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SUFFICIENT CONDITION OVER THE CERTAIN DIFFERENTIAL SUBORDINATIONS

Skender Avdiji[1] and Nikola Tuneski

ABSTRACT. Let f be analytic in the unit disk and normalized by $f(0) = f'(0) - 1 = 0$. In this paper using a method from the theory of first order differential subordination we investigate the sufficient conditions over the differential subordination

$$\frac{zp'(z)}{p(z)^a} \prec \frac{(A-B)z\left(\frac{1+Bz}{1+Az}\right)^a}{(1+Bz)^2}$$

that implies $p(z) \prec \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, and further use it for obtaining inequalities over the function f.

1. INTRODUCTION AND PRELIMINARIES

Analytic function f defined in the domain D is univalent if it is injective. Let $\mathcal{A}$ denotes the class of functions f that are analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, i.e., such that $f(z) = z + a_2 z^2 + \cdots$.

A function $f \in \mathcal{A}$ is said to be *starlike* if, and only if

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad (z \in \mathbb{D}).$$

[1]*corresponding author*

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We denote by $\mathcal{S}^*$ the class of all such functions which are at the same time univalent. Their geometrical characterisations is the following: f is starlike if, and only if, $t\omega \in f(\mathbb{D})$ for all $\omega \in f(\mathbb{D})$ and all $t \in [0,1]$, i.e., for all $z \in \mathbb{D}$, $f(z)$ is visible from the origin. For details see [1,8].

A special subclass of $\mathcal{S}^*$ is the class of *starlike function of order* α with $0 \le \alpha < 1$, given by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left[\frac{zf'(z)}{f(z)} > \alpha \right],\, z \in \mathbb{D} \right\}.$$

Further, a function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if, and only if, $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$. For details see [2].

Using subordination, another generalisation is defined by

$$\mathcal{S}^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\, z \in \mathbb{D} \right\},$$

$-1 \le B < A \le 1$. Geometrically, this means that the image of $\mathbb{D}$ by $zf'(z)/f(z)$ is inside the open disk centered on the real axis with diameter endpoints $(1-A)/(1-B)$ and $(1+A)/(1+B)$. In [5] it is given that special selections of A and B lead us to the following:

- $\mathcal{S}^*[1,-1] \equiv \mathcal{S}^*$;
- $\mathcal{S}^*[1-2\alpha,-1] \equiv \mathcal{S}^*(\alpha)$, $0 \le \alpha < 1$.

Next, we denote by $\mathcal{K}$ the class of *convex functions*, i.e., the class of function $f(z) \in \mathcal{A}$ for which

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)} \right] > 0, \quad (z \in \mathbb{D}).$$

and its generalization, the class of *convex functions of order* α, with $0 \le \alpha < 1$, given by

$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)} \right] > \alpha,\, z \in \mathbb{D} \right\}.$$

Both these classes ($\mathcal{S}^*$ and $\mathcal{K}$) are subclasses of univalent function in $\mathbb{D}$ and even more $\mathcal{K} \subset \mathcal{S}^*$. For details see [1,8].

In this paper we study the differential subordination of the form

$$\frac{zp'(z)}{p^a(z)} \prec \frac{(A-B)z\left(\frac{1+Bz}{1+Az}\right)^a}{(1+Bz)^2}$$

and conditions when it implies the subordination $p(z) \prec (1+Az)/(1+Bz)$, where $p(z)$ is analytic function on $\mathbb{D}$ and $p(0)=1$.

For that purpose we will use a method from the theory of first order differential subordinations. If $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is analytic in the domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then we say that $p(z)$ satisfies the *(first-order) differential subordination*

$$\psi\left(p(z), zp'(z)\right) \prec h(z). \tag{1}$$

The function $p(z)$ is called the *solution of differential subordination* (1). The univalent function $q(z)$ is called *dominant* of the solution of differential equation (1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1). The dominant $\widetilde{q}(x)$ that satisfies $\widetilde{q}(x) \prec q(z)$ for all dominants $q(z)$ of (1) is said to be *the best dominant* of (1).

From this theory we will make use of the following lemma due to Miller and Mocanu [2].

Lemma 1. *Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that:*

(i) *Q is starlike in the unit disk $\mathbb{D}$,*

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right] > 0$, $z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{2}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2).

2. Main results and consequences

Firs we will prove a lemma that will later lead to the main result.

Lemma 2. *Let $p(z)$ be analytic in the unit disk $\mathbb{D}$, $p(0) = 1$, $0 \notin p(\mathbb{D})$. Also, let A, B and a be real number with $-1 \leq B < A \leq 1$ and*

$$\begin{cases} a \geq 0 & \textit{if} \quad |A| < |B|, \\ a \leq \frac{(1+|A|)(1-|B|)}{|A|-|B|} & \textit{if} \quad |A| > |B|, \\ a \in \mathbb{R} & \textit{if} \quad |A| = |B|. \end{cases} \tag{3}$$

If

$$\frac{zp'(z)}{p(z)^a} \prec \frac{(A-B)z\left(\frac{1+Bz}{1+Az}\right)^a}{(1+Bz)^2}, \tag{4}$$

then $p(z) \prec q(z) = \frac{1+Az}{1+Bz}$*, and* $q(z)$ *is the best dominant of* (4).

Proof. In Lemma 1 we choose $\theta(\omega) = 0$ and $\phi(\omega) = \frac{1}{\omega^a}$. Then $q(z) = \frac{1+Az}{1+Bz}$ is univalent in $\mathbb{D}$ and $\phi(\omega)$ and $\theta(\omega)$ are analytic in domain $D = \mathbb{C} \setminus \{0\}$ containing $q(z) = \frac{1+Az}{1+Bz}$ with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Further, set

$$Q(z) = zq'(z)\phi(q(z)) = \frac{(A-B)z\left(\frac{1+Bz}{1+Az}\right)^a}{(1+Bz)^2},$$

which is starlike because

$$\frac{zQ'(z)}{Q(z)} = -1 + \frac{a}{1+Az} + \frac{2-a}{1+Bz}$$

and for all $z = e^{it}$, $t \in [0, 2\pi]$,

$$\mathrm{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} = \frac{1}{2}\left(\frac{a(1-A^2)}{|1+Ae^{it}|^2} + \frac{(a-2)(1-B^2)}{|1+Be^{it}|^2}\right) \geq 0.$$

The last inequality holds because it is equivalent to

$$a(|B| - |A|) \geq -(1+|A|)(1-|B|),$$

having in mind the condition over $|A|, |B|$ and a.

Also

$$h(z) = \theta(q(z)) + Q(z) = 0 + Q(z) = Q(z),$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{zQ'(z)}{Q(z)} = -1 + \frac{a}{1+Az} + \frac{2-a}{1+Bz}.$$

For $z = e^{it}$, $t \in [0, 2\pi]$ we have

$$\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} \geq 0$$

So, from $p(0) = q(0) = 1$ and from (2) we receive that $p(z) \prec q(z)$ and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant od (4). □

Theorem 1. *Let $f \in \mathcal{A}$, and let A, B and a be real numbers such that $-1 \leq B < A \leq 1$. If a, A and B satisfy* (3), *and*

$$\left(\frac{zf'(z)}{f(z)}\right)^{-a}\left[\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}\right)+\frac{z^2f''(z)}{f(z)}\right] \prec \frac{(A-B)z\left(\frac{1+Bz}{1+Az}\right)^a}{(1+Bz)^2} \equiv h(z), \tag{5}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \tag{6}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (5).

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{1+Az}{1+Bz}$. Then,

$$\frac{zp'(z)}{p(z)^a} = \left(\frac{zf'(z)}{f(z)}\right)^{-a}\left[\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}\right)+\frac{z^2f''(z)}{f(z)}\right],$$

and the rest follows from Lema 1. □

Corollary 1. *Let $f \in \mathcal{A}$, $-1 \leq B < A \leq 1$ and a be real numbers such that* (3) *holds. If*

$$\left|\frac{zf'(z)}{f(z)}\right|^{-a} \cdot \left|\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}\right)+\frac{z^2f''(z)}{f(z)}\right| < b$$

for all $z \in \mathbb{D}$, where

$$b \equiv (A-B)\begin{cases} \frac{(1-|A|)^{-a}}{(1+|B|)^{2-a}}, & a \leq 0, \\ \frac{1}{(1+|A|)^a(1+|B|)^{2-a}}, & 0 < a \leq 2, \\ \frac{(1-|B|)^{a-2}}{(1+|A|)^a}, & a > 2, \end{cases} \tag{7}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{A-B}{1-|B|}, \quad (z \in \mathbb{D}). \tag{8}$$

Proof. First let note that by the definition of subordination (6) implies (8). Indeed,

$$\frac{zf'(z)}{f(z)} - 1 \prec \frac{(A-B)z}{1+Bz}$$

implies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{A-B}{1-|B|}, \quad (z \in \mathbb{D}).$$

So, in order to complete the proof it is enough to show that inequality (7) implies subordination (5). Again, by the definition of subordination, we need to show that

$$\min_{|z|=1} |h(z)| = b, \tag{9}$$

where $|h(z)| = (A-B) \cdot |z| \cdot |1+Az|^{-a} \cdot |1+Bz|^{a-2}$.

Expression (9) holds since $|z|=1$ and $-1 \le B < A \le 1$. □

In a similar way as in Theorem 1, if we choose $p(z) = \frac{f(z)}{z}$ and $p(x) = f'(z)$ we receive the following two results.

Theorem 2. *Let $f \in \mathcal{A}$, and let A, B and a be real numbers such that $-1 \le B < A \le 1$ and* (3) *holds. If*

$$\left[\frac{f(z)}{z}\right]^{-a} \cdot \left[f'(z) - \frac{f(z)}{z}\right] \prec h(z),$$

then $\frac{f(z)}{z} \prec \frac{1+Az}{1+Bz}$, with $\frac{1+Az)}{1+Bz}$ being best dominant.

Theorem 3. *Let $f \in \mathcal{A}$, and let A, B and a be real numbers such that $-1 \le B < A \le 1$ and* (3) *holds. If*

$$\frac{zf''(z)}{(f'(z))^a} \prec h(z),$$

then $f'(z) \prec \frac{1+Az}{1+Bz}$, with $\frac{1+Az}{1+Bz}$ being best dominant.

Applying similar technique as in Corollary 1, from Theorem 2 and Theorem 3 we receive the following results

Corollary 2. *Let $f \in \mathcal{A}$, and let A, B and a be real numbers such that $-1 \le B < A \le 1$ and* (3) *holds. If*

$$\left|\frac{f(z)}{z}\right|^{-a} \cdot \left|f'(z) - \frac{f(z)}{z}\right| < b \quad (z \in \mathbb{D}),$$

where b is defined in Corollary 1, then

$$\left|\frac{f(z)}{z} - 1\right| < \frac{A-B}{1-|B|} \quad (z \in \mathbb{D}).$$

Corollary 3. *Let $f \in \mathcal{A}$, and let A, B and a be real numbers such that $-1 \le B < A \le 1$ and* (3) *holds. If*

$$\left|\frac{zf''(z)}{(f'(z))^a}\right| < b \quad (z \in \mathbb{D}),$$

where b is defined in Corollary 1, then

$$|f'(z)| < \frac{A-B}{1-|B|} \quad (z \in \mathbb{D}).$$

All three corollaries are sharp, i.e., value of b is the largest possible so that the corresponding conclusions hold in the following cases:

(i) $B \leq 0 \leq A$ and $a < 0$;
(ii) $AB \geq 0$ and $0 < a \leq 2$;
(iii) $AB \leq 0$ and $a > 2$.

The extremal function for Corollary 1, 2 and 3 are f_1, f_2 and f_3 defined respectively by

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1+Az}{1+Bz}, \quad \frac{f_2(z)}{z} = \frac{1+Az}{1+Bz}, \quad f_3'(z) = \frac{1+Az}{1+Bz}.$$

In the remaining cases, the three corollaries can be improved, but it involves tremendous calculations, so we leave it as an open problem.

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Email address: skender.av@gmail.com

DEPARTMENT OF MATHEMATICS AND INFORMATICS,
FACULTY OF MECHANICAL ENGINEERING,SS. CYRIL AND METHODIUS UNIVERSITY IN SKOPJE, KARPOŠ II B.B., 1000 SKOPJE, REPUBLIC OF NORTH MACEDONIA.
Email address: nikola.tuneski@mf.edu.mk