 2a$ is even, then $d-1$ (odd), $f'(-1) = (-1)^d f'(-1) - f(-1)d(-1)^{d-1} \Rightarrow f(-1) = 0$. If $d \neq 2a$ is odd, then $d-1$ is even, $f'(-1) = -f'(-1) - f(-1)d \Rightarrow f'(-1) = -df(-1)/2$.

□

Definition 11.2. *A integral self reciprocal polynomial $p(t) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$ is of Littlewood type if every coefficient non-zero p_i has modulus 1. A polynomial $p(t)$ of Littlewood type with all $p_i \neq 0$, for $i = 0, 1, 2, \ldots, n$ is said to be Littlewood.*

12. Cyclotomic field

Definition 12.1. *A field is said to be cyclotomic if it is of the form $Q[x]/(\Phi_m(x))$ for some $m \geq 1$. Cyclotomic field is isomorphic to $Q(\zeta_m)$. $[Q(\zeta_m) : Q] = \varphi(m)$. The ring of algebraic integers of cyclotomic field is $Z[\zeta_m] = \{a_0 + a_1\zeta_m + \cdots + a_{m-1}\zeta_m^{m-1}, a_0, a_1, \ldots a_{m-1} \in Z\}$. $1, \zeta_m, \ldots, \zeta_m^{\varphi(m)-1}$ are a basis for $Q(\zeta_m)$ over Q.*

Definition 12.2. *A prime p is called regular if the class number of $Z[\zeta_m]$ is not divisible by p otherwise p is irregular. The irregular primes are 37, 59, 67, 101, 103, etc.*

Properties 12.1.

(1) The elements of $Z[\zeta_m]$ are cyclotomic integers.

(2) The elements of $Q[\zeta_m]$ are cyclotomic numbers.

(3) The unit in the cyclotomic field is cyclotomic unit.

(4) When $m = 3$, the cyclotomic field is a quadratic field.

(5) Every abelian extension of Q is contained in a cyclotomic field.

(6) The discriminant of $Z[\zeta_m]$ is $(-1)^{(p-1)/2}p^{p-2}$.

13. APPLICATIONS

(1) A Kronecker polynomial is a monic polynomial with integer coefficients having all of its roots on or inside the unit disc. Kronecker relates the his polynomials with cyclotomic polynomial.
(2) Cyclotomic polynomial are used to prove the Gauss-Wanzel theorem, Dirichlet theorem and Wedderburn theorem.
(3) The applications of self-reciprocal polynomial in coding theory and reduction theory. It is the study of the relationship between the self reciprocal polynomials of the form $J_{n,H}(x)$ and the subgroup H.
(4) The application of cyclotomic polynomials that we will explore is when a regular n-gon is constructible with a straightedge and compass.
(5) The application to the special case Dirichlet's theorem on primes in arithmetic progression.

14. CONCLUSION

Hence conclude that, we have found the companion matrix, flat, jump, gap of the cyclotomic and inverse cyclotomic polynomial for the binary, ternary, quaternary and quinary polynomial. Some properties for these polynomials are found. Few applications of cyclotomic polynomial are determined and to develop the further research on finding flat, jump, gap for the modified and unitary cyclotomic polynomial.

REFERENCES

[1] D.ANDRICA, O.BAGDASAR: *On cyclotomic polynomial coefficients*, Malasian Journal of Mathematical Sciences **14(3)**(2020), 389–402.
[2] ANDREW ARNOLD, MICHAEL MONAGAN: *Calculating cyclotomic polynomials*, Mathematics of Computation **80(276)**(2011), 2359–2379.
[3] G.BACHMAN : *Flat cyclotomic polynomials of order three*, Bull. London Math. Soc. **38** (2006), 53–60.
[4] BARTLOMIEJ BZDEGA, ANDRES HERRERA-POYATOS AND PIETER MOREE: *Cyclotomic polynomials at roots of unity*, Acta Arithmetica **184** (2018), 215– 230.
[5] B. BZDEGA: *Products of cyclotomic polynomials on unit circle*, International Journal of number theory **13**(10) (2017), 2515–2530.
[6] B. BZDEGA: *Jumps of ternary cylotomic coefficients*, Acta Arithmetica **163** (2014), 203–213.

[7] B. BZDEGA: *On the height of cyclotomic polynomials*, Acta Arithmetica **152** (2012), 349–359.

[8] B. ZHANG: *The Height of a class of ternary cyclotomic polynomials*, Bull. Korean Math. Soc. **54(1)** (2017), 43–50.

[9] O. CAMBURU, E. CIOLAN, P. MOREE, F. LUCA, E. SHAPARLINSKI: *Cyclotomic coefficients: gaps and jumps*, Journal of number theory **163** (2016), 211–237.

[10] Y. GALLOT, P. MOREE: *Neighbouring ternary cyclotomic coefficients differ by at most one*, Journal of mathematical Society **24** (2009), 235–248.

[11] G. BROOKFIELD: *The coefficients of cyclotomic polynomials* **89(3)** (2016), 179–188.

[12] H. HONG, E. LEE, H.-S.LEE, C.-M.PARK: *Maximum gap in(inverse) cyclotomic polynomial*,Journal of Number theory **132** (2012), 2297–2315.

[13] N. KAPLAN: *Flat cyclotomic polynomials of order three*, Journal of Number theory **127** (2007), 118–126.

[14] P. MOREE: *Inverse cyclotomic polynomials*, Journal of Number theory **129** (2009), 667–680.

[15] R. THANGADURAI: *On the coefficients of cyclotomic polynomials*,Cyclotomic fields and related topics, Bhaskaracharya Pratishthana, Pune (2000), 311–322.

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