ON DUAL SLANT HELICES IN $\mathbb{D}^3$

Derya Sağlam

ABSTRACT. In this paper, by using the method in [6] we study dual tangent indicatrix and dual binormal indicatrix of a dual slant helix. Moreover we obtain the relationship between the dual slant helices and dual general helices in $\mathbb{D}^3$. We get some characterizations of a dual slant helix in $\mathbb{D}^3$.

1. INTRODUCTION

In 1873, William Clifford introduced dual numbers. Later, Eduard Study defined the dual angle. The dual angle $\tilde{\omega} = \omega + \varepsilon \omega^*$ is defined by

$$\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{b}} \rangle = \cos \tilde{\omega} = \cos \omega - \varepsilon \omega^* \sin \omega.$$ 

$\omega$ is the real angle between two lines corresponding to the dual unit vectors $\overrightarrow{\tilde{a}}, \overrightarrow{\tilde{b}}$ and $\omega^*$ is the shortest distance between the lines. Hermann Grassmann generalized dual numbers and defined the Grassmann number at the end of the 19th century.

Each straightline in $\mathbb{R}^3$ corresponds to unique a point of a dual unit sphere. The dual points of dual unit sphere match the lines in $\mathbb{R}^3$ and in general two different
points on $\mathbb{D}^3$ represent two skew-lines in $\mathbb{R}^3$. A dual space curve on dual unit sphere in $\mathbb{D}^3$ represents a ruled surface in $\mathbb{R}^3$ \cite{1,2}.

A curve is called a general helix if the tangent vector make a constant angle with a fixed direction \cite{4,6}. A curve is called a slant helix in $\mathbb{R}^3$ if the principal normal vector makes a constant angle with a fixed direction \cite{3}. Moreover, a curve is a slant helix iff the geodesic curvature of the principal normal vector of the curve

\[ \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left( \frac{\tau}{\kappa} \right)' \]

is constant.

In \cite{4}, the tangent indicatrix and the binormal indicatrix of a slant helix of $\mathbb{R}^3$ were studied. In \cite{6} the relations was investigated between a general helix and a slant helix.

In this paper, by using the method in \cite{6} we study dual tangent indicatrix and dual binormal indicatrix of a dual slant helix. Moreover we obtain the relationship between the dual slant helices and dual general helices in $\mathbb{D}^3$. We get some characterizations of a dual slant helix in $\mathbb{D}^3$.

2. Preliminaries

The dual numbers set is defined with $\mathbb{D} = \mathbb{R} \times \mathbb{R}$. The dual number $(0, 1)$ is denoted by $\varepsilon$. From the definition of the multiplication operation, $\varepsilon^2 = 0$. Also the dual number $\tilde{a} = (a, a^*) \in \mathbb{D}$ can be written as $\tilde{a} = a + \varepsilon a^*$. The set of

$\mathbb{D} = \{ \tilde{a} = a + \varepsilon a^* \mid a, a^* \in \mathbb{R} \}$

of dual numbers is a commutative ring following with the operations

(i) $\tilde{a} + \tilde{b} = (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$,

(ii) $\tilde{a} \tilde{b} = (a + \varepsilon a^*)(b + \varepsilon b^*) = ab + \varepsilon(ab^* + a^*b)$.

The division is defined by

\[ \frac{\tilde{a}}{\tilde{b}} = \frac{a + \varepsilon a^*}{b + \varepsilon b^*} = \frac{a}{b} + \varepsilon \frac{a^*b - ab^*}{b^2} \quad b \neq 0. \]

We define a dual vector with a vector of dual numbers $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ and the set

$\mathbb{D}^3 = \{ \tilde{a} \mid \tilde{a} = (a_1 + \varepsilon a_1^*, a_2 + \varepsilon a_2^*, a_3 + \varepsilon a_3^*) = \tilde{a}^* + \varepsilon \tilde{a}^* \subseteq \mathbb{R}^3 \}$,
where \( \overrightarrow{a} = (a_1, a_2, a_3), \overrightarrow{a^*} = (a_1^*, a_2^*, a_3^*) \), a module on the ring \( D \). The scalar product and the vector product of \( \overrightarrow{a} \) and \( \overrightarrow{b} \) are defined by

\[
\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left( \left\langle \overrightarrow{a}, \overrightarrow{a^*} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle \right)
\]

and

\[
\overrightarrow{a} \times \overrightarrow{b} = (\tilde{a}_2b_3 - \tilde{a}_3b_2, \tilde{a}_3b_1 - \tilde{a}_1b_3, \tilde{a}_1b_2 - \tilde{a}_2b_1),
\]

respectively. The norm is defined by

\[
\left\| \overrightarrow{a} \right\| = \sqrt{\left\langle \overrightarrow{a}, \overrightarrow{a} \right\rangle} = \left\| \overrightarrow{a} \right\| + \varepsilon \left\| \overrightarrow{a^*} \right\|, \quad a \neq 0.
\]

A dual vector \( \overrightarrow{a} \) with norm 1 is called a dual unit vector. Let \( \overrightarrow{a} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*} \in \mathbb{D}^3 \).

The set

\[
S^2 = \{ \overrightarrow{a} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*} | \left\| \overrightarrow{a} \right\| = (1, 0); \quad \overrightarrow{a}, \overrightarrow{a^*} \in \mathbb{R}^3 \}
\]

is called the dual unit sphere in \( \mathbb{D}^3 \).

E. Study mapped dual unit vectors one-to-one with directed lines by dual unit vectors. Moreover a ruled surface is represented by the dual curves on the dual unit sphere in \( \mathbb{D}^3 \).

The dual angle \( \tilde{\omega} = \omega + \varepsilon \omega^* \) is defined by

\[
\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle = \cos \tilde{\omega} = \cos \omega - \varepsilon \omega^* \sin \omega.
\]

\( \omega \) is the real angle between two lines corresponding to the dual unit vectors \( \overrightarrow{a}, \overrightarrow{b} \) and \( \omega^* \) is the shortest distance between the lines.

The dual space curve \( \tilde{\psi} : I \subset \mathbb{R} \rightarrow \mathbb{D}^3 \) is differentiable, if every \( \psi_i, \psi_i^* : I \rightarrow \mathbb{R}, \) for \( 1 \leq i \leq 3 \), are differentiable. The dual curve \( \tilde{\psi} \) is

\[
\tilde{\psi}(t) = (\psi_1(t) + \varepsilon \psi_1^*(t), \psi_2(t) + \varepsilon \psi_2^*(t), \psi_3(t) + \varepsilon \psi_3^*(t)) = \overrightarrow{\psi}(t) + \varepsilon \overrightarrow{\psi^*}(t),
\]

where \( \overrightarrow{\psi}(t) = (\psi_1(t), \psi_2(t), \psi_3(t)) \) and \( \overrightarrow{\psi^*}(t) = (\psi_1^*(t), \psi_2^*(t), \psi_3^*(t)) \). The dual arclength of the curve \( \tilde{\psi} \) from \( t_1 \) to \( t \) is defined following

\[
(2.1) \quad \tilde{s} = \int_{t_1}^{t} \left\| \tilde{\psi}'(t) \right\| dt = \int_{t_1}^{t} \left\| \tilde{\psi}'(t) \right\| dt + \varepsilon \int_{t_1}^{t} \left\langle \overrightarrow{T(t), \tilde{\psi}^*(t)} \right\rangle dt = s + \varepsilon s^*,
\]
where $\vec{T}$ is a unit tangent of $\vec{\psi}$. We define the dual unit tangent vector of $\vec{\psi}$ following

$$\vec{T} = \frac{d\vec{\psi}}{d\tilde{s}} = \frac{d\tilde{\psi}}{d\tilde{s}} = \frac{d\tilde{s}}{ds}.$$

The function $\tilde{\kappa} : I \to \mathbb{D}$, $\tilde{\kappa} = \left| \frac{d\vec{T}}{d\tilde{s}} \right|$ is called the dual curvature function of $\vec{\psi}$.

We assume that $\tilde{\kappa} : I \to \mathbb{D}$ is never pure dual. Then, we can define the principal normal of $\vec{\psi}$ with the dual unit vector $\vec{N} = \frac{1}{\tilde{\kappa}} \frac{d\vec{T}}{d\tilde{s}}$. The dual vector $\vec{B} = \vec{T} \times \vec{N}$ is called the binormal of $\vec{\psi}$. Also one obtain that

$$d\frac{d\vec{\psi}}{d\tilde{s}} = \left[ \begin{array}{c} \vec{T} \\ \vec{N} \\ \vec{B} \end{array} \right] = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -\tilde{\kappa} & 0 & \tilde{\tau} \\ 0 & -\tilde{\tau} & 0 \end{array} \right] \left[ \begin{array}{c} \vec{T} \\ \vec{N} \\ \vec{B} \end{array} \right],$$

where $\tilde{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\tilde{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion [1].

A dual unit speed curve $\vec{\psi} : I \to \mathbb{D}^3$ is a dual general helix if $\left\langle \vec{T}, \vec{U} \right\rangle = \cos \tilde{\omega}$ is dual constant for some dual constant vector $\vec{U}$. The function $\frac{\tilde{\tau}}{\tilde{\kappa}}$ is a dual constant iff the dual curve $\vec{\psi}$ is a dual general helix. The dual curve is a dual circular helix, if the curvatures $\tilde{\kappa}$ and $\tilde{\tau}$ are a dual constant number (never pure dual).

The dual unit tangent vectors of the dual unit regular curve $\vec{\psi}$ define a dual curve $\vec{\alpha}$ on $\mathbb{S}^2$. We call that $\vec{\alpha}$ is dual tangent indicatrix of the dual curve $\vec{\psi}$. Similarly we define the dual principal normal indicatrix $\vec{\beta} = \vec{N}$ and dual binormal indicatrix $\vec{\gamma} = \vec{B}$.

Let $\{\vec{\alpha}, \vec{N}, \vec{B}\}$ be the dual Frenet frame of the dual tangent indicatrix $\vec{\alpha}$ of a dual curve $\vec{\psi}$, then

$$d\frac{d\vec{\alpha}}{d\tilde{s}} = \left[ \begin{array}{c} \vec{T} \\ \vec{N} \\ \vec{B} \end{array} \right] = \left[ \begin{array}{ccc} 0 & \tilde{\kappa} & 0 \\ -\tilde{\kappa} & 0 & \tilde{\tau} \\ 0 & -\tilde{\tau} & 0 \end{array} \right] \left[ \begin{array}{c} \vec{T} \\ \vec{N} \\ \vec{B} \end{array} \right].$$
where
\[
\overrightarrow{T_{\tilde{\alpha}}} = \overrightarrow{N} \\
\overrightarrow{N_{\tilde{\alpha}}} = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(-\tilde{\kappa} \overrightarrow{T} + \tilde{\tau} \overrightarrow{B}) \\
\overrightarrow{B_{\tilde{\alpha}}} = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(\tilde{\tau} \overrightarrow{T} + \tilde{\kappa} \overrightarrow{B})
\]
and the dual curvatures of \(\tilde{\alpha}\) are
\[
\tilde{\kappa}_{\tilde{\alpha}} = \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa}, \quad \tilde{\tau}_{\tilde{\alpha}} = \frac{\kappa \kappa' - \kappa''}{\kappa(\kappa^2 + \tau^2)}.
\]
Let \(\{\overrightarrow{T_{\tilde{\beta}}}, \overrightarrow{N_{\tilde{\beta}}}, \overrightarrow{B_{\tilde{\beta}}}\}\) be the dual Frenet frame of the dual principal normal indicatrix \(\tilde{\psi}\) of a dual curve \(\tilde{\psi}\), then
\[
\frac{d}{ds} \begin{bmatrix} \overrightarrow{T_{\tilde{\beta}}} \\ \overrightarrow{N_{\tilde{\beta}}} \\ \overrightarrow{B_{\tilde{\beta}}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\kappa}_{\tilde{\beta}} & 0 \\ -\tilde{\kappa}_{\tilde{\beta}} & 0 & \tilde{\tau}_{\tilde{\beta}} \\ 0 & -\tilde{\tau}_{\tilde{\beta}} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T_{\tilde{\beta}}} \\ \overrightarrow{N_{\tilde{\beta}}} \\ \overrightarrow{B_{\tilde{\beta}}} \end{bmatrix},
\]
where
\[
\overrightarrow{T_{\tilde{\beta}}} = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(-\tilde{\kappa} \overrightarrow{T} + \tilde{\tau} \overrightarrow{B}) \\
\overrightarrow{N_{\tilde{\beta}}} = \frac{1}{\sqrt{(\kappa^2 + \tau^2)(\kappa \kappa' - \kappa'' \tau)^2 + (\kappa^2 + \tau^2)^4}} \cdot [(\kappa \kappa' - \kappa'' \tau)(\tilde{\tau} \overrightarrow{T} + \tilde{\kappa} \overrightarrow{B}) - (\kappa^2 + \tau^2)^2 \overrightarrow{N}] \\
\overrightarrow{B_{\tilde{\beta}}} = \frac{1}{\sqrt{(\kappa \kappa' - \kappa'' \tau)^2 + (\kappa^2 + \tau^2)^3}} \cdot [(\kappa^2 + \tau^2)(\tilde{\tau} \overrightarrow{T} + \tilde{\kappa} \overrightarrow{B}) + (\kappa \kappa' - \kappa'' \tau) \overrightarrow{N}],
\]
and the dual curvatures of \(\tilde{\beta}\) are
\[
\tilde{\kappa}_{\tilde{\beta}} = \sqrt{(\kappa \kappa' - \kappa'' \tau)^2 + (\kappa^2 + \tau^2)^3}, \\
\tilde{\tau}_{\tilde{\beta}} = \frac{[(\kappa \kappa'' - \kappa'' \tau)(\kappa^2 + \tau^2) - 3(\kappa \kappa' - \kappa'' \tau) (\kappa \kappa' + \kappa'' \tau)]}{(\kappa^2 + \tau^2)^3 + (\kappa \kappa' - \kappa'' \tau)^2}. 
\]
Let \( \overrightarrow{T_\tilde{\gamma}}, \overrightarrow{N_\tilde{\gamma}}, \overrightarrow{B_\tilde{\gamma}} \) be the dual Frenet frame of the dual binormal indicatrix \( \tilde{\gamma} \) of a dual curve \( \tilde{\psi} \), then

\[
\frac{d}{ds} \begin{bmatrix} \overrightarrow{T_\tilde{\gamma}} \\ \overrightarrow{N_\tilde{\gamma}} \\ \overrightarrow{B_\tilde{\gamma}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\kappa}_\tilde{\gamma} & 0 \\ -\tilde{\kappa}_\tilde{\gamma} & 0 & \tilde{\tau}_\tilde{\gamma} \\ 0 & -\tilde{\tau}_\tilde{\gamma} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T_\tilde{\gamma}} \\ \overrightarrow{N_\tilde{\gamma}} \\ \overrightarrow{B_\tilde{\gamma}} \end{bmatrix},
\]

where

\[
\overrightarrow{T_\tilde{\gamma}} = -\overrightarrow{N_\tilde{\gamma}}, \quad \overrightarrow{N_\tilde{\gamma}} = \frac{1}{\sqrt{\tilde{\kappa}^2 + \tilde{\tau}^2}}(\tilde{\kappa}\overrightarrow{T} - \tilde{\tau}\overrightarrow{B}), \quad \overrightarrow{B_\tilde{\gamma}} = \frac{1}{\sqrt{\tilde{\kappa}^2 + \tilde{\tau}^2}}(\tilde{\tau}\overrightarrow{T} + \tilde{\kappa}\overrightarrow{B})
\]

and the dual curvatures of \( \tilde{\gamma} \) are

\[
\tilde{\kappa}_\tilde{\gamma} = \frac{\sqrt{\tilde{\kappa}^2 + \tilde{\tau}^2}}{\tilde{\tau}}, \quad \tilde{\tau}_\tilde{\gamma} = -\frac{(\tilde{\tau}\tilde{\kappa}' - \tilde{\kappa}\tilde{\tau}')}{\tilde{\tau}(\tilde{\kappa}^2 + \tilde{\tau}^2)}.
\]

In this paper we will assume that dual curvatures \( \tilde{\kappa} \) and \( \tilde{\tau} \) of the dual curve \( \tilde{\psi} \) are nowhere pure dual.

3. The Dual Spherical Indicatrices of Dual Slant Helices

As in Euclidean space, the necessary and sufficient condition for a dual unit curve \( \tilde{\psi} \) to be a dual slant helix is shown that the following the dual function

\[
\frac{\tilde{\kappa}^2}{(\tilde{\kappa}^2 + \tilde{\tau}^2)^{3/2}} \left(\frac{\tilde{\tau}}{\tilde{\kappa}}\right)' = \text{dual constant}.
\]

From now on, let’s denote the function \( \frac{\tilde{\kappa}^2}{(\tilde{\kappa}^2 + \tilde{\tau}^2)^{3/2}} \left(\frac{\tilde{\tau}}{\tilde{\kappa}}\right)' \) with \( \tilde{\sigma} \).

**Theorem 3.1.** The dual tangent indicatrix \( \tilde{\alpha} \) of a dual unit speed slant helix \( \tilde{\psi} \) is a dual spherical helix.

**Proof.** From the equations in (2.2), we obtain that

\[
\frac{\tilde{\tau}_{\tilde{\alpha}}}{\tilde{\kappa}_{\tilde{\alpha}}} = \tilde{\sigma}.
\]
According to (3.1), the dual function $\tilde{\tau}/\tilde{\kappa}$ is a dual constant. This completes the proof.

**Theorem 3.2.** The dual binormal indicatrix $\tilde{\gamma}$ of a dual unit speed slant helix $\tilde{\psi}$ is a dual spherical helix.

**Proof.** From the equations in (2.4) we have

$$\frac{\tilde{\tau}}{\tilde{\kappa}} = -\tilde{\sigma}.$$  

According to (3.1), the dual function $\tilde{\tau}/\tilde{\kappa}$ is a dual constant. This completes the proof.

---

4. SOME CHARACTERIZATIONS OF DUAL SLANT HELICES

**Theorem 4.1.** $\tilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff

$$\frac{d^2 \tilde{\vec{N}}_\beta}{ds^2} + \tilde{\kappa}_\beta^2 \tilde{\vec{N}}_\beta = 0.$$  

**Proof.** Let $\tilde{\psi}$ be a dual slant helix. From the equations in (2.8) the dual curvatures of $\tilde{\beta}$ are

$$\tilde{\kappa}_\beta = \sqrt{1 + \tilde{\sigma}^2}$$  

and

$$\tilde{\tau}_\beta = \frac{(\tilde{k}^2 + \tilde{\tau}^2)^{5/2}}{(\tilde{k}^2 + \tilde{\tau}^2)^{3/2} + (\kappa^2 + \tilde{\tau}^2)^{3/2}}.$$  

Since $\tilde{\sigma} = \frac{\tilde{k}^2}{(\tilde{k}^2 + \tilde{\tau}^2)^{3/2}} (\tilde{\tau}/\tilde{\kappa})'$ is a nowhere pure dual constant function, we get $\tilde{\kappa}_\beta = 0$. Also the dual principal normal indicatrix of $\tilde{\psi}$ is a dual circle. From (2.7), we have

$$\frac{d^2 \tilde{\vec{N}}_\beta}{ds^2} + \tilde{\kappa}_\beta^2 \tilde{\vec{N}}_\beta = 0.$$  

Conversely, let the equation (4.1) be provided. From (2.7)
Theorem 4.2. The dual curve \( \tilde{\psi} : I \rightarrow \mathbb{D}^3 \) is a dual slant helix iff
\[
\frac{d^3 \tilde{T}_{\tilde{\alpha}}}{ds^3} - 3 \tilde{\kappa}_{\tilde{\alpha}} \frac{d^2 \tilde{T}_{\tilde{\alpha}}}{ds^2} + \left( \tilde{\kappa}_{\tilde{\alpha}} \right)^2 - 3 \left( \frac{\tilde{\kappa}_{\tilde{\alpha}}}{\tilde{\kappa}_{\tilde{\alpha}}} \right)^2 \frac{d \tilde{T}_{\tilde{\alpha}}}{ds} = 0,
\]
where \( \tilde{\lambda}_1 \neq 1 \) is dual constant (\( \tilde{\lambda}_1 = 1 + \frac{1}{\tilde{c}_1} \) and \( \tilde{c}_1 \) is nowhere pure dual number).

Proof. We assume that \( \tilde{\psi} \) is a dual slant helix. Also the dual tangent indicatrix \( \tilde{\alpha} \) of \( \tilde{\psi} \) is a dual general helix. From (2.3), we have \( \frac{d \tilde{T}_{\tilde{\alpha}}}{ds} = \tilde{\kappa}_{\tilde{\alpha}} \tilde{N}_{\tilde{\alpha}} \). By differentiating this equation, we get
\[
\frac{d^3 \tilde{T}_{\tilde{\alpha}}}{ds^3} = -2 \tilde{\kappa}_{\tilde{\alpha}} \tilde{\kappa}'_{\tilde{\alpha}} \tilde{T}_{\tilde{\alpha}} - 2 \tilde{\kappa}'_{\tilde{\alpha}} \tilde{N}_{\tilde{\alpha}} + 2 \tilde{\kappa}''_{\tilde{\alpha}} \tilde{B}_{\tilde{\alpha}} + \tilde{\kappa}_{\tilde{\alpha}} \tilde{\tau}_{\tilde{\alpha}} \frac{d \tilde{B}_{\tilde{\alpha}}}{ds} = 0.
\]
By using (2.3), we get (4.5).

Conversely, we suppose that (4.5) is valid. From (2.3), we obtain
\[
\tilde{B}_{\tilde{\alpha}} = \frac{1}{\tilde{\tau}_{\tilde{\alpha}}} \frac{d \tilde{N}_{\tilde{\alpha}}}{ds} + \frac{\tilde{\kappa}_{\tilde{\alpha}}}{\tilde{\tau}_{\tilde{\alpha}}} \tilde{T}_{\tilde{\alpha}}.
\]
Differentiating the last equality, we have
\[
\frac{d \tilde{B}_{\tilde{\alpha}}}{ds} = \frac{1}{\tilde{\kappa}_{\tilde{\alpha}} \tilde{\tau}_{\tilde{\alpha}}} \left\{ d^3 \tilde{T}_{\tilde{\alpha}} - 3 \tilde{\kappa}'_{\tilde{\alpha}} \frac{d^2 \tilde{T}_{\tilde{\alpha}}}{ds^2} \right\} \left( \tilde{\kappa}_{\tilde{\alpha}} \right)^2 - 3 \left( \frac{\tilde{\kappa}_{\tilde{\alpha}}}{\tilde{\kappa}_{\tilde{\alpha}}} \right)^2 \frac{d \tilde{T}_{\tilde{\alpha}}}{ds}
\]
\[
+ \frac{1}{\tilde{\kappa}_{\tilde{\alpha}} \tilde{\tau}_{\tilde{\alpha}}} \left( \tilde{\kappa}_{\tilde{\alpha}} \right)' \frac{d^2 \tilde{T}_{\tilde{\alpha}}}{ds^2} - \left( \tilde{\kappa}_{\tilde{\alpha}} \right)' \tilde{A}_{\tilde{\alpha}} + \tilde{\kappa}'_{\tilde{\alpha}} \left( \tilde{\kappa}_{\tilde{\alpha}} \right)' \tilde{B}_{\tilde{\alpha}} + \left( \tilde{\kappa}_{\tilde{\alpha}} \right)' \frac{d \tilde{T}_{\tilde{\alpha}}}{ds}.
\]
By using (2.3) and (4.5), we get
\[
\left( \frac{\tilde{\kappa}_{\tilde{\alpha}}}{\tilde{\tau}_{\tilde{\alpha}}} \right)' = 0 \quad \text{and} \quad \frac{\tilde{\kappa}_{\tilde{\alpha}}}{\tilde{\tau}_{\tilde{\alpha}}} = \sqrt{\frac{1}{\lambda_1 - 1}} = \tilde{c}_1 \quad \text{(nowhere pure dual constant)}.
\]
Thus from (3.1), we obtain \( \tilde{\sigma}_\beta = \frac{\tilde{\tau}_\alpha}{\tilde{\kappa}_\alpha} \) = dual constant, hence \( \tilde{\psi} \) is a dual slant helix.

\[ \Box \]

**Theorem 4.3.** The dual curve \( \tilde{\psi} : I \to \mathbb{D}^3 \) is a dual slant helix iff

\[(4.9) \quad \frac{d^3 \tilde{T}_\tilde{\alpha}}{ds^3} - 3 \frac{\tilde{\tau}_\alpha^2}{\tilde{\kappa}_\alpha} \frac{d^2 \tilde{T}_\tilde{\alpha}}{ds^2} - \left[ \frac{\tilde{\tau}_\alpha^2}{\tilde{\kappa}_\alpha} - 3 \left( \frac{\tilde{\tau}_\alpha}{\tilde{\kappa}_\alpha} \right)^2 - \tilde{\mu}_1 \tilde{\tau}_\alpha^2 \right] \frac{d \tilde{T}_\tilde{\alpha}}{ds} = 0, \]

where \( \tilde{\mu}_1 \neq 1 \) is a dual constant (\( \tilde{\mu}_1 = 1 + \tilde{c}_1^2 \) and \( \tilde{c}_1 \) is nowhere pure dual number).

**Theorem 4.4.** The dual curve \( \tilde{\psi} : I \to \mathbb{D}^3 \) is a dual slant helix iff

\[(4.10) \quad \frac{d^2 \tilde{N}_\tilde{\alpha}}{ds^2} - \frac{\tilde{\kappa}'_\alpha}{\tilde{\kappa}_\alpha} \frac{d \tilde{N}_\tilde{\alpha}}{ds} + \tilde{\lambda}_1 \tilde{\tau}_\alpha^2 \frac{d \tilde{T}_\tilde{\alpha}}{ds} = 0, \]

where \( \tilde{\lambda}_1 \neq 1 \) is dual constant (\( \tilde{\lambda}_1 = 1 + \frac{1}{\tilde{c}_1} \) and \( \tilde{c}_1 \) is nowhere pure dual number).

**Proof.** Let \( \tilde{\psi} \) be a dual slant helix. Hence the dual tangent indicatrix \( \tilde{\alpha} \) of \( \tilde{\psi} \) is a dual general helix. We differentiate \( \frac{d \tilde{\beta}}{ds} = -\tilde{\kappa}_\alpha \tilde{T}_\tilde{\alpha} + \tilde{\tau}_\alpha \tilde{B}_\tilde{\alpha}, \) we get

\[(4.11) \quad \frac{d^2 \tilde{T}_\tilde{\alpha}}{ds^2} = -\tilde{\kappa}'_\alpha \tilde{T}_\tilde{\alpha} + \tilde{\tau}'_\alpha \tilde{B}_\tilde{\alpha} - \left( \tilde{\kappa}_\alpha^2 + \tilde{\tau}_\alpha^2 \right) \tilde{N}_\tilde{\alpha}. \]

From the equation (2.3) and (4.11), we obtain (4.10).

Conversely, let the equation (4.10) be provided. According to (2.3), we obtain

\[(4.12) \quad \frac{d \tilde{N}_\tilde{\alpha}}{ds} = -\frac{1}{\tilde{\kappa}_\alpha} \frac{d \tilde{N}_\tilde{\alpha}}{ds} + \tilde{\tau}_\alpha \tilde{B}_\tilde{\alpha}. \]

Differentiating the last equality, we have

\[(4.13) \quad \frac{d^2 \tilde{N}_\tilde{\alpha}}{ds^2} = \left[ \frac{d^2 \tilde{N}_\tilde{\alpha}}{ds^2} - \frac{\tilde{\kappa}'_\alpha}{\tilde{\kappa}_\alpha} \frac{d \tilde{N}_\tilde{\alpha}}{ds} + \left( \tilde{\kappa}_\alpha^2 + \tilde{\tau}_\alpha^2 \right) \tilde{N}_\tilde{\alpha} \right] + \tilde{\kappa}_\alpha \tilde{N}_\tilde{\alpha} + \left( \frac{\tilde{\tau}_\alpha}{\tilde{\kappa}_\alpha} \right)' \tilde{B}_\tilde{\alpha}. \]

By using (2.3) and (4.10), we get

\[ \left( \frac{\tilde{\tau}_\alpha}{\tilde{\kappa}_\alpha} \right)' = 0 \quad \text{and} \quad \frac{\tilde{\kappa}_\alpha}{\tilde{\tau}_\alpha} = \sqrt{\frac{1}{\tilde{\lambda}_1} - 1} = \tilde{c}_1 \quad \text{(nowhere pure dual constant)}. \]

Thus from (3.1), we obtain \( \tilde{\sigma}_\beta = \frac{\tilde{\tau}_\alpha}{\tilde{\kappa}_\alpha} \) = dual constant. Also \( \tilde{\psi} \) is a dual slant helix.

\[ \Box \]
Theorem 4.5. The dual curve $\tilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff

$$
\frac{d^2 \overrightarrow{N}}{ds^2} - \frac{\tilde{r}_0'}{\tilde{r}_0} \frac{d \overrightarrow{N}}{ds} + \tilde{\mu}_1 \tilde{r}_0^2 \overrightarrow{N} = 0,
$$

where $\tilde{\mu}_1 \neq 1$ is dual constant ($\tilde{\mu}_1 = 1 + \frac{1}{\tilde{c}_1^2}$ and $\tilde{c}_1$ is nowhere pure dual number).

Theorem 4.6. The dual curve $\tilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff

$$
\frac{d^3 \overrightarrow{B}}{ds^3} - 3 \frac{\tilde{r}_0'}{\tilde{r}_0} \frac{d^2 \overrightarrow{B}}{ds^2} - \left[\frac{\tilde{r}_0''}{\tilde{r}_0} - 3 \left(\frac{\tilde{r}_0'}{\tilde{r}_0}\right)^2 - \tilde{\lambda}_1 \tilde{r}_0^2\right] \frac{d \overrightarrow{B}}{ds} = 0,
$$

where $\tilde{\lambda}_1 \neq 1$ is dual constant ($\tilde{\lambda}_1 = 1 + \frac{1}{\tilde{c}_1^2}$ and $\tilde{c}_1$ is nowhere pure dual number).

Theorem 4.7. The dual curve $\tilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff

$$
\frac{d^3 \overrightarrow{T}}{ds^3} - 3 \frac{\tilde{r}_0'}{\tilde{r}_0} \frac{d^2 \overrightarrow{T}}{ds^2} - \left[\frac{\tilde{r}_0''}{\tilde{r}_0} - 3 \left(\frac{\tilde{r}_0'}{\tilde{r}_0}\right)^2 - \tilde{\lambda}_2 \tilde{r}_0^2\right] \frac{d \overrightarrow{T}}{ds} = 0,
$$

where $\tilde{\lambda}_2 \neq 1$ is dual constant ($\tilde{\lambda}_2 = 1 + \frac{1}{\tilde{c}_2^2}$ and $\tilde{c}_2$ is nowhere pure dual number).

Proof. We suppose that $\tilde{\psi}$ is a dual slant helix. Also the dual binormal indicatrix $\tilde{T}$ of $\tilde{\psi}$ is a dual general helix. If we differentiate following equation $\frac{d \overrightarrow{T}}{ds} = \tilde{\kappa}_\gamma \overrightarrow{N}_\gamma$, then we get

$$
\frac{d^3 \overrightarrow{T}}{ds^3} = -2\tilde{\kappa}_\gamma \tilde{\kappa}_\gamma \overrightarrow{T}_\gamma - \tilde{\kappa}_\gamma^2 \overrightarrow{N}_\gamma + \tilde{\kappa}_\gamma^3 \overrightarrow{N}_\gamma + 2\tilde{\kappa}_\gamma \overrightarrow{B}_\gamma + \tilde{\kappa}_\gamma \overrightarrow{B}_\gamma = 0.
$$

By using (2.7), we get (4.17).

Conversely let us assume that (4.17) holds. From (2.7), we have

$$
\overrightarrow{B}_\gamma = \frac{1}{\tilde{\kappa}_\gamma} \frac{d \overrightarrow{N}_\gamma}{ds} + \frac{\tilde{\kappa}_\gamma}{\tilde{\kappa}_\gamma} \overrightarrow{T}_\gamma.
$$
Differentiating the last equality, we have

\[
\frac{d\tilde{T}}{ds} = \frac{1}{\kappa_T^2} \left\{ \frac{d^3\tilde{T}}{ds^3} - 3 \tilde{\kappa_T}^2 \frac{d^2\tilde{T}}{ds^2} - \left[ \tilde{\kappa_T}^2 - 3 \left( \frac{\tilde{\gamma_T}}{\tilde{\kappa_T}} \right)^2 - \tilde{\tau_T}^2 \right] \frac{d\tilde{T}}{ds} \right\}
\]

(4.20)

\[+ \frac{1}{\kappa_T^2} \left( \frac{\tilde{\kappa_T}}{\tilde{\tau_T}} \right)^2 \frac{d^2\tilde{T}}{ds^2} - \left( \frac{\tilde{\gamma_T}}{\tilde{\kappa_T}} + \frac{\tilde{\kappa_T}}{\tilde{\tau_T}} \left( \frac{\tilde{\kappa_T}}{\tilde{\tau_T}} \right) \right) \frac{d\tilde{T}}{ds} + \left( \frac{\tilde{\gamma_T}}{\tilde{\tau_T}} \right) \frac{d\tilde{T}}{ds} = 0.
\]

By using (2.7) and (4.17), we get

\[
\frac{\tilde{\gamma_T}}{\tilde{\tau_T}} = 0 \quad \text{and} \quad \frac{\tilde{\kappa_T}}{\tilde{\tau_T}} = \sqrt{\frac{1}{\lambda_2} - 1} = \tilde{c}_2 \quad \text{(nowhere pure dual constant)}.
\]

Thus from (3.1), we obtain \(\tilde{\beta} = -\frac{\tilde{c}_2}{\tilde{c}_2} = \text{dual constant}. \) Also \(\tilde{\psi}\) is a dual slant helix.

**Theorem 4.9.** The dual curve \(\tilde{\psi} : I \to \mathbb{D}^3\) is a dual slant helix iff

\[
\frac{d^3\tilde{T}}{ds^3} - 3 \tilde{\kappa_T}^2 \frac{d^2\tilde{T}}{ds^2} - \left[ \tilde{\kappa_T}^2 - 3 \left( \frac{\tilde{\gamma_T}}{\tilde{\kappa_T}} \right)^2 - \tilde{\tau_T}^2 \right] \frac{d\tilde{T}}{ds} = 0,
\]

where \(\tilde{\mu}_2 \neq 1\) is a dual constant \((\tilde{\mu}_2 = 1 + \tilde{c}_2^2\) and \(\tilde{c}_2\) is nowhere pure dual number).

**Theorem 4.10.** The dual curve \(\tilde{\psi} : I \to \mathbb{D}^3\) is a dual slant helix iff

\[
\frac{d^2\tilde{N}}{ds^2} - \frac{\tilde{\gamma_T}}{\tilde{\tau_T}} \frac{d\tilde{N}}{ds} + \tilde{\mu}_2 \tilde{c}_2 \tilde{N} = 0,
\]

where \(\tilde{\mu}_2 \neq 1\) is dual constant \((\tilde{\mu}_2 = 1 + \tilde{c}_2^2\) and \(\tilde{c}_2\) is nowhere pure dual number).

**Proof.** Let \(\tilde{\psi}\) be a dual slant helix. Thus the dual binormal indicatrix \(\tilde{\gamma}\) of \(\tilde{\psi}\) is a dual general helix. If we differentiate following the equation \(\frac{d\tilde{N}}{ds} = -\tilde{\kappa_T} \tilde{T} + \tilde{\tau_T} \tilde{B}\),

then we get

\[
\frac{d^2\tilde{N}}{ds^2} = -\tilde{\kappa_T} \frac{d\tilde{T}}{ds} + \tilde{\tau_T} \frac{d\tilde{N}}{ds} - (\tilde{\kappa_T}^2 + \tilde{\tau_T}^2) \tilde{N}.
\]

From (2.7) and (4.23), we get (4.22).

Conversely, let the equation (4.22) be provided. According to (2.7), we obtain

\[
\frac{d\tilde{N}}{ds} = -\frac{\tilde{\tau_T}}{\tilde{\kappa_T}} \frac{d\tilde{T}}{ds} + \frac{\tilde{\tau_T} \tilde{B}}{\tilde{\kappa_T}}.
\]
Differentiating the last equality, we have
\[
\frac{dT_\tilde{\gamma}}{ds} = -\frac{1}{\kappa_0} \left[ \frac{d^2N_\tilde{\gamma}}{ds^2} - \frac{\tilde{\tau}_0^2}{\kappa_0} \frac{dN_\tilde{\gamma}}{ds} + (\tilde{\kappa}_0^2 + \tilde{\tau}_0^2) \frac{d^2N_\tilde{\gamma}}{ds^2} \right] + \tilde{\kappa}_0 \frac{dN_\tilde{\gamma}}{ds} + \left( \frac{\tilde{\tau}_0}{\kappa_0} \right) '= \tilde{B}_\tilde{\gamma}.
\]

By using (2.7) and (4.22), we get
\[
\left( \frac{\tilde{\tau}_0}{\kappa_0} \right) ' = 0 \quad \text{and} \quad \frac{\tilde{\tau}_0}{\kappa_0} = \sqrt{\mu_2 - 1} = \tilde{c}_2 \quad \text{(nowhere pure dual constant)}.
\]

Thus from (3.1), we obtain \( \tilde{\sigma}_\beta = -\frac{\tilde{\tau}_0}{\kappa_0} = \text{dual constant} \). Also \( \tilde{\psi} \) is a dual slant helix. \( \square \)

**Theorem 4.11.** The dual curve \( \tilde{\psi} : I \to \mathbb{D}^3 \) is a dual slant helix iff
\[
\frac{d^2N_\tilde{\gamma}}{ds^2} - \frac{\tilde{\kappa}_0^2}{\kappa_0} \frac{dN_\tilde{\gamma}}{ds} + \tilde{\lambda}_2 \tilde{\kappa}_0^2 \frac{dN_\tilde{\gamma}}{ds} = 0,
\]
where \( \tilde{\lambda}_2 \neq 1 \) is dual constant \( (\tilde{\lambda}_2 = 1 + \frac{1}{\tilde{c}_2} \) and \( \tilde{c}_2 \) is nowhere pure dual number).

**Theorem 4.12.** The dual curve \( \tilde{\psi} : I \to \mathbb{D}^3 \) is a dual slant helix iff
\[
\frac{d^3B_\tilde{\gamma}}{ds^3} - 3 \frac{\tilde{\tau}_0}{\kappa_0} \frac{d^2B_\tilde{\gamma}}{ds^2} - \left[ \frac{\tilde{\tau}_0''}{\kappa_0} - 3 \left( \frac{\tilde{\tau}_0'}{\kappa_0} \right)^2 - \tilde{\mu}_2 \tilde{\tau}_0^2 \right] \frac{d^2B_\tilde{\gamma}}{ds^2} = 0,
\]
where \( \tilde{\mu}_2 \neq 1 \) is a dual constant \( (\tilde{\mu}_2 = 1 + \tilde{c}_2 \) and \( \tilde{c}_2 \) is nowhere pure dual number).

**Theorem 4.13.** The dual curve \( \tilde{\psi} : I \to \mathbb{D}^3 \) is a dual slant helix iff
\[
\frac{d^3B_\tilde{\gamma}}{ds^3} - 3 \frac{\tilde{\tau}_0}{\kappa_0} \frac{d^2B_\tilde{\gamma}}{ds^2} - \left[ \frac{\tilde{\kappa}_0''}{\kappa_0} - 3 \left( \frac{\tilde{\kappa}_0'}{\kappa_0} \right)^2 - \tilde{\lambda}_2 \tilde{\kappa}_0^2 \right] \frac{d^2B_\tilde{\gamma}}{ds^2} = 0,
\]
where \( \tilde{\lambda}_2 \neq 1 \) is dual constant \( (\tilde{\lambda}_2 = 1 + \frac{1}{\tilde{c}_2} \) and \( \tilde{c}_2 \) is nowhere pure dual number).

**References**


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KIRIKKALE
YAHŞIhan, KIRIKKALE,
TURKEY.
Email address: deryasaglam@kku.edu.tr