Advances in Mathematics: Scientific Journal **11** (2022), no.6, 557–564
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.6.4

QUALITATIVE UNCERTAINTY PRINCIPLE FOR THE FOURIER TRANSFORM ON SEMISIMPLE LIE GROUPS: APPLICATION TO LINEAR SPECIAL GROUP $SL(2,\mathbb{R})$

Taki N'dah Jean Pierre Auguste[1], Brou Kouakou, and Kangni kinvi

ABSTRACT. It is known that if the support of a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform have finite measure then $f = 0$ almost everywhere. We study generalizations of this property for semisimple Lie groups.

1. INTRODUCTION

Let λ be the Lebesque measure on $\mathbb{R}^n$ and let f be an $\lambda-$integrable function. Let put :

$$A_f = \{x \in \mathbb{R}^n : f(x) \neq 0\} \text{ and } B_f = \{x \in \mathbb{R}^n : \widehat{f}(x) \neq 0\},$$

where $\widehat{f}$ is the Fourier transform of f. We want to give a generalization of the following result of Benedicks [8].

Theorem 1.1. *if $f \in L^1(\mathbb{R}^n)$ verifies: $\lambda(A_f) < \infty$ and $\lambda(B_f) < \infty$ then f is null $\lambda-$almost everywhere.*

This result has been proved in the cases of nilpotent and solvable Lie Groups [3], [2].

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 43A30, 22E27, 22E30.
Key words and phrases. Induced representation, Fourier transform, uncertainty principle.
Submitted: 03.06.2022; *Accepted:* 19.06.2022; *Published:* 24.06.2022.

Let G be a locally topological compact group, dm be the Haar measure on G and $\widehat{G}$ its unitary dual, $\mu_{\widehat{G}}$ denote the Plancherel measure on $\widehat{G}$.

For $f \in L^1(G)$, we put:

$$A_f = \{x \in G : f(x) \neq 0\} \text{ and } B_f = \{\pi \in \widehat{G} : \widehat{f}(\pi) \neq 0\}.$$

The qualitative uncertainty principle for Fourier Transform can be states as follows: For $f \in L^1(G)$, if $m(A_f) < \infty$ and $\mu(B_f) < \infty$, then $f = 0$.

In this paper, we prove that any semisimple Lie group verifies the qualitative uncertainty principle.

We considered an Iwasawa decomposition of $G(G = KAN)$ and we will use the fact that any unitary representation of G is induced by a representation of the solvable subgroup AN of G. [4]

For other results of the qualitative uncertainty principle of certain groups, refer to articles [1], [5], [6] and [7].

2. Qualitative Uncertainty Principle on semisimple Lie groups

Let's consider a semisimple Lie group G with finite center and it Iwasawa decomposition $G = KAN$ where K is a maximal compact subgroup of G, A a closed abelian subgroup of G, N a nilpotent subgroup of G. We denote by dm, a Haar measure on G.

Let us put $G_1 = AN$, then G_1 is a solvable Lie subgroup of G. Let's set $\pi = Ind_{G_1}^{G}\pi_1$, $\pi_1 \in \widehat{G_1}$, where the carrier space is denoted by $L(G_1)$.

For $\phi \in L(G_1)$, $\displaystyle\int_{G/G_1} \|\phi(x)\|^2 d\dot{x} < \infty$ where $d\dot{x}$ is a G-invariant measure on G/G_1.

We consider a Haar measure on K and a Haar measure on G_1 denoted respectively by dk and dm_1. The Haar measure on G relative to the Iwasawa decomposition is given by the formula

$$dx = \left(\frac{\Delta_{G_1}(m_1)}{\Delta_G(m_1)}\right)^{-1} dk dm_1 \text{ if } x = km_1$$

with $m_1 = an \in G_1$ where Δ is the module function on G.

Let us put $\Delta(m_1) = \frac{\Delta_{G_1}(m_1)}{\Delta_G(m_1)} = \Delta_{G_1}(m_1)$ because G is semisimple ($\Delta_G(m_1) = 1$).

For all $f \in L^1(G)$ and $\pi \in \widehat{G}$, we define the Fourier transform operator by:

$$\widehat{f}(\pi) = \pi(f) = \int_G f(x)\pi(x)dx.$$

If G is a semisimple Lie group, we have for all $f \in L^1(G)$, $\pi \in \widehat{G}$ and $\phi \in L(G_1)$,

$$\widehat{f}(\pi)\phi(g) = \int_G f(x)\pi(x)\phi(g)dx.$$

for all $k \in K$, we consider the function defined by $f^k(m_1) = \Delta(m_1)^{-1/2} f(km_1^{-1})$, for all $m_1 \in G_1$. We have $f^k \in L^1(G_1)$. Let us put

$$A_f = \{x \in G : f(x) \neq 0\} \text{ and } B_f = \{\pi \in \widehat{G} : \widehat{f}(\pi) \neq 0\}$$
$$A_{f^k} = \{m_1 \in G_1 : f^k(m_1) \neq 0\} \text{ and } B_{f^k} = \{\pi_1 \in \widehat{G_1} : \widehat{f^k}(\pi_1) \neq 0\}.$$

We have the following result.

Theorem 2.1. *$\pi(f) = 0$ if and only if $\pi_1(f^k) = 0$ for all $k \in K$.*

Proof. For all $\phi \in L(G_1)$, we have

$$\begin{aligned}
\widehat{f}(\pi)\phi(g) &= \int_G f(x)\pi(x)\phi(g)dx \\
&= \int_G f(x)\phi(x^{-1}g)dx \\
&= \int_K \int_{G_1} f(km_1)\phi[(km_1)^{-1}g]\Delta(m_1)^{-1}dkdm_1 \\
&= \int_K \int_{G_1} f^k(m_1^{-1})\phi[(km_1)^{-1}g]\Delta(m_1)^{-3/2}dkdm_1, \\
\widehat{f}(\pi)\phi(g) &= \int_K \int_{G_1} f^k(m_1^{-1})\pi_1(m_1^{-1})\Delta(m_1^{-1})\phi(k^{-1}g)dkdm_1 \\
&= \int_K \int_{G_1} f^k(m_1^{-1})\pi_1(m_1^{-1})\Delta(m_1^{-1})\phi(k^{-1}g)dkdm_1 \\
&= \int_K \left(\int_{G_1} f^k(m_1^{-1})\pi_1(m_1^{-1})\Delta(m_1^{-1})dm_1\right)\phi(k^{-1}g)dk \\
&= \int_K \widehat{f^k}(\pi_1)\phi(k^{-1}g)dk.
\end{aligned}$$

- if $\pi(f) = 0$ then for all $\phi \in L(G_1)$ we have $\widehat{f}(\pi)\phi(g) = \pi(f)\phi(g) = 0$ and $\int_K \widehat{f^k}(\pi_1)\phi(k^{-1}g)dk = 0$. This implies that for all $k \in K$, $\widehat{f^k}(\pi_1) = 0$, for all $g \in G$, and so $\pi_1(f^k) = 0$.
- Conversely for all $k \in K$ and for all $g \in G$ if $\pi_1(f^k) = 0$, then $\widehat{f}(\pi) = \pi(f) = 0$.

□

Remark 2.1. *We can translate the previous assert in terms of B_f and B_{f^k} as follows: $\pi \notin B_f \Longleftrightarrow \pi_1 \notin B_{f^k}, \forall k \in K$.*

Theorem 2.2. *Any semisimple Lie group G satisfies the qualitative uncertainty principle.*

Proof. We assume that $m(A_f)$ and $\mu_G(B_f)$ are finite. Let us show that if $m(A_f)$ is finite then $m_1(A_{f^k})$ is finite for all $k \in K$. Indeed,

$$\begin{aligned} m(A_f) &= \int_G 1_{A_f}(x)dx \\ &= \int_K \int_{G_1} \Delta(m_1)^{-1} 1_{A_f}(km_1)dkdm_1. \end{aligned}$$

But

$$\begin{aligned} 1_{A_f}(km_1) = 1 &\Longleftrightarrow f(km_1) \neq 0 \\ &\Longleftrightarrow \Delta(m_1)^{-1/2} f^k(m_1^{-1}) \neq 0 \\ &\Longleftrightarrow f^k(m_1^{-1}) \neq 0 \\ &\Longleftrightarrow 1_{A_{f^k}}(m_1^{-1}) = 1. \end{aligned}$$

In the same way,

$$1_{A_f}(km_1) = 0 \Longleftrightarrow 1_{A_{f^k}(m_1^{-1})} = 0.$$

So,

$$\begin{aligned} m(A_f) &= \int_K \left(\int_{G_1} \Delta(m_1)^{-1} 1_{A_{f^k}}(m_1^{-1})dm_1 \right) dk \\ &= \int_K \left(\int_{G_1} 1_{A_{f^k}}(m_1)dm_1 \right) dk \\ &= \int_K m_1(A_{f^k})dk. \end{aligned}$$

If $m(A_f) < \infty$, then $m_1(A_{f^k}) < \infty$ for m_1-almost everywhere and for all k in K. In the previous remark, $1_{B_{f^k}}(\pi_1) = 0$ for all $k \in K$ is equivalent to $1_{B_f}(\pi) = 0$. For all $k \in K$, if $1_{B_{f^k}}(\pi_1) = 1$ then $1_{B_f}(\pi) = 1$. In any case, we deduce that for all $k \in K$,

$$1_{B_{f^k}}(\pi_1) \leq 1_{B_f}(\pi).$$

It follows that

$$\int_{\widehat{G_1}} 1_{B_{f^k}}(\pi_1)d\mu_{\widehat{G_1}}(\pi_1) \leq \int_{\widehat{G}} 1_{B_f}(\pi)d\mu_G(\pi).$$

So

$$\mu_{\widehat{G_1}}(B_{f^k}) \leq \mu_{\widehat{G}}(B_f).$$

Since $\mu_{\widehat{G}}(B_f) < \infty$, then $\mu_{\widehat{G_1}}(B_{f^k}) < \infty$.

We have $m_1(A_{f^k}) < \infty$ and $\mu_{\widehat{G_1}}(B_{f^k}) < \infty$ since G_1 verifies qualitative uncertainty principle then $f^k = 0$ for all $k \in K$ and hence $f = 0$. Thus G verifies the qualitative uncertainty principle. □

Remark 2.2. *In the paper [9], the authors used the conditions: $m(KA_fK) < \infty$ and $\mu_G(B_f) < \infty$ to obtain the result. In our paper, we proved that if $\mu_G(B_f) < \infty$, and $m(A_f) < \infty$ then $f = 0$.*

3. Application to linear special group $SL(2,\mathbb{R})$

We consider the linear special group

$$G = SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a, b, c \text{ and } d \in \mathbb{R} \text{ such that: } ad - bc = 1 \right\}.$$

Here, G is a semisimple Lie group. Let's consider $G = KAN$ with $K = \{u_\theta, \theta \in \mathbb{R}\} \approx \mathbb{R}/2\pi\mathbb{Z}$, $A = \{a_t, t \in \mathbb{R}\} \approx \mathbb{R}$ et $N = \{n_\lambda / \lambda \in \mathbb{R}\}$,

$$u_\theta = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix},\ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \text{ and } n_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

and for all element $g \in SL(2,\mathbb{R})$ there exists three reals θ, t and λ such that $g = u_\theta a_t n_\lambda$. We set:

$$G_1 = AN = \left\{ \begin{pmatrix} e^t & \lambda e^t \\ 0 & e^{-t} \end{pmatrix}, t, \lambda \in \mathbb{R} \right\} = \{a_t n_\lambda, t, \lambda \in \mathbb{R}\}.$$

Next, G_1 is a solvable subgroup of G.

If dx, dt, $d\lambda$ are respectively the Haar measures on $SL(2,\mathbb{R})$, A, N and $\frac{d\theta}{2\pi}$ the Haar measure on K, then for any integrable fonction f on $SL(2,\mathbb{R})$ we have

$$\int_{SL(2,\mathbb{R})} f(x)dx = \frac{1}{2\pi} \int_N \int_A \int_K f(u_\theta a_t n_\lambda) e^{2t} d\theta dt d\lambda.$$

Let dm_1 the Haar measure on G_1. We have $dm_1 = dtd\lambda$ and every unitary irreducible representation of G is induced by an unitary irreducible representation of G_1. We have $\pi = Ind_{G_1}^G \pi_1$, $\pi_1 \in \widehat{G_1}$ with $\pi : G \longrightarrow GL(L(G_1))$ where $L(G_1)$ is a Hilbert space defined above.

For $f \in L^1(G)$, we have: $\widehat{f}(\pi) = \pi(f) = \displaystyle\int_G f(x)\pi(x)dx$.

Let $f \in L^1(G)$ and $\phi \in L(G_1)$. Then we have

$$\begin{aligned}
\widehat{f}(\pi)\phi(g) &= \int_G f(x)\pi(x)\phi(g)dx \\
&= \int_G f(x)\phi(x^{-1}g)dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f(u_\theta a_t n_\lambda)\phi[(u_\theta a_t n_\lambda)^{-1}g]e^{2t}d\theta dt d\lambda \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f[u_\theta(a_t n_\lambda)]\phi[(a_t n_\lambda)^{-1}(u_\theta^{-1}g)]e^{2t}d\theta dt d\lambda \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f[u_\theta(a_t n_\lambda)]\check{\phi}[(g^{-1}u_\theta)(a_t n_\lambda)]e^{2t}d\theta dt d\lambda \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f[u_\theta(a_t n_\lambda)]\pi_1[(a_t n_\lambda)^{-1}]\check{\phi}(g^{-1}u_\theta)e^{2t}d\theta dt d\lambda \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f[u_\theta(a_t n_\lambda)]\pi_1[(a_t n_\lambda)^{-1}]\phi(u_\theta^{-1}g)e^{2t}d\theta dt d\lambda.
\end{aligned}$$

We set: $f^\theta(a_t n_\lambda) = e^{2t}f([u_\theta(a_t n_\lambda)^{-1}]$ and we have $f^\theta \in L^1(G_1)$. Then

$$\begin{aligned}
\widehat{f}(\pi)\phi(g) &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f^\theta[(a_t n_\lambda)^{-1}]\pi_1[(a_t n_\lambda)^{-1}]\phi(u_\theta^{-1}g)d\theta dt d\lambda \\
&= \frac{1}{2\pi}\int_0^{2\pi} \widehat{f^\theta}(\pi_1)\phi(u_\theta^{-1}g)d\theta.
\end{aligned}$$

Now we set:

$$A_f = \{x \in G : f(x) \neq 0\}, \quad B_f = \{\pi \in \widehat{G} : \widehat{f}(\pi) \neq 0\},$$

$$A_{f^\theta} = \{m_1 \in G_1 : f^\theta(m_1) \neq 0\}, \quad B_{f^\theta} = \{\pi_1 \in \widehat{G_1} : \widehat{f^\theta}(\pi_1) \neq 0\}.$$

By the result above , $\widehat{f}(\pi) = 0$ if and only if $\widehat{f^\theta}(\pi_1) = 0$ for all $\theta \in \mathbb{R}$ and for all $g \in G$ i.e. $\pi \notin B_f \Longleftrightarrow \pi_1 \notin B_{f^\theta}$, for all $\theta \in \mathbb{R}$.

Theorem 3.1. *The group $SL(2,\mathbb{R})$ satisfies the qualitative uncertainty principle.*

Proof. Let us assume that $m(A_f)$ and $\mu_G(B_f)$ are finite. We have:

$$\begin{aligned}
m(A_f) &= \int_G 1_{A_f}(x)dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} 1_{A_f}(u_\theta a_t n_\lambda)e^{2t}d\theta dt d\lambda.
\end{aligned}$$

But,

$$\begin{aligned} 1_{A_f}(u_\theta a_t n_\lambda) = 1 &\iff f[u_\theta(a_t n_\lambda)] \neq 0 \\ &\iff e^{-2t} f^\theta[(a_t n_\lambda)^{-1}] \neq 0 \\ &\iff f^\theta[(a_t n_\lambda)^{-1} \neq 0 \\ &\iff 1_{A_{f^\theta}}([(a_t n_\lambda)^{-1}] = 1. \end{aligned}$$

In the same way, $1_{A_f}(u_\theta a_t n_\lambda) = 0 \iff 1_{A_{f^\theta}}[(a_t n_\lambda)^{-1}] = 0$. So,

$$\begin{aligned} m(A_f) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{2\pi} 1_{A_{f^\theta}}[(a_t n_\lambda)^{-1}] e^{2t} dt d\lambda d\lambda \\ &= \int_K \left(\int_{G_1} 1_{A_{fk(m_1)}} dm_1 \right) dk \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} m_1(A_{f^\theta}) d\theta. \end{aligned}$$

If $m(A_f) < \infty$ then $m_1(A_{f^\theta}) < \infty$ for m_1-almost everywhere and for all θ in $\mathbb{R}$. We have proved that $1_{B_{f^\theta}}(\pi_1) = 0$ for all $\theta \in \mathbb{R}$ that is equivalent to $1_{B_f}(\pi) = 0$.

And for all $\theta \in \mathbb{R}, 1_{B_{f^\theta}}(\pi_1) = 1$ implies that $1_{B_f}(\pi) = 1$.

We deduce that for all $\theta \in \mathbb{R}$,

$$1_{B_{f^\theta}}(\pi_1) \leq 1_{B_f}(\pi).$$

It follows that

$$\int_{\widehat{G_1}} 1_{B_{f^\theta}}(\pi_1) d\mu_{\widehat{G_1}}(\pi_1) \leq \int_{\widehat{G}} 1_{B_f}(\pi) d\mu_G(\pi).$$

Then

$$\mu_{\widehat{G_1}}(B_{f^\theta}) \leq \mu_{\widehat{G}}(B_f).$$

Since $\mu_{\widehat{G}}(B_f) < \infty$ then $\mu_{\widehat{G_1}}(B_{f^\theta}) < \infty$. We have $m_1(A_{f^\theta}) < \infty$ and $\mu_{\widehat{G_1}}(B_{f^\theta}) < \infty$ since G_1 satisfies qualitative uncertainty principle then $f^\theta = 0$ for all $\theta \in \mathbb{R}$ and hence $f = 0$. Thus G satisfies the qualitative uncertainty principle. $\square$

REFERENCES

[1]] Bansal, A. Kumar ,A Qualitative Uncertainty Principle for Gabor Transform, Bull. Korean Math. Soc. **54**(2017) 71-84.

[2] B. BOUAL: *A qualitative uncertainty principle for completely solvable Lie groups*, Revista de la union matematica argentina, **48(1) (2007), 41–46.**

[3] B. BOUALI, M. HEMADAOUI: *Principe d'incertitude qualitatif pour les groupes de Lie nilpotents*, **Revista Matematica complutense, 17(2) (2004), 277–285**

[4] G. WARNER: *Harmonic Analysis on Semi-Simple Lie group II*, **Springer-Verlag, Berlin, Heidelberg, New York, 1972.**

[5] H. MEJJAOLI: *Qualitative Uncertainty Principles for the Inverse of the Hypergeometric Fourier Transform*, **Korean J. Math. 23(1) (2015), 129–151.**

[6] J. SHARMA, A. KUMAR: *Qualitative uncertainty principle for Gabor transform on certain locally compact groups*, **Bulletin of the Korean Mathematical Society, 54(1) (2017), 71–84.**

[7] E. KANIUTH: *Qualitative Uncertainty Principles for Group with Finite Dimensional Irreducible Representation*, **J. Funct. Anal. 257 (2009), 340–356.**

[8] M. BENEDICKS: *On Fourier transforms of functions supported on sets of finite Lebesque maesure*, **Journal of mathematical analysis and applications. 106(1) (1985), 180–183.**

[9] M. COWLING, J.F. PRICE, A. SITARAM: *Uncertainty Principle for semisimple Lie Groups*, **J. Austral. Math. Soc. (serieA), 45 (1988), 127–132.**

DEPARTMENT OF MATHEMATICS AND INFORMATIC
UNIVERSITY OF NANGUI ABROGOUA
ABIDJAN, CÔTE D'IVOIRE.
Email address: takiauguste@yahoo.fr

DEPARTMENT OF MATHEMATICS AND INFORMATIC
UNIVERSITY OF MAN
MAN, CÔTE D'IVOIRE.
Email address: broukouakou320@yahoo.fr

DEPARTMENT OF MATHEMATICS AND INFORMATIC
UNIVERSITY OF FELIX HOUPHOUET BOIGNY
ABIDJAN, CÔTE D'IVOIRE.
Email address: kangnikinvi@yahoo.fr