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DESIGN OF MULTISTABLE SYSTEM OF COUPLED DIFFERENT LORENZ AND NUCLEAR SPIN GENERATOR SYSTEMS

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ABSTRACT. In this paper, we propose a new theoretical scheme design of multistable system of coupled Nuclear spin generator and Lorenz systems. In the system coupled Nuclear spin generator and Lorenz systems reduces to a single modified Lorenz system. We derive the existence conditions of fixed points and the conditions of local stability of the modified system is also derived. To obtain multistable behaviour maximum lyapunov exponent of the system and bifurcation analysis are analyzed. Dynamical behaviour with respect to multistable parameter using MATCONT software are also analyzed. The main observation is that: In coupling two m-dimensional dynamical systems multistable behaviour can be obtained if i number of variables of the two systems are completely synchronized and j number of variables keep a constant difference between them, where $i + j = m$ and $1 \leq i, j \leq m-1$. Numerical simulation results are presented to verify the proposed schemes.

1. INTRODUCTION

Multistability is the property whereby the solutions of a dynamical system can alternate between two or more exclusive lyapunov stable and convergent equilibrium states under asymptotically slowly changing inputs or system parameters.

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Multistable systems are observed in laser physics [1], condensed matter physics [2] and electronic oscillators [3] etc. and biological system namely population dynamics [4], neuroscience [5] and climate dynamics [6]. The dynamics of multistable systems are extremely sensitive to the initial state due to the coexistence of different attractors and as a result very small perturbations of the initial state might cause a large change in the final state. The mechanisms behind multistable behaviour of many natural system's are not completely known. Understanding the rules behind multistability behaviour of a dynamical system remains one of the fundamental problem of dynamical systems theory. In extreme multistability the number of coexisting attractors is infinite. Techniques for designing extreme multistable systems had been reported by Sun et.al. [7]. In their technique the choice of coupling plays the vital role. Synchronization of two or more coupled nonlinear systems are fundamental concept of nonlinear dynamics. Many synchronization techniques were proposed since the pioneer work of Pecora and Carroll [8]. In 1997, Feudel et.al.[9] have studied the behaviour of multistable systems that one obtained from conservative ones by adding a small amount of damping. Layton et.al. [10] have studied multistability in tubuloglomerular feedback and spectral complexity in spontaneously hypertensive rats in 2006. In 2011, Geltrude et.al.[11] have discussed multistability of chaotic systems to explore a complexity deterministic closed loop mechanism to control bursting phenomenon. In 2015, Li et.al. [12] have studied multistability in symmetric chaotic systems using amplitude control techniques. Since multistability and amplitude control sometimes involved in dynamical systems with involutional symmetry. Hens et.al.[13] have shown that the coexistence of infinitely many attractors in two coupled m dimensional systems will be possible if $m-1$ of the variable of the two systems are completely synchronized and one of them keeps a constant difference between them and Pal et.al.[14] observed the coupling two $m-$ dimensional dynamical systems in multistable nature by obtaining i number of variables of the two systems are completely synchronized and j number of variables keep a constant difference between them, where $i+j=m$ and $1 \leq i, j \leq m-1$. Very recently, in 2017, Bao et.al.[15] have illustrated that the long term dynamical behaviour closely depends on memristor initial conditions, thus leading to the immergence of hidden extreme multistability in the memristive hyper chaotic systems. In the same year

2017, Khan et. al.[16] have introduced a generalized scheme for designing multistable systems by coupling two different dynamical systems. The basic idea of the scheme is to design partial synchronization of states between the coupled systems and finding some completely initial condition-dependent constants of motion.

The basic idea of the scheme is to design multistability of states between the coupled systems and finding some completely initial condition-dependent constants of motion. We discuss our scheme coupling two different Lorenz and Nuclear spin generator systems. The bifurcation diagrams of the system with respect to multistability parameters are shown here.

The paper is organized as follows: In Section 2, a generalized scheme for designing multistability system is proposed and discussed taking two coupled Lorenz system and Nuclear spin generator system. The existence conditions of local stability is discussed in Section 3. Numerical simulation results are presented in Section 4. Finally, a conclusion is drawn in Section 5.

2. Generalised scheme for designing multistable systems

Consider the coupled two dynamical systems in the following way

$$
\begin{aligned}
\dot{x_1} &= f_1(x_1, x_2, x_3, \ldots, x_n) + u_1(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n) \\
\dot{x_2} &= f_2(x_1, x_2, x_3, \ldots, x_n) + u_2(x_1, x_3, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n) \\
\dot{x_3} &= f_3(x_1, x_2, x_3, \ldots, x_n) + u_3(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n) \\
&\ldots \\
\dot{x_n} &= f_n(x_1, x_2, x_3, \ldots, x_n) + u_n(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n)
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
\dot{y_1} &= g_1(y_1, y_2, y_3, \ldots, y_n) + v_1(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n) \\
\dot{y_2} &= g_2(y_1, y_2, y_3, \ldots, y_n) + v_2(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n) \\
\dot{y_3} &= g_3(y_1, y_2, y_3, \ldots, y_n) + v_3(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n) \\
&\ldots \\
\dot{y_n} &= g_n(y_1, y_2, y_3, \ldots, y_n) + v_n(x_1, x_2, x_3, \ldots, x_n; y_1, y_2, y_3, \ldots, y_n)
\end{aligned} \tag{2.2}
$$

where $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ are the controllers. We define the error as $e_i = y_i - x_i, i = 1, 2, \ldots, n$. Now we obtain the error dynamical system

$$\begin{aligned} \dot{e_1} &= g_1(y_1, y_2, y_3, \ldots, y_n) - f_1(x_1, x_2, x_3, \ldots, x_n) + v_1 - u_1 \\ \dot{e_2} &= g_2(y_1, y_2, y_3, \ldots, y_n) - f_2(x_1, x_2, x_3, \ldots, x_n) + v_2 - u_2 \\ \dot{e_3} &= g_3(y_1, y_2, y_3, \ldots, y_n) - f_3(x_1, x_2, x_3, \ldots, x_n) + v_3 - u_3 \\ &\ldots \\ \dot{e_n} &= g_n(y_1, y_2, y_3, \ldots, y_n) - f_n(x_1, x_2, x_3, \ldots, x_n) + v_n - u_n \end{aligned} \tag{2.3}$$

We choose the controllers $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ suitable such that the above system become multistable. Hens et al.[13] propose that the coupled systems (1)and (2) have multistable behaviour if $(n-1)$ states of the two systems synchronize and one state variable keeps constant difference with corresponding state variable of the other system. They choose controllers $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ in such way that

$$\begin{aligned} \dot{e_1} &= 0 \\ \dot{e_2} &= -e_2 \\ \dot{e_3} &= -e_3 \\ &\ldots \\ \dot{e_n} &= -e_n \end{aligned} \tag{2.4}$$

Here, we generalize the results of Hens et al.[13] and conjecture that "multistable systems can be designed choosing $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ in such way that i $(1 \leq i \leq n-1)$ number of state variables synchronize and $(n-i)$ number of state variables keeps constant difference". Therefore according to our scheme we choose $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ in such way that

$$\begin{aligned} \dot{e_1} &= 0 \\ \dot{e_2} &= 0 \\ \dot{e_3} &= 0 \\ &\ldots \\ \dot{e_i} &= 0 \\ \dot{e_{i+1}} &= -e_{i+1} \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
\dot{e_{i+2}} &= -e_{i+2} \\
&\dots \\
\dot{e_n} &= -e_n
\end{aligned}
$$

where $1 \leq i \leq n-1$. Then for such choice the system formed by coupling (1) and (2) may show multistability.

Now we choose the function $L = (e_{i+1}^2 + e_{i+2}^2 + e_{i+3}^2 + \ldots + e_n^2)/2$ is a lyapunov function for the above system because

$$
\begin{aligned}
\dot{V} &= e_{i+2}\dot{e_{i+2}} + e_{i+3}\dot{e_{i+3}} + \ldots + e_n\dot{e_n} \\
&= -e_{i+1}^2 - e_{i+2}^2 - e_{i+3}^2 - \ldots - e_n^2.
\end{aligned}
$$

Hence the errors $e_{i+1}, e_{i+2}, e_{i+3}, \ldots, e_n$ must tend to zero i.e., $y_{i+1} = x_{i+1}$, $y_{i+2} = x_{i+2}$,...,$y_n = x_n$, as t tends to $\to \infty$ and $e_1, e_2, e_3, \ldots, e_i$ remains constant in time.

Therefore $y_1 = x_1 + c_1, y_2 = x_2 + c_2, y_3 = x_3 + c_3, \ldots, y_i = x_i + c_i$ and $y_{i+1} = x_{i+1}$, $y_{i+2} = x_{i+2}$,...,$y_n = x_n$.

Now the dynamics of the coupled system(1) and (2) is equivalent to the following system:

$$
\begin{aligned}
\dot{x_1} &= f_1(x_1, \ldots, x_n) + u_1(x_1, \ldots, x_n; x_1 + c_1, \ldots, x_i + c_i, x_{i+1} \ldots, x_n) \\
\dot{x_2} &= f_2(x_1, \ldots, x_n) + u_2(x_1, \ldots, x_n; x_1 + c_1, \ldots, x_i + c_i, x_{i+1} \ldots, x_n) \\
\dot{x_3} &= f_3(x_1, \ldots, x_n) + u_3(x_1, \ldots, x_n; x_1 + c_1, \ldots, x_i + c_i, x_{i+1} \ldots, x_n) \\
&\dots \\
\dot{x_n} &= f_n(x_1, \ldots, x_n) + u_n(x_1, \ldots, x_n; x_1 + c_1, \ldots x_i + c_i, x_{i+1} \ldots, x_n)
\end{aligned} \tag{2.6}
$$

where $c_1, c_2, c_3, \ldots, c_i' s$ are initial condition dependent constants. The system (6) shows multistable behaviour if its dynamics changes qualitatively with varition of $c_1, c_2, c_3, \ldots, c_i' s$. Notice that we have chosen $\dot{e_1} = \dot{e_2} = \cdots = \dot{e_i} = 0$ in general $\dot{e_1}, \dot{e_2}, \ldots \dot{e_i}$ may be chosen as any polynomial functions of $e_{i+1}, e_{i+2}, e_{i+3}, \ldots, e_n$.

In the following section, we shall discuss our scheme coupling two different Lorenz and Nuclear spin generator systems. Example of a proposition.

3. Construction of multistable systems using Lorenz and Nuclear spin generator systems

We consider the coupled Lorenz system [17] and Nuclear spin generator system [18] in the following form:

$$\begin{aligned} \dot{x_1} &= \sigma(x_2 - x_1) + u_1 \\ \dot{x_2} &= rx_1 - x_2 - x_1x_3 + u_2 \\ \dot{x_3} &= x_1x_2 - bx_3 + u_3 \\ \dot{y_1} &= -\beta y_1 + y_2 + v_1 \\ \dot{y_2} &= -y_1 - \beta y_2(1 - \kappa y_3) + v_2 \\ \dot{y_3} &= \beta[\alpha(1 - y_3) - \kappa y_2^2] + v_3 \end{aligned} \tag{3.1}$$

where $u_1, u_2, u_3, v_1, v_2, v_3$ are controllars and we choose $u_1 = \sigma(x_1 - y_1)$, $u_2 = x_2 - y_2$, $u_3 = 0$ and $v_1 = (\sigma - 1)y_2 + \beta y_1$, $v_2 = y_1 + (\beta - 1)y_2 + rx_1 - y_2 - y_3(\beta\kappa y_2 + x_1)$, $v_3 = \beta\kappa y_2^2 + x_1x_2 - \beta\alpha(1 - y_3) - by_3$ in such way that the above system reduces to

$$\begin{aligned} \dot{x_1} &= \sigma(x_2 - y_1) \\ \dot{x_2} &= rx_1 - y_2 - x_1x_3 \\ \dot{x_3} &= x_1x_2 - bx_3 \\ \dot{y_1} &= \sigma y_1 \\ \dot{y_2} &= rx_1 - y_2 - x_1y_3 \\ \dot{y_3} &= x_1x_2 - by_3 \end{aligned} \tag{3.2}$$

We now show that the six dimensional dynamical system is a multistable system. Following Sun et al.[19] we construct the governing equations for the synchronization errors $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$ and $e_3 = y_3 - x_3$ as

$$\begin{aligned} \dot{e_1} &= \sigma e_2 \\ \dot{e_2} &= -x_1e_3 \\ \dot{e_3} &= -be_3 \end{aligned} \tag{3.3}$$

It follows that e_3 must tend to zero with time i.e., $y_3 = x_3$. Since x_1 is a bounded physical quantity therefore

$$\begin{aligned} \dot{e_1} &= 0 \\ \dot{e_2} &= 0 \end{aligned} \tag{3.4}$$

which implies e_1= constant=c_1 and e_2= constant=c_2. Hence, $y_1 = x_1 + c_1$ and $y_2 = x_2 + c_2$ where c_1, c_2 are some constants (dependent on the initial condition of the full system). Each new set of initial conditions gives rise to different value of c_1 and c_2. Therefore, the dynamics of the system of equations (7) is equivalent to following three dimensional system

$$\begin{aligned} \dot{x_1} &= \sigma(x_2 - x_1 - c_1) \\ \dot{x_2} &= rx_1 - x_2 - c_2 - x_1x_3 \\ \dot{x_3} &= x_1x_2 - bx_3. \end{aligned} \tag{3.5}$$

The system (7) is a multistable system if the dynamical behaviour of the system (11) varies with the variation of the value of c_1 and c_2.

4. Preliminaries

4.1. **Dissipativity and existence of attractor.** For the above system(11), we observe that $\nabla V = \frac{\partial \dot{x_1}}{\partial x_1} + \frac{\partial \dot{x_2}}{\partial x_2} + \frac{\partial \dot{x_3}}{\partial x_3} = -(\sigma+b+1) < 0$, as $\sigma > 0$ and $b > 0$. So, the above system is dissipative, with an exponential contraction rate $\frac{dV}{dt} = -(\sigma + b + 1)V$. That is, a volume element V_0 is contracted by the flow into a volume element $V_0 e^{-(\sigma+b+1)t}$ in time t. This means that each volume containing the system trajectory shrinks to zero as $t \to +\infty$ at an exponential rate,$-(\sigma + b + 1)$. Therefore, system orbits are ultimately confined to a specific subset of zero volume, and the asymptotic motion settles onto an attractor.

4.2. **Equilibrium Points.** We first study the nature of equilibrium points of the system (11). An equilibrium point (x_1, x_2, x_3) is such that the solution of a system does not change in time. The equilibrium point of the system (11) is the point $E^* \equiv (x_1^*, x_2^*, x_3^*)$, where $x_2^* = x_1^* + c_1$, $x_3^* = b(rx_1^* - x_1^* - c_1 - c_2)/x_1^*(x_1^* + c_1)$ and x_1^* is the real root of the cubic equation $x_1^3 + c_1 x_1^2 + b(1-r)x_1 + b(c_1 + c_2) = 0$. Therefore, existence of non-trivial equilibrium points depend on the parameter value c_1 and

c_2. E^* exist when $\triangle = 19b^2c_1^2 + 18b^2c_1c_2(1-r) + 4b^3r^3 + r^2b^2c_1^2 - 4b^3 + 12b^3r(1-r) - 27b(c_1+c_2) - 18b^2c_1r - 4bc_1^4 - 4bc_1^3c_2 - 2rb^2c_1^2 > 0$.

The Jacobian matrix of the system (11) at the equilibrium point $E^* = (x_1^*, x_2^*, x_3^*)$ is given by

$$J(E^*) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - x_3^* & -1 & -x_1^* \\ x_2^* & x_1^* & -b \end{pmatrix}. \tag{4.1}$$

The eigenvalues of the Jacobian matrix are the roots of the following equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

where, $a_1 = \sigma(x_1^*x_2^* + bx_3^* - br) + \sigma(x_1^{*2} + b)$, $a_2 = x_1^{*2} + b + b\sigma + \sigma x_3^* + \sigma - \sigma r$, $a_3 = b + 1 + \sigma$.

The equilibrium point $E^* = (x_1^*, x_2^*, x_3^*)$ is stable if $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$, otherwise E^* is unstable.

5. Numerical results

We perform the dynamical behaviours of the system (11) through numerical analysis with the parameter values which are taken form Lorenz system [17]. We have varied the vital parameters c_1 and c_2 throughout the whole numerical simulations.

First, we discuss the simulation results of the system (11) with bifurcation and maximum lyapunov exponent. The bifurcation and maximum lyapunov exponent of the system(11) for different values of c_1 and c_2 are plotted for fixed $\sigma = 10$, $r = 28$ and $b = 8/3$. The bifurcation and maximum lyapunov exponent diagram with respect to c_1 of the system(11) are plotted in figures 1(a),(b) to 4(a),(b) for $c_2 = -2$, $c_2 = -1$, $c_2 = 1$ and $c_2 = 2$ respectively. Also the bifurcation and maximum lyapunov exponent diagrams with respect to c_2 of the system(11) are plotted in figures 5(a),(b) to 7(a),(b) for $c_1 = -1$, $c_1 = 1$ and $c_1 = 2$ respectively. The multistable behaviour of the system(11) is established from these diagrams. In figure 8 extinction region of c_1 and c_2 of the system (11) are plotted for $\sigma = 10$, $r = 28$ and $b = 8/3$. Figure 8 depicts regions of stable state, unstable state and stable state. Notice that in Figure 8 the boundaries between the different dynamical regions are not perfectly distinct. Because, in Figure 8 that the high periodic oscillations and chaotic region there are small areas. This occurs because there is some degree of sensitivity to small changes in parameter values resulting in sharp transitions between different dynamical outcomes. For internal bifurcation scenarios of the system (11), we study a pattern of bifurcation sequences in next Section.

6. Hopf bifurcation and continuation

Our main aim of this section is to investigate the bifurcation scenarios of the system(11) with respect to the parameter c_1 and c_2. These are done by studying the change in the eigenvalues of the Jacobian matrix and following the continuation algorithm. We choose initial points $x_{10} = 9.5043676$, $x_{20} = 15.565189$, $x_{30} = 18.166588$ fixing the parameter values $\sigma = 10$, $r = 28$ and $b = 8/3$. The characteristics of Hopf point, limit cycle and the general bifurcation nature are explored using the software package MATCONT2.5.1. In this package we use prediction-correction continuation algorithm based on the Moore-Penrose matrix pseudo inverse for computing the curves of equilibria, limit point (LP) and its continuation curves.

The continuation curves from the equilibrium point of x_3 with respect to c_1 for $c_2 = -5$(Red line), $c_2 = 0$(Blue line) and $c_2 = 5$(Magenta line) for the fixed parameter values $\sigma = 10$, $r = 28$, $b = 8/3$ are presented in figure 9. Existence of two Hopf points (H_1, H_2), two limit points (LP_1, LP_2) are observed in figure 9 for all those cases. The Hopf points H_1 and H_2 for $c_2 = -5$ are located at $(x_1, x_2, x_3, c_1, c_2)$ ≡(9.5043676,15.565189,18.166588,-0.022349081,-5.0) and $(x_1, x_2, x_3, c_1, c_2)$ ≡(9.5043676,15.565189,18.166588,2.293176,-5.000) with first Lyapunov coefficient is to be 0.002076872, indicating a sub critical Hopf bifurcation. Therefore, there are two complex eigenvalues of the equilibrium with real $\lambda_{2,3} \approx 0$ at the parameter. First Lyapunov coefficient is positive implies that a unstable limit cycle appears from the equilibrium point. The limit points LP_1 and LP_2 occur at $(x_1, x_2, x_3, c_1, c_2) \equiv$ (9.5043676,15.565189,18.166588,18.741569, -5.00) with normal form of coefficient $a = -0.2615149$ and $(x_1, x_2, x_3, c_1, c_2) \equiv$ (9.5043676, 15.565189,18.166588,-24.076094,-5.00) with normal form of coefficient $a = -0.2585899$. The Hopf points H_1 and H_2 for $c_2 = 0$ are located at $(x_1, x_2, x_3, c_1, c_2) \equiv$ (9.5043676,15.565189,18.166588,-1.2610557,0.000) and $(x_1, x_2, x_3, c_1, c_2) \equiv$

(9.5043676,15.565189,18.166588,1.2610557,0.000) with first Lyapunov coefficient is to be 0.002029014, indicating a sub critical Hopf bifurcation. Therefore, there are two complex eigenvalues of the equilibrium with real $\lambda_{2,3} \approx 0$ at the parameter. First Lyapunov coefficient is positive implies that a unstable limit cycle appears from the equilibrium point. The limit points LP_1 and LP_2 occur at $(x_1,\ x_2,\ x_3,\ c_1,c_2)$≡(9.5043676, 15.565189,18.166588,-21.280859,0.00) with normal form of coefficient $a = -0.2599334$ and $(x_1,\ x_2,\ x_3,c_1,c_2)$≡ (9.5043676, 15.565189, $21.280859$18.166588,21.280859,0.00) with normal form of coefficient $a = -0.2599334$. The Hopf points H_1 and H_2 for $c_2 = 5$ are located at $(x_1,\ x_2,\ x_3,\ c_1,\ c_2)$≡ (9.5043676,15.565189,18.166588,-2.293173,5.00) with first Lyapunov coefficient is to be 0.002076873 and $(x_1,\ x_2,x_3,\ c_1,c_2)$ ≡ (9.5043676, 15.565189, 18.166588, 0.022354, 5.0) with first Lyapunov coefficient is to be 0.001964840, indicating a sub critical Hopf bifurcation. Therefore, there are two complex eigenvalues of the equilibrium with real $\lambda_{2,3} \approx 0$ at the parameter. First Lyapunov coefficient is positive implies that a unstable limit cycle appears from the equilibrium point. The limit points LP_1 and LP_2 occur at $(x_1,\ x_2,\ x_3,\ c_1,c_2)$≡ (9.5043676, 15.565189, 18.166588, -24.076094, 5.0) with normal form of coefficient $a = -0.2585901$ and (x_1, x_2, x_1,c_1,c_2)≡(9.5043676, 15.565189, 18.166588, 18.741569, 5.0) with normal form of coefficient $a = 0.2615148$. The bifurcation results together with normal coefficients with respect to c_1 are listed in Table 1.

The continuation curves from the equilibrium point of x_3 with respect to c_2 for $c_1 = -5$ (Red line), $c_1 = 0$ (Blue line) and $c_1 = 5$ (Magenta line) for the fixed parameter values $\sigma = 10,\ r = 28,\ b = 8/3$ are also presented in figure 10. Existence of two Hopf points (H_1, H_2), two limit points (LP_1, LP_2) are also observed in figure 10 for all these cases. The bifurcation results together with normal coefficients with respect to c_2 are also listed in Table 2.

Table 1: Bifurcation points of the system(11) in figure 9, togather with first Lyapunov coefficients, normal form coefficients and eigenvalues for parameters $\sigma = 10$, r=28 $b = 8/3$. H_1, H_2- Hopf points; LP_1, LP_2-Limit Points.

c_2	c_1	Label	First Lyapunov coefficients/ Normal	Eigenvalues

			form coefficient	
-5.0000	-0.022349081	H_1	$l_1 = 0.002076872$	$-13.666,\ \pm i10.43590$
-5.0000	2.293176	H_2	$l_1 = 0.002076872$	$-13.666,\ \pm i10.9509$
-5.000	-18.741569	LP_1	$a = -0.2615149$	$-19.2776, 0.0000, 5.61162$
-5.0000	24.076094	LP_2	$a = -0.2585899$	$-19.2436, -0.000001148, 5.57762$
0.0000	-1.2610557	H_1	$l_1 = 0.002029014$	$-13.666,\ \pm i10.7053$
0.0000	1.2610557	H_2	$l_1 = 0.002029013$	$-13.666,\ \pm i10.7053$
0.0000	-21.280859	LP_1	$a = -0.2599334$	$-19.259, 0.0000, 5.59298$
0.0000	21.280859	LP_2	$a = -0.2599334$	$-19.259, 0.0000, 5.59298$
5.0000	-2.293173	H_1	$l_1 = 0.002076873$	$-13.666,\ \pm i10.8509$
5.0000	0.022354	H_2	$l_1 = 0.001964840$	$-13.666,\ \pm i10.4359$
5.0000	-24.076094	LP_1	$a = -0.2585901$	$-19.2436, 0.0000, 5.57761$
5.0000	18.741569	LP_2	$a = 0.2615148$	$-19.2776, 0.0000, 5.61162$

TABLE 2. Bifurcation points of the system(11) in figure 10, togather with first Lyapunov coefficients, normal form coefficients and eigenvalues for parameters $\sigma = 10$, r=28 $b = 8/3$. H_1, H_2- Hopf points; LP_1, LP_2-Limit Points.

c_2	c_1	Label	First Lyapunov coefficients/ Normal form coefficient	Eigenvalues
-5.0000	-17.569229	H_1	$l_1 = 0.001737604$	$-13.666,\ \pm i9.60221$
-5.0000	22.684046	H_2	$l_1 = 0.002156435$	$-13.666,\ \pm i11.6893$
-5.000	-50.443686	LP_1	$a = 0.2822183$	$-19.6054, 0.00000, 5.9394$
-5.0000	157.38987	LP_2	$a = 0.2981529$	$-20.7374, -0.00000, 7.07142$
0.0000	-5.0813	H_1	$l_1 = 0.001963648$	$-13.666,\ \pm i10.4313$
0.0000	5.0813	H_2	$l_1 = 0.001963647$	$-13.666,\ \pm i10.434$
0.0000	-88.170607	LP_1	$a = 0.2946126$	$-20.00, 0.0000, 6.334$
0.0000	88.170607	LP_2	$a = -0.2946126$	$-20., 0.00000, 6.334$
5.0000	-17.569224	H_1	$l_1 = 0.001737605$	$-13.666,\ \pm i9.60221$
5.0000	-22.684046	H_2	$l_1 = 0.002156435$	$-13.666,\ \pm i11.6893$
5.0000	-50.443686	LP_1	$a = 0.2822182$	$-19.6054, 0.0000, 5.9394$
5.0000	157.38987	LP_2	$a = -0.2981532$	$-20.7374, -0.00000, 7.07142$

7. Conclusion

We introduce a generalised scheme for designing multistable coupling Lorenz system. In this scheme the two state variables of the coupled systems synchronize and other state variables keep constant difference. In this scheme the coupled Lorenz system reduces to a single modified Lorenz system. Equilibrium points of the proposed system are determined and the local stability criteria is derived. Multistable nature of the coupled Lorenz system is described through bifurcation and maximum lyapunov exponent diagrams. One and two parameter bifurcation analysis is done using MATCONT software. Our investigation and predictions may be very useful for designing multistable systems in different branches such as biology, physics and engineering sciences.

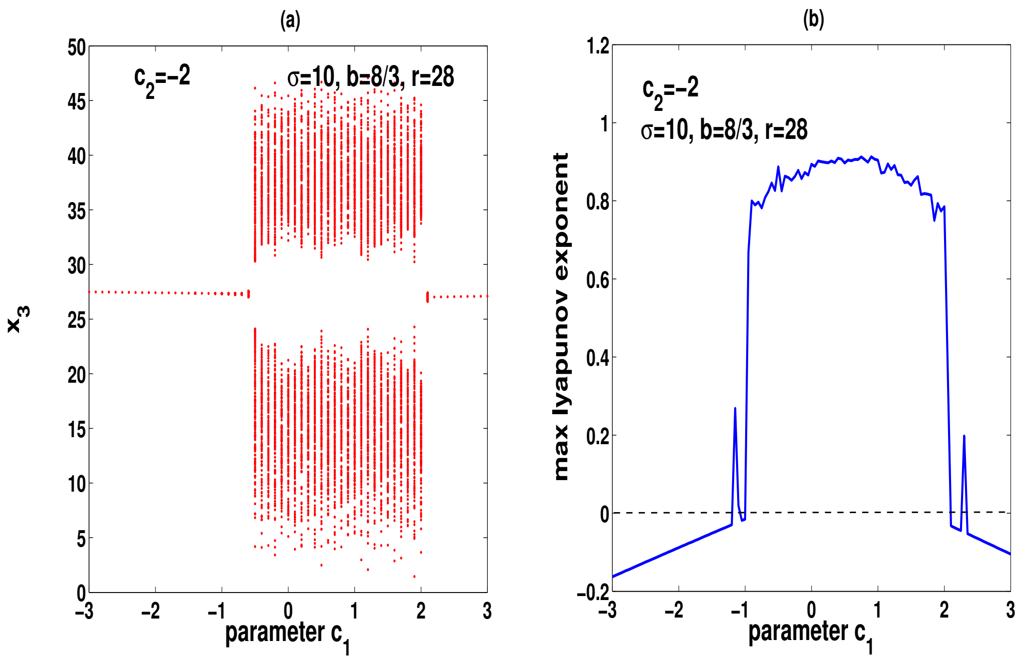

FIGURE 1. Figure depicts in(a)Bifurcation of x_3 with respect to c_1 and in (b) Maximu lyapunov exponent with respect to c_1 for fixed $c_2 = -2$ and $\sigma = 10, r = 28, b = 8/3$

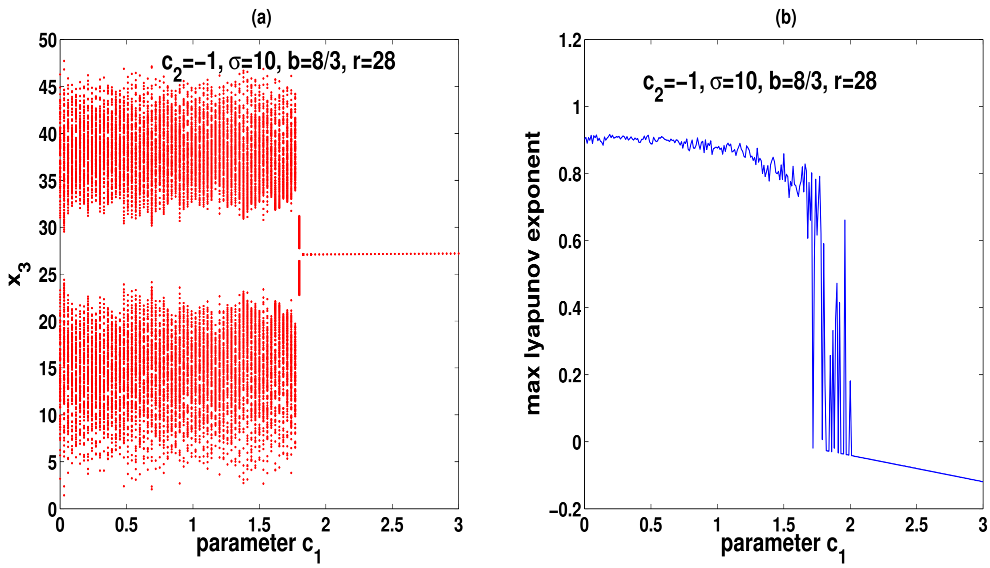

FIGURE 2. Figure depicts in(a)Bifurcation of x_3 with respect to c_1 and in (b) Maximu lyapunov exponent with respect to c_1 for fixed $c_2 = -1$ and $\sigma = 10, r = 28, b = 8/3$

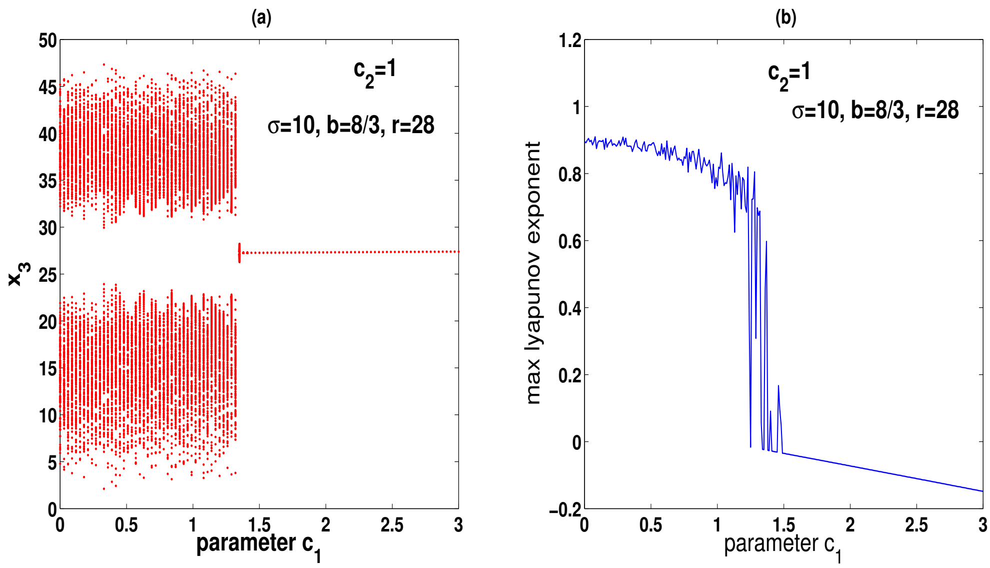

FIGURE 3. Figure depicts in(a)Bifurcation of x_3 with respect to c_1 and in (b) Maximu lyapunov exponent with respect to c_1 for fixed $c_2 = 1$ and $\sigma = 10, r = 28, b = 8/3$

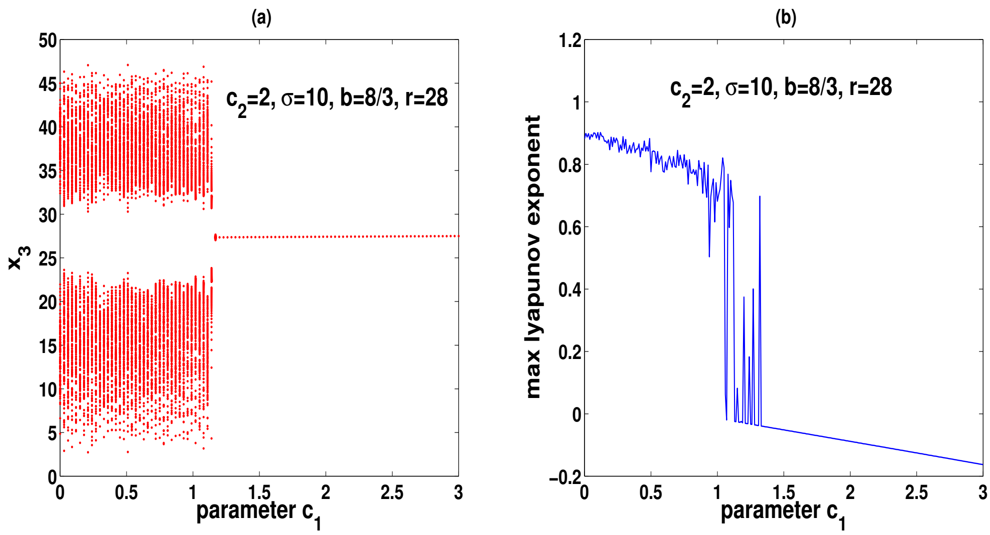

FIGURE 4. Figure depicts in(a)Bifurcation of x_3 with respect to c_1 and in (b) Maximu lyapunov exponent with respect to c_1 for fixed $c_2 = 2$ and $\sigma = 10, r = 28, b = 8/3$

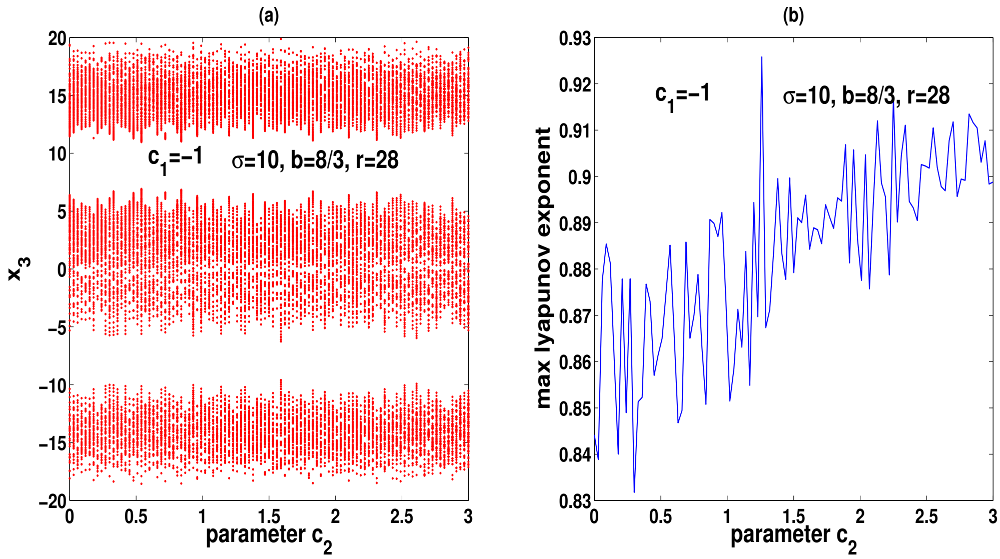

FIGURE 5. Figure depicts in(a)Bifurcation of x_3 with respect to c_2 and in (b) Maximu lyapunov exponent with respect to c_2 for fixed $c_1 = -1$ and $\sigma = 10, r = 28, b = 8/3$

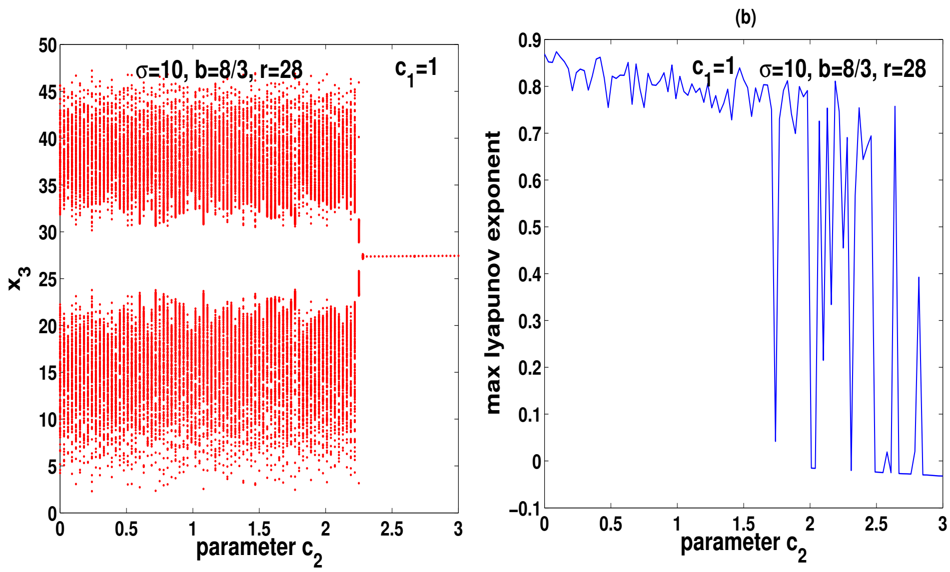

FIGURE 6. Figure depicts in(a)Bifurcation of x_3 with respect to c_2 and in (b) Maximu lyapunov exponent with respect to c_2 for fixed $c_1 = 1$ and $\sigma = 10, r = 28, b = 8/3$

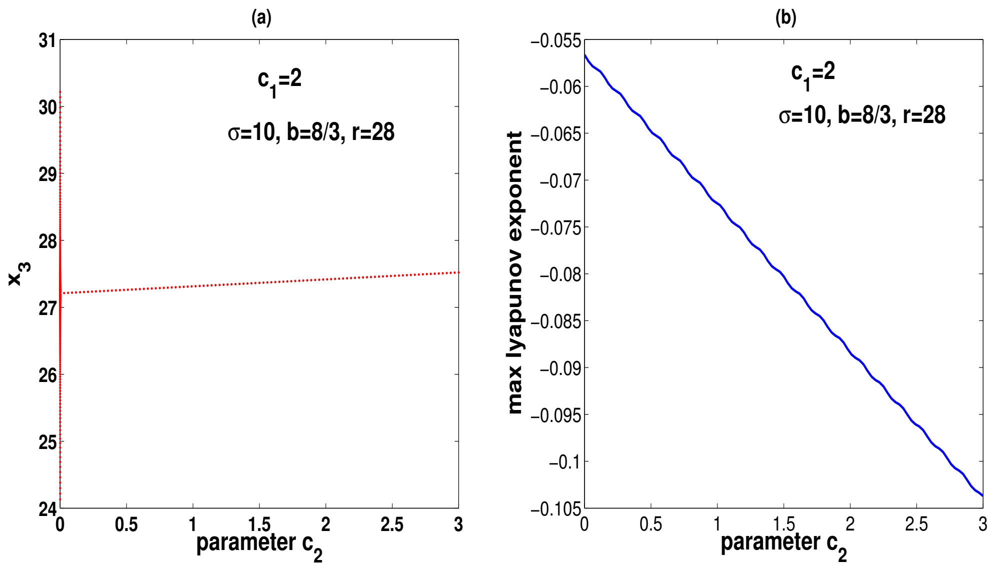

FIGURE 7. Figure depicts in(a)Bifurcation of x_3 with respect to c_2 and in (b) Maximu lyapunov exponent with respect to c_2 for fixed $c_1 = 2$ and $\sigma = 10, r = 28, b = 8/3$

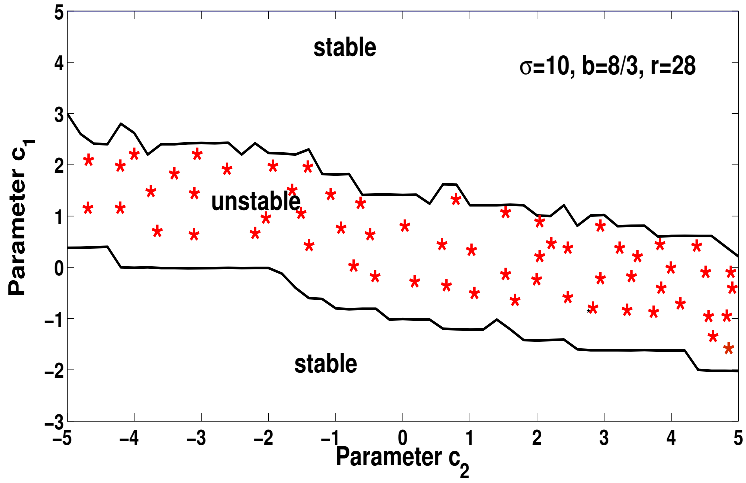

FIGURE 8. Dynamics of the system with respect to c_2 as a function of c_1. It divides the plane as the stable and unstable region for $\sigma = 10, r = 28, b = 8/3$.

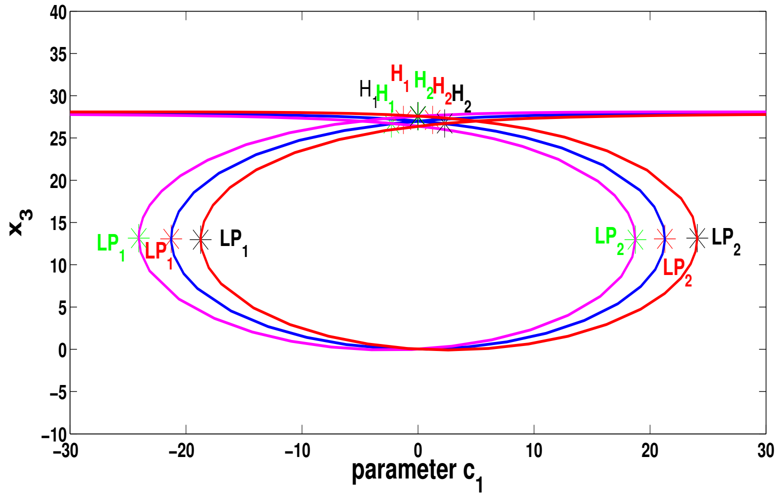

FIGURE 9. Continuation curves of equilibrium with the variation of the parameter c_1 for c_2= -5(red line), 0 (blue line), 5 (magenta line) of the system (11) for $\sigma = 10.0, r = 28, b = 8/3$: $H_1,\ H_2$-Hopf point, $LP_1,\ LP_2$-limit point.

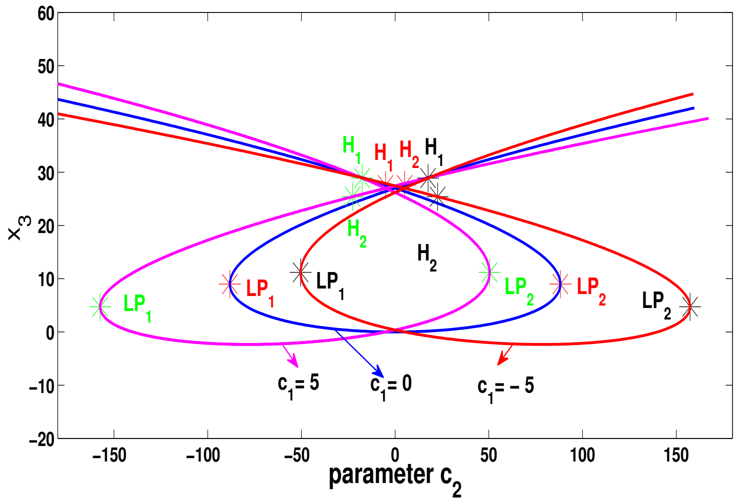

FIGURE 10. Continuation curves of equilibrium with the variation of the parameter c_2 for c_1 = -5(red line), 0 (blue line), 5 (magenta line) of the system (11) for $\sigma = 10.0, r = 28, b = 8/3$: $H_1,\ H_2$-Hopf point, $LP_1,\ LP_2$-limit point.

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