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BLOW UP OF THE WAVE EQUATION WITH NONLINEAR FIRST ORDER PERTURBATION TERM

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ABSTRACT. In this work, we study the wave equation with damping, source and nonlinear first order perturbation terms. Our aim is to prove that if the damping terms dominated the first order perturbation term then the energy is decreasing and the solutions with sufficiently negative initial energy blow up in finite time.

1. INTRODUCTION

In this paper, we consider the following system

$$(1.1)\quad \begin{aligned} &u_{tt} - \Delta u + g * \Delta u + a u_t + F(t, \nabla u) = |u|^{p-2} u \quad \text{in } \Omega \times (0,T), \\ &u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ &u(.,0) = u_0(.) \quad \text{and} \quad u_t(.,0) = u_1(.) \quad \text{in } \Omega, \end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$ with a smooth boundary $\partial\Omega$, $T > 0$, $p > 2$, $a > 0$ are constants, g and F are a functions satisfying some conditions to be specified later. Noting that $(g * v)(t) = \int_0^t g(t-\tau)v(\tau)d\tau$ for all $t \geq 0$.

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When $F = 0$, the problem of existence and nonexistence of global solution has been extensively studied by many researches. **In the absence of the polynomial source term** $|u|^{p-2}u$, Messaoudi [14] in 2005 considered system (1.1) with $a = 0$. He obtained an exponential decay result of the global solution under some conditions on the relaxation function g. In 1988 and 1989, Haraux, Zuazua and Kopackova [9, 10] proved that if $g = 0$, then a nonlinear damping term of polynomial or arbitrary growth assured the global estimates for arbitrary initial data. Cavalcanti et al [4] in 2002 proved that the global solution of the semilinear viscoelastic wave equation with localized damping term decays exponentially to zero. **In the presence of the polynomial source term** $|u|^{p-2}u$, Ball [1] in 1977 proved that if the damping terms are absent, that is for $a = 0$ and $g = 0$, then the solutions blow up when the energy of the initial data is negative. Berrimi and Messaoudi [2] in 2006 considered the case of $a = 0$ and $g \neq 0$, they proved that the solutions decay exponentially or polynomially depending on the relaxation function g. The case of $a \neq 0$ and $g = 0$ was considered by Levine [11] in 1974, he showed that the solutions blow up in finite time under some assumptions on the initial energy. Messaoudi [12] in 2001 proved that if $g = 0$ and the source term dominated the polynomial damping term, then the solutions with negative initial energy blow up in finite time. In 2003, the same author [13] considered, the wave equation with damping terms (polynomial and viscoelastic). Under some assumptions on g, he proved that if the source term dominated the polynomial damping term then the solutions with negative initial energy blow up in finite time and if the polynomial damping term dominated the source term then for any initial data the global solution exists. In 2006, he considered the same system and proved that under some conditions on the relaxation function g, damping and sources terms the solutions with positive initial energy blow up too [15].

When $F \neq 0$ and the polynomial source term is absent, the systems of the second order hyperbolic equation with linear or nonlinear first order perturbation term have been considered in [3, 5–7]. Noting that, the inclusion of this term produce serious additional difficulties since we do not have any information about their influence on the energy of the solution, specially, about the signal of the derivative of the energy. In 2008, Hamchi [8] considered the case of linear first order perturbation term, she introduced a new multiplier to remove the condition of smallness imposed in the literature on the linear perturbation term.

Our aim in this work is to prove that if the damping terms (linear and viscoelastic) dominated the nonlinear first order perturbation term then the energy is decreasing. So, we can define the auxiliary functional L. After that, we show that the solutions with sufficiently negative initial energy blow up in finite time.

This paper consists of two sections in addition to introduction. In section 2, we present some preliminary results needed for our work. In section 3, we give the proof of main result of this work.

2. Preliminary results

In this section, we shall give some preliminary results which will be used throughout this work.

The existence and uniqueness result for system (1.1) is given in the following theorem

Theorem 2.1. *Suppose that*

$$p \leq \frac{2(n-1)}{n-2} \quad \textit{if} \quad n \geq 3.$$

g is a $C^1(\mathbb{R}^+)$ positive decreasing function satisfying

$$1 - \int_0^\infty g(s)ds > 0$$

and F is a $C^1(\mathbb{R}^+ \times \mathbb{R}^n)$ function.

If $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ then there exists a unique maximal strong solution u in $[0, T)$ of system (1.1). *Moreover, the following alternatives hold:*

(1) $T = +\infty$,

or

(2) $T < +\infty$ *and* $\lim_{t \to T} (\| \nabla u(t) \|_2^2 + \| u_t(t) \|_2^2) = +\infty$.

Proof. As in [16]. □

Now, we consider the energy functional for the local solution u of (1.1) defined for all $t \in [0, T)$ by

$$E(t) = \frac{1}{2} \parallel u_t(t) \parallel_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right) \parallel \nabla u(t) \parallel_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p} \parallel u(t) \parallel_p^p,$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau) \parallel v(\tau) - v(t) \parallel_2^2 d\tau.$$

We have

Lemma 2.1. *Assume that the hypotheses of the Theorem* (2.1) *are verified and suppose that*

$$| F(t,U) |^2 \leq 2ag(t) | U |^2, \quad \forall\, t \geq 0, \quad \forall\, U \in \mathbb{R}^n. \tag{2.1}$$

Then E is a decreasing function.

Proof. We multiply the first equation in (1.1) by $u_t(t)$, integrate it over Ω and use Green formula to obtain for all $t \in [0,T)$

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - a \parallel u_t(t) \parallel_2^2 - \frac{1}{2} g(t) \parallel \nabla u(t) \parallel_2^2 - \int_\Omega F(t, \nabla u) u_t(t) dx.$$

Since $g' \leq 0$ then

$$E'(t) \leq -a \parallel u_t(t) \parallel_2^2 - \frac{1}{2} g(t) \parallel \nabla u(t) \parallel_2^2 + \int_\Omega | F(t,\nabla u) || u_t(t) | dx.$$

If we use the following Young inequality

$$XY \leq \frac{\delta^\mu}{\mu} X^\mu + \frac{\delta^{-\theta}}{\theta} Y^\theta, \quad \text{for all} \quad X,Y \geq 0, \quad \delta > 0 \quad \text{and} \quad \frac{1}{\mu} + \frac{1}{\theta} = 1, \tag{2.2}$$

with

$$X = | F(t,\nabla u) |, \; Y = | u_t(t) | \quad \text{and} \quad \mu = \theta = 2$$

we find

$$E'(t) \;\; \leq \;\; \frac{1}{2} \int_\Omega \left[\delta \mid F(t,\nabla u) \mid^2 - g(t) \mid \nabla u(t) \mid^2\right] dx + \left(\frac{1}{2\delta} - a\right) \parallel u_t(t) \parallel_2^2 .$$

If we take $\delta = \dfrac{1}{2a}$ we obtain

$$E'(t) \;\; \leq \;\; \frac{1}{2} \int_\Omega [\frac{1}{2a} \mid F(t,\nabla u) \mid^2 - g(t) \mid \nabla u(t) \mid^2] dx.$$

By (2.1), we find

$$E'(t) \leq 0, \quad \forall t \in [0, T).$$

□

Consider the following functional

$$H(t) = -E(t), \qquad \forall\, t \in [0, T).$$

Lemma 2.2. *Suppose the conditions of Lemma* (2.1) *hold. Assume further that* $E(0) < 0$*. Then*

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \parallel u(t) \parallel_p^p, \qquad \forall\, t \in [0, T).$$

Proof. From the definition of E, H and the decreasing of E. □

3. Main result

In this section, we shall discuss the blow up question of system (1.1).

Theorem 3.1. *Under the assumptions of Lemma* (2.2) *and assume that* a *and* g *verify*

$$\alpha := \frac{p-2}{2} - \frac{3C_* \sqrt{ag(0)}}{2} - (\frac{p-2}{2} + \frac{1}{p}) \int_0^\infty g(\tau) d\tau > 0, \tag{3.1}$$

where C_* *is the best constant of the Poincare inequality then the solution of problem* (1.1) *blows up in finite time.*

Proof. We proceed in 4 steps:
Step 1 Since H is positive then we can define for all $\epsilon > 0$ the auxiliary functional L as follow

$$L(t) = \mathrm{e}^{at} H^{\frac{p+2}{2p}}(t) + \epsilon \mathrm{e}^{at} \int_\Omega u(t) u_t(t) dx, \quad \forall t \in [0, T).$$

If we derive the functional L with respect to t we obtain

$$\begin{aligned} L'(t) &= a\mathrm{e}^{at} H^{\frac{p+2}{2p}}(t) + \frac{p+2}{2p} \mathrm{e}^{at} H^{\frac{2-p}{2p}}(t) H'(t) + \epsilon a \mathrm{e}^{at} \int_\Omega u(t) u_t(t) dx \\ &+ \epsilon \mathrm{e}^{at} \parallel u_t(t) \parallel_2^2 + \epsilon \mathrm{e}^{at} \int_\Omega u(t) u_{tt}(t) dx. \end{aligned}$$

Since H and H' are positive then

$$L'(t) \geq \epsilon a e^{at} \int_\Omega u(t)u_t(t)dx + \epsilon e^{at} \| u_t(t) \|_2^2 + \epsilon e^{at} \int_\Omega u(t)u_{tt}(t)dx. \tag{3.2}$$

If we multiply the first equation of (1.1) by $u(t)$ and integrate it over Ω we obtain

$$\begin{aligned} \int_\Omega u(t)u_{tt}(t)dx &= \int_\Omega \Delta u(t)u(t)dx - \int_\Omega \int_0^t g(t-\tau)\Delta u(\tau)u(t)d\tau dx \\ &- a\int_\Omega u_t(t)u(t)dx - \int_\Omega F(t,\nabla u)u(t)dx + \| u(t) \|_p^p . \end{aligned}$$

If we use Green formula and boundary conditions we find

$$\begin{aligned} \int_\Omega u(t)u_{tt}(t)dx &= -\| \nabla u(t) \|_2^2 + \int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau \\ &- a\int_\Omega u_t(t)u(t)dx - \int_\Omega F(t,\nabla u)u(t)dx + \| u(t) \|_p^p . \end{aligned} \tag{3.3}$$

First, we have

$$\begin{aligned} \int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau &= \int_0^t g(t-\tau)\int_\Omega \nabla u(t).[\nabla u(\tau)-\nabla u(t)]dxd\tau \\ &+ \int_0^t g(\tau)d\tau \| \nabla u(t) \|_2^2 . \end{aligned}$$

By Schwart inequality, we find

$$\begin{aligned} &\int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau \\ \geq & -\int_0^t g(t-\tau) \| \nabla u(t) \|_2 \| \nabla u(\tau)-\nabla u(t) \|_2 \, d\tau \\ + & \int_0^t g(\tau)d\tau \| \nabla u(t) \|_2^2 . \end{aligned}$$

If we exploit Young inequality (2.2) with

$$X = \sqrt{g(t-\tau)} \| \nabla u(\tau)-\nabla u(t) \|_2, \; Y = \sqrt{g(t-\tau)} \| \nabla u(t) \|_2$$

and

$$\mu = \theta = 2,$$

we obtain for all $\beta_1 > 0$

$$\begin{aligned}\int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau &\geq -\beta_1(g\circ\nabla u)(t)\\ &+ \left(1-\frac{1}{4\beta_1}\right)\int_0^t g(\tau)d\tau \parallel \nabla u(t)\parallel_2^2. \end{aligned}\tag{3.4}$$

On the other hand, if we use Young and Poincare inequalities we find for all $\beta_2 > 0$

$$-\int_\Omega F(t,\nabla u)u(t)dx \geq -\frac{1}{2\beta_2}\int_\Omega \mid F(t,\nabla u)\mid^2 dx - \frac{\beta_2}{2}C_*^2 \parallel \nabla u(t)\parallel_2^2, \quad \forall t\in[0,T).$$

By (2.1), we find

$$-\int_\Omega F(t,\nabla u)u(t)dx \geq -\left(\frac{ag(t)}{\beta_2}+\frac{\beta_2}{2}C_*^2\right)\parallel \nabla u(t)\parallel_2^2, \quad \forall t\in[0,T).$$

Since $g' \leq 0$ then

$$-\int_\Omega F(t,\nabla u)u(t)dx \geq -\left(\frac{ag(0)}{\beta_2}+\frac{\beta_2}{2}C_*^2\right)\parallel \nabla u(t)\parallel_2^2, \quad \forall t\in[0,T).\tag{3.5}$$

Now, by the definition of H we have

$$\begin{aligned}\parallel u(t)\parallel_p^p &= \frac{p}{2}\parallel u_t(t)\parallel_2^2 + \frac{p}{2}\left(1-\int_0^t g(\tau)d\tau\right)\parallel \nabla u(t)\parallel_2^2\\ &+ \frac{p}{2}(g\circ\nabla u)(t)+pH(t).\end{aligned}\tag{3.6}$$

Replacing (3.4), (3.5) and (3.6) in (3.3) to obtain

$$\begin{aligned}&\int_\Omega u(t)u_{tt}(t)dx\\ \geq\ & -\parallel \nabla u(t)\parallel_2^2 - \beta_1(g\circ\nabla u)(t) + \left(1-\frac{1}{4\beta_1}\right)\int_0^t g(\tau)d\tau \parallel \nabla u(t)\parallel_2^2\\ -\ & a\int_\Omega u_t(t)u(t)dx - \left(\frac{ag(0)}{\beta_2}+\frac{\beta_2}{2}C_*^2\right)\parallel \nabla u(t)\parallel_2^2 + \frac{p}{2}\parallel u_t(t)\parallel_2^2\\ +\ & \frac{p}{2}\left(1-\int_0^t g(\tau)d\tau\right)\parallel \nabla u(t)\parallel_2^2 + \frac{p}{2}(g\circ\nabla u)(t)+pH(t)\\ =\ & (\frac{p}{2}-\beta_1)(g\circ\nabla u)(t) - a\int_\Omega u_t(t)u(t)dx + \frac{p}{2}\parallel u_t(t)\parallel_2^2 + pH(t)\\ +\ & \left[\frac{p-2}{2} - \frac{ag(0)}{\beta_2} - \frac{\beta_2 C_*^2}{2} - (\frac{p-2}{2}+\frac{1}{4\beta_1})\int_0^t g(\tau)d\tau\right]\parallel \nabla u(t)\parallel_2^2 .\end{aligned}$$

If we take $\beta_1 = \frac{p}{4}$ and $\beta_2 = \frac{\sqrt{ag(0)}}{C_*}$ we find

$$\begin{aligned}\int_\Omega u(t)u_{tt}(t)dx &\geq \frac{p}{4}(g\circ\nabla u)(t) - a\int_\Omega u_t(t)u(t)dx + \frac{p}{2}\parallel u_t(t)\parallel_2^2 + pH(t)\\ &+ \left[\frac{p-2}{2} - \frac{3C_*\sqrt{ag(0)}}{2} - (\frac{p-2}{2}+\frac{1}{p})\int_0^t g(\tau)d\tau\right]\parallel \nabla u(t)\parallel_2^2 .\end{aligned}$$

Since $g \geq 0$ then

$$\begin{aligned}\int_\Omega u(t)u_{tt}(t)dx &\geq \frac{p}{4}(g\circ\nabla u)(t) - a\int_\Omega u_t(t)u(t)dx + \frac{p}{2}\parallel u_t(t)\parallel_2^2 + pH(t)\\ &+ \alpha\parallel \nabla u(t)\parallel_2^2 .\end{aligned} \tag{3.7}$$

Replacing (3.7) in (3.2) to find

$$\begin{aligned}L'(t) &\geq \epsilon e^{at}(1+\frac{p}{2})\parallel u_t(t)\parallel_2^2 + \epsilon e^{at}\frac{p}{4}(g\circ\nabla u)(t) + \epsilon e^{at}pH(t)\\ &+ \epsilon e^{at}\alpha\parallel \nabla u(t)\parallel_2^2 .\end{aligned}$$

Let $\beta > 0$. By writing $p = 2\beta + (p-2\beta)$ and since

$$H(t) \geq \frac{1}{p}\parallel u(t)\parallel_p^p - \frac{1}{2}\parallel u_t(t)\parallel_2^2 - \frac{1}{2}\parallel \nabla u(t)\parallel_2^2 - \frac{1}{2}(g\circ\nabla u)(t)$$

we obtain

$$\begin{aligned}L'(t) &\geq \epsilon e^{at}(1+\frac{p}{2}-\beta)\parallel u_t(t)\parallel_2^2 + \epsilon e^{at}(\frac{p}{4}-\beta)(g\circ\nabla u)(t) + \epsilon e^{at}(p-2\beta)H(t)\\ &+ \epsilon e^{at}(\alpha-\beta)\parallel \nabla u(t)\parallel_2^2 + \epsilon e^{at}\frac{2\beta}{p}\parallel u(t)\parallel_p^p .\end{aligned} \tag{3.8}$$

If we take $\beta < \min\left(\frac{p}{4},\alpha\right)$, inequality (3.8) takes the form

$$L'(t) \geq Ce^{at}[H(t) + \parallel u_t(t)\parallel_2^2 + (g\circ\nabla u)(t) + \parallel u(t)\parallel_p^p], \quad \forall t\in[0,T]. \tag{3.9}$$

Step 2 We have

$$L(0) = H^{\frac{p+2}{2p}}(0) + \epsilon\int_\Omega u_0u_1dx.$$

If

$$\int_\Omega u_0u_1dx \geq 0,$$

then

$$L(0) \geq 0.$$

If

$$\int_\Omega u_0 u_1 dx < 0,$$

then, if we take

$$\epsilon < \frac{-H^{\frac{p+2}{2p}}(0)}{\int_\Omega u_0 u_1 dx}$$

we obtain

$$L(0) \geq 0.$$

then from the increase of L, we find that

$$L(t) \geq 0, \quad \forall t \in [0, T).$$

Step 3 By the definition of L and the following inequality

$$(\mu + \theta)^m \leq 2^m(\mu^m + \theta^m), \quad \text{for all} \quad \mu, \theta \geq 0 \quad \text{and} \quad m > 0$$

with

$$\mu = H(t), \quad \theta = \epsilon \int_\Omega | u(t) || u_t(t) | dx \quad \text{and} \quad m = \frac{2p}{p+2}$$

we obtain

$$\begin{aligned} L^{\frac{2p}{p+2}}(t) &\leq 2^{\frac{2p}{p+2}} \mathrm{e}^{\frac{2pat}{p+2}} \left[H(t) + \epsilon^{\frac{2p}{p+2}} (\int_\Omega | u(t) || u_t(t) | dx)^{\frac{2p}{p+2}} \right] \\ &\leq C \mathrm{e}^{\frac{2pat}{p+2}} \left[H(t) + \left(\int_\Omega | u(t) || u_t(t) | dx \right)^{\frac{2p}{p+2}} \right], \end{aligned}$$

where C is a generic positive constant.

If we use Schwarz inequality and the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$ we find

$$L^{\frac{2p}{p+2}}(t) \leq C \mathrm{e}^{\frac{2pat}{p+2}} \left[H(t) + \| u(t) \|_p^{\frac{2p}{p+2}} \| u_t(t) \|_2^{\frac{2p}{p+2}} \right].$$

If we use Young inequality (2.2) with

$$X = \| u(t) \|_p^{\frac{2p}{p+2}},\ Y = \| u_t(t) \|_2^{\frac{2p}{p+2}},\ \mu = \frac{p+2}{2} \quad \text{and} \quad \theta = \frac{p+2}{p}$$

we find

$$L^{\frac{2p}{p+2}}(t) \leq C e^{\frac{2pat}{p+2}} \left[H(t) + \| u_t(t) \|_2^2 + \| u(t) \|_p^p \right]. \tag{3.10}$$

Step 4 We proceed by contradiction, we assume that $T = +\infty$. By combining (3.9) and (3.10), we arrive at

$$L'(t) L^{-\frac{2p}{p+2}}(t) \geq C e^{\frac{(2-p)at}{p+2}}.$$

A simple integration over $(0, t)$ gives

$$L(t) \geq \frac{1}{\left[L^{\frac{2-p}{p+2}}(0) - C \left(1 - e^{\frac{(2-p)at}{p+2}} \right) \right]^{\frac{p-2}{p+2}}}.$$

This leads to a contradiction. □

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