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ON DUAL SLANT HELICES IN $\mathbb{D}^3$

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ABSTRACT. In this paper, by using the method in [6] we study dual tangent indicatrix and dual binormal indicatrix of a dual slant helix. Moreover we obtain the relationship between the dual slant helices and dual general helices in $\mathbb{D}^3$. We get some characterizations of a dual slant helix in $\mathbb{D}^3$.

1. INTRODUCTION

In 1873, William Clifford introduced dual numbers. Later, Eduard Study defined the dual angle. The dual angle $\widetilde{\omega} = \omega + \varepsilon\omega^*$ is defined by

$$\left\langle \overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}} \right\rangle = \cos\widetilde{\omega} = \cos\omega - \varepsilon\,\omega^* \sin\omega.$$

ω is the real angle between two lines corresponding to the dual unit vectors $\overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}}$ and ω^* is the shortest distance between the lines. Hermann Grassmann generalized dual numbers and defined the Grassmann number at the end of the 19th century.

Each straightline in $\mathbb{R}^3$ corresponds to unique a point of a dual unit sphere. The dual points of dual unit sphere match the lines in $\mathbb{R}^3$ and in general two different

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points on $\mathbb{D}^3$ represent two skew-lines in $\mathbb{R}^3$. A dual space curve on dual unit sphere in $\mathbb{D}^3$ represents a ruled surface in $\mathbb{R}^3$ [1,2].

A curve is called a general helix is if the tangent vector make a constant angle with a fixed direction [4,6]. A curve is called a slant helix in $\mathbb{R}^3$ if the principal normal vector makes a constant angle with a fixed direction [3]. Moreover, a curve is a slant helix iff the geodesic curvature of the principal normal vector of the curve

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is constant.

In [4], the tangent indicatrix and the binormal indicatrix of a slant helix of $\mathbb{R}^3$ were studied. In [6] the relations was investigated between a general helix and a slant helix.

In this paper, by using the method in [6] we study dual tangent indicatrix and dual binormal indicatrix of a dual slant helix. Moreover we obtain the relationship between the dual slant helices and dual general helices in $\mathbb{D}^3$. We get some characterizations of a dual slant helix in $\mathbb{D}^3$.

2. Preliminaries

The dual numbers set is defined with $\mathbb{D}=\mathbb{R}\times\mathbb{R}$. The dual number $(0,1)$ is denoted by ε. From the definition of the multiplication operation, $\varepsilon^2=0$. Also the dual number $\widetilde{a}=(a,a^*)\in\mathbb{D}$ can be written as $\widetilde{a}=a+\varepsilon a^*$. The set of

$$\mathbb{D}=\{\widetilde{a}=a+\varepsilon a^* \mid a,a^*\in\mathbb{R}\}$$

of dual numbers is a commutative ring following with the operations

(i) $\widetilde{a}+\widetilde{b}=(a+\varepsilon a^*)+(b+\varepsilon b^*)=(a+b)+\varepsilon(a^*+b^*)$,

(ii) $\widetilde{a}\widetilde{b}=(a+\varepsilon a^*)(b+\varepsilon b^*)=ab+\varepsilon(ab^*+a^*b)$.

The division is defined by

$$\frac{\widetilde{a}}{\widetilde{b}}=\frac{a+\varepsilon a^*}{b+\varepsilon b^*}=\frac{a}{b}+\varepsilon\frac{a^*b-ab^*}{b^2},\ b\neq 0.$$

We define a dual vector with a vector of dual numbers $(\widetilde{a}_1,\widetilde{a}_2,\widetilde{a}_3)$. Also the set

$$\mathbb{D}^3=\{\overrightarrow{\widetilde{a}}\mid \overrightarrow{\widetilde{a}}=(a_1+\varepsilon a_1^*,a_2+\varepsilon a_2^*,a_3+\varepsilon a_3^*)=\overrightarrow{a}+\varepsilon\overrightarrow{a^*},\ \ \overrightarrow{a},\overrightarrow{a^*}\in\mathbb{R}^3\},$$

where $\overrightarrow{a} = (a_1, a_2, a_3)$, $\overrightarrow{a^*} = (a_1^*, a_2^*, a_3^*)$, a module on the ring $\mathbb{D}$. The scalar product and the vector product of $\overrightarrow{\widetilde{a}}$ and $\overrightarrow{\widetilde{b}}$ are defined by

$$\left\langle \overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle \right)$$

and

$$\overrightarrow{\widetilde{a}} \times \overrightarrow{\widetilde{b}} = (\widetilde{a}_2\widetilde{b}_3 - \widetilde{a}_3\widetilde{b}_2, \widetilde{a}_3\widetilde{b}_1 - \widetilde{a}_1\widetilde{b}_3, \widetilde{a}_1\widetilde{b}_2 - \widetilde{a}_2\widetilde{b}_1),$$

respectively. The norm is defined by

$$\left\| \overrightarrow{\widetilde{a}} \right\| = \sqrt{\left\langle \overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{a}} \right\rangle} = \|\overrightarrow{a}\| + \varepsilon \frac{\left\langle \overrightarrow{a}, \overrightarrow{a^*} \right\rangle}{\|\overrightarrow{a}\|}, \; a \neq 0.$$

A dual vector $\overrightarrow{\widetilde{a}}$ with norm 1 is called a dual unit vector. Let $\overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*} \in \mathbb{D}^3$. The set

$$\mathbb{S}^2 = \{ \overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*} \mid \left\| \overrightarrow{\widetilde{a}} \right\| = (1, 0); \;\; \overrightarrow{a}, \overrightarrow{a^*} \in \mathbb{R}^3 \}$$

is called the dual unit sphere in $\mathbb{D}^3$.

E. Study mapped dual unit vectors one-to-one with directed lines by dual unit vectors. Moreover a ruled surface is represented by the dual curves on the dual unit sphere in $\mathbb{D}^3$.

The dual angle $\widetilde{\omega} = \omega + \varepsilon\omega^*$ is defined by

$$\left\langle \overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}} \right\rangle = \cos\widetilde{\omega} = \cos\omega - \varepsilon\,\omega^* \sin\omega.$$

ω is the real angle between two lines corresponding to the dual unit vectors $\overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}}$ and ω^* is the shortest distance between the lines.

The dual space curve $\widetilde{\psi} : I \subset \mathbb{R} \to \mathbb{D}^3$ is differentiable, if every ψ_i, $\psi_i^* : I \to \mathbb{R}$, for $1 \leq i \leq 3$, are differentiable. The dual curve $\widetilde{\psi}$ is

$$\widetilde{\psi}(t) = (\psi_1(t) + \varepsilon\psi_1^*(t), \psi_2(t) + \varepsilon\psi_2^*(t), \psi_3(t) + \varepsilon\psi_3^*(t)) = \overrightarrow{\psi}(t) + \varepsilon\overrightarrow{\psi^*}(t),$$

where $\overrightarrow{\psi}(t) = (\psi_1(t), \psi_2(t), \psi_3(t))$ and $\overrightarrow{\psi^*}(t) = (\psi_1^*(t), \psi_2^*(t), \psi_3^*(t))$. The dual arclength of the curve $\widetilde{\psi}$ from t_1 to t is defined following

$$\widetilde{s} = \int_{t_1}^{t} \left\| \widetilde{\psi}'(t) \right\| dt = \int_{t_1}^{t} \left\| \overrightarrow{\psi'}(t) \right\| dt + \varepsilon \int_{t_1}^{t} \left\langle \overrightarrow{T(t)}, \overrightarrow{\psi^{*\prime}}(t) \right\rangle dt = s + \varepsilon s^*, \tag{2.1}$$

where $\overrightarrow{T}$ is a unit tangent of $\overrightarrow{\psi}$. We define the dual unit tangent vector of $\widetilde{\psi}$ following

$$\overrightarrow{\widetilde{T}} = \frac{d\widetilde{\psi}}{d\widetilde{s}} = \frac{d\widetilde{\psi}}{ds}\frac{ds}{d\widetilde{s}}.$$

The function $\widetilde{\kappa} : I \to \mathbb{D}$, $\widetilde{\kappa} = \left\| \frac{d\overrightarrow{\widetilde{T}}}{d\widetilde{s}} \right\|$ is called the dual curvature function of $\widetilde{\psi}$. We assume that $\widetilde{\kappa} : I \to \mathbb{D}$ is never pure dual. Then, we can define the principal normal of $\widetilde{\psi}$ with the dual unit vector $\overrightarrow{\widetilde{N}} = \frac{1}{\widetilde{\kappa}}\frac{d\overrightarrow{\widetilde{T}}}{d\widetilde{s}}$. The dual vector $\overrightarrow{\widetilde{B}} = \overrightarrow{\widetilde{T}} \times \overrightarrow{\widetilde{N}}$ is called the binormal of $\widetilde{\psi}$. Also one obtain that

$$\frac{d}{d\widetilde{s}}\begin{bmatrix} \overrightarrow{\widetilde{T}} \\ \overrightarrow{\widetilde{N}} \\ \overrightarrow{\widetilde{B}} \end{bmatrix} = \begin{bmatrix} 0 & & 0 \\ -\widetilde{\kappa} & 0 & \widetilde{\tau} \\ 0 & -\widetilde{\tau} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{\widetilde{T}} \\ \overrightarrow{\widetilde{N}} \\ \overrightarrow{\widetilde{B}} \end{bmatrix}, \tag{2.2}$$

where $\widetilde{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\widetilde{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion [1].

A dual unit speed curve $\widetilde{\psi} : I \to \mathbb{D}^3$ is a dual general helix if $\left\langle \overrightarrow{\widetilde{T}}, \widetilde{U} \right\rangle = \cos\widetilde{\omega}$ is dual constant for some dual constant vector $\widetilde{U}$. The function $\frac{\widetilde{\tau}}{\widetilde{\kappa}}$ is a dual constant iff the dual curve $\widetilde{\psi}$ is a dual general helix. The dual curve is a dual circular helix, if the curvatures $\widetilde{\kappa}$ and $\widetilde{\tau}$ are a dual constant number (never pure dual).

The dual unit tangent vectors of the dual unit regular curve $\widetilde{\psi}$ define a dual curve $\widetilde{\alpha}$ on $\mathbb{S}^2$. We call that $\widetilde{\alpha}$ is dual tangent indicatrix of the dual curve $\widetilde{\psi}$. Similarly we define the dual principal normal indicatrix $\widetilde{\beta} = \overrightarrow{\widetilde{N}}$ and dual binormal indicatrix $\widetilde{\gamma} = \overrightarrow{\widetilde{B}}$.

Let $\{\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}, \overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}, \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}\}$ be the dual Frenet frame of the dual tangent indicatrix $\widetilde{\alpha}$ of a dual curve $\widetilde{\psi}$, then

$$\frac{d}{d\widetilde{s}}\begin{bmatrix} \overrightarrow{\widetilde{T}}_{\widetilde{\alpha}} \\ \overrightarrow{\widetilde{N}}_{\widetilde{\alpha}} \\ \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}} \end{bmatrix} = \begin{bmatrix} 0 & \widetilde{\kappa}_{\widetilde{\alpha}} & 0 \\ -\widetilde{\kappa}_{\widetilde{\alpha}} & 0 & \widetilde{\tau}_{\widetilde{\alpha}} \\ 0 & -\widetilde{\tau}_{\widetilde{\alpha}} & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{\widetilde{T}}_{\widetilde{\alpha}} \\ \overrightarrow{\widetilde{N}}_{\widetilde{\alpha}} \\ \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}} \end{bmatrix}, \tag{2.3}$$

where

$$\begin{aligned} \overrightarrow{\widetilde{T}}_{\widetilde{\alpha}} &= \overrightarrow{\widetilde{N}} \\ \overrightarrow{\widetilde{N}}_{\widetilde{\alpha}} &= \frac{1}{\sqrt{\widetilde{\kappa}^2+\widetilde{\tau}^2}}(-\widetilde{\kappa}\overrightarrow{\widetilde{T}}+\widetilde{\tau}\overrightarrow{\widetilde{B}}) \\ \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}} &= \frac{1}{\sqrt{\widetilde{\kappa}^2+\widetilde{\tau}^2}}(\widetilde{\tau}\overrightarrow{\widetilde{T}}+\widetilde{\kappa}\overrightarrow{\widetilde{B}}) \end{aligned} \tag{2.4}$$

and the dual curvatures of $\widetilde{\alpha}$ are

$$\widetilde{\kappa}_{\widetilde{\alpha}} = \frac{\sqrt{\widetilde{\kappa}^2+\widetilde{\tau}^2}}{\widetilde{\kappa}},\ \widetilde{\tau}_{\widetilde{\alpha}} = \frac{\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau}}{\widetilde{\kappa}(\widetilde{\kappa}^2+\widetilde{\tau}^2)}. \tag{2.5}$$

Let $\{\overrightarrow{\widetilde{T}}_{\widetilde{\beta}},\ \overrightarrow{\widetilde{N}}_{\widetilde{\beta}},\ \overrightarrow{\widetilde{B}}_{\widetilde{\beta}}\}$ be the dual Frenet frame of the dual principal normal indicatrix $\widetilde{\beta}$ of a dual curve $\widetilde{\psi}$, then

$$\frac{d}{d\widetilde{s}}\begin{bmatrix} \overrightarrow{\widetilde{T}}_{\widetilde{\beta}} \\ \overrightarrow{\widetilde{N}}_{\widetilde{\beta}} \\ \overrightarrow{\widetilde{B}}_{\widetilde{\beta}} \end{bmatrix} = \begin{bmatrix} 0 & \widetilde{\kappa}_{\widetilde{\beta}} & 0 \\ -\widetilde{\kappa}_{\widetilde{\beta}} & 0 & \widetilde{\tau}_{\widetilde{\beta}} \\ 0 & -\widetilde{\tau}_{\widetilde{\beta}} & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{\widetilde{T}}_{\widetilde{\beta}} \\ \overrightarrow{\widetilde{N}}_{\widetilde{\beta}} \\ \overrightarrow{\widetilde{B}}_{\widetilde{\beta}} \end{bmatrix}, \tag{2.6}$$

where

$$\begin{aligned} \overrightarrow{\widetilde{T}}_{\widetilde{\beta}} &= \frac{1}{\sqrt{\widetilde{\kappa}^2+\widetilde{\tau}^2}}(-\widetilde{\kappa}\overrightarrow{\widetilde{T}}+\widetilde{\tau}\overrightarrow{\widetilde{B}}) \\ \overrightarrow{\widetilde{N}}_{\widetilde{\beta}} &= \frac{1}{\sqrt{(\widetilde{\kappa}^2+\widetilde{\tau}^2)(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})^2+(\widetilde{\kappa}^2+\widetilde{\tau}^2)^4}} \\ &\quad \cdot [(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})(\widetilde{\tau}\overrightarrow{\widetilde{T}}+\widetilde{\kappa}\overrightarrow{\widetilde{B}})-(\widetilde{\kappa}^2+\widetilde{\tau}^2)^2\overrightarrow{\widetilde{N}}] \\ \overrightarrow{\widetilde{B}}_{\widetilde{\beta}} &= \frac{1}{\sqrt{(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})^2+(\widetilde{\kappa}^2+\widetilde{\tau}^2)^3}}\cdot[(\widetilde{\kappa}^2+\widetilde{\tau}^2)(\widetilde{\tau}\overrightarrow{\widetilde{T}}+\widetilde{\kappa}\overrightarrow{\widetilde{B}})+(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})\overrightarrow{\widetilde{N}}], \end{aligned} \tag{2.7}$$

and the dual curvatures of $\widetilde{\beta}$ are

$$\begin{aligned} \widetilde{\kappa}_{\widetilde{\beta}} &= \frac{\sqrt{(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})^2+(\widetilde{\kappa}^2+\widetilde{\tau}^2)^3}}{(\widetilde{\kappa}^2+\widetilde{\tau}^2)^{3/2}}, \\ \widetilde{\tau}_{\widetilde{\beta}} &= \frac{[(\widetilde{\kappa}\widetilde{\tau}''-\widetilde{\kappa}''\widetilde{\tau})(\widetilde{\kappa}^2+\widetilde{\tau}^2)-3(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})(\widetilde{\kappa}\widetilde{\kappa}'+\widetilde{\tau}'\widetilde{\tau})]}{(\widetilde{\kappa}^2+\widetilde{\tau}^2)^3+(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})^2}. \end{aligned} \tag{2.8}$$

Let $\{\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}},\ \overrightarrow{\widetilde{N}}_{\widetilde{\gamma}},\ \overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}\}$ be the dual Frenet frame of the dual binormal indicatrix $\widetilde{\gamma}$ of a dual curve $\widetilde{\psi}$, then

$$\frac{d}{d\widetilde{s}}\begin{bmatrix}\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}\\ \overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}\\ \overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}\end{bmatrix}=\begin{bmatrix}0 & \widetilde{\kappa}_{\widetilde{\gamma}} & 0\\ -\widetilde{\kappa}_{\widetilde{\gamma}} & 0 & \widetilde{\tau}_{\widetilde{\gamma}}\\ 0 & -\widetilde{\tau}_{\widetilde{\gamma}} & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}\\ \overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}\\ \overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}\end{bmatrix}, \tag{2.9}$$

where

$$\begin{aligned}\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}} &= -\overrightarrow{\widetilde{N}}\\ \overrightarrow{\widetilde{N}}_{\widetilde{\gamma}} &= \frac{1}{\sqrt{\widetilde{\kappa}^2+\widetilde{\tau}^2}}(\widetilde{\kappa}\overrightarrow{\widetilde{T}}-\widetilde{\tau}\overrightarrow{\widetilde{B}})\\ \overrightarrow{\widetilde{B}}_{\widetilde{\gamma}} &= \frac{1}{\sqrt{\widetilde{\kappa}^2+\widetilde{\tau}^2}}(\widetilde{\tau}\overrightarrow{\widetilde{T}}+\widetilde{\kappa}\overrightarrow{\widetilde{B}})\end{aligned} \tag{2.10}$$

and the dual curvatures of $\widetilde{\gamma}$ are

$$\widetilde{\kappa}_{\widetilde{\gamma}}=\frac{\sqrt{\widetilde{\kappa}^2+\widetilde{\tau}^2}}{\widetilde{\tau}},\ \widetilde{\tau}_{\widetilde{\gamma}}=\frac{-(\widetilde{\kappa}\widetilde{\tau}'-\widetilde{\kappa}'\widetilde{\tau})}{\widetilde{\tau}(\widetilde{\kappa}^2+\widetilde{\tau}^2)}. \tag{2.11}$$

In this paper we will assume that dual curvatures $\widetilde{\kappa}$ and $\widetilde{\tau}$ of the dual curve $\widetilde{\psi}$ are nowhere pure dual.

3. The Dual Spherical Indicatricies of Dual Slant Helices

As in Euclidean space, the necessary and sufficient condition for a dual unit curve $\widetilde{\psi}$ to be a dual slant helix is shown that the following the dual function

$$\frac{\widetilde{\kappa}^2}{(\widetilde{\kappa}^2+\widetilde{\tau}^2)^{3/2}}\left(\frac{\widetilde{\tau}}{\widetilde{\kappa}}\right)' = \text{dual constant.} \tag{3.1}$$

From now on, let's denote the function $\frac{\widetilde{\kappa}^2}{(\widetilde{\kappa}^2+\widetilde{\tau}^2)^{3/2}}\left(\frac{\widetilde{\tau}}{\widetilde{\kappa}}\right)'$ with $\widetilde{\sigma}$.

Theorem 3.1. *The dual tangent indicatrix $\widetilde{\alpha}$ of a dual unit speed slant helix $\widetilde{\psi}$ is a dual spherical helix.*

Proof. From the equations in (2.2), we obtain that

$$\frac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}=\widetilde{\sigma}.$$

Acoording to (3.1), the dual function $\dfrac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}$ is a dual constant. This complete the proof. □

Theorem 3.2. *The dual binormal indicatrix $\widetilde{\gamma}$ of a dual unit speed slant helix $\widetilde{\psi}$ is a dual spherical helix.*

Proof. From the equations in (2.4) we have

$$\frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}} = -\widetilde{\sigma}.$$

According to (3.1), the dual function $\dfrac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}$ is a dual constant. This complete the proof. □

4. Some Characterizations of Dual Slant Helices

Theorem 4.1. $\widetilde{\psi} : I \to \mathbb{D}^3$ *is a dual slant helix iff*

$$\frac{d^2 \overrightarrow{\widetilde{N}}_{\widetilde{\beta}}}{d\widetilde{s}^2} + \widetilde{\kappa}^2_{\widetilde{\beta}} \overrightarrow{\widetilde{N}}_{\widetilde{\beta}} = 0. \tag{4.1}$$

Proof. Let $\widetilde{\psi}$ be a dual slant helix. From the equations in ((2.8) the dual curvatures of $\widetilde{\beta}$ are

$$\widetilde{\kappa}_{\widetilde{\beta}} = \sqrt{1 + \widetilde{\sigma}^2} \tag{4.2}$$

and

$$\widetilde{\tau}_{\widetilde{\beta}} = \frac{(\widetilde{\kappa}^2 + \widetilde{\tau}^2)^{5/2}}{(\widetilde{\kappa}^2 + \widetilde{\tau}^2)^3 + (\widetilde{\kappa}\widetilde{\tau}' - \widetilde{\kappa}'\widetilde{\tau})^2} \widetilde{\sigma}'. \tag{4.3}$$

Since $\widetilde{\sigma} = \dfrac{\widetilde{\kappa}^2}{(\widetilde{\kappa}^2 + \widetilde{\tau}^2)^{3/2}} \left(\frac{\widetilde{\tau}}{\widetilde{\kappa}}\right)'$ is a nowhere pure dual constant function, we get $\widetilde{\kappa}_{\widetilde{\beta}} =$ nowhere pure dual constant and $\widetilde{\tau}_{\widetilde{\beta}} = 0$. Also the dual principal normal indicatrix of $\widetilde{\psi}$ is a dual circle. From (2.7), we have

$$\frac{d^2 \overrightarrow{\widetilde{N}}_{\widetilde{\beta}}}{d\widetilde{s}^2} + \widetilde{\kappa}^2_{\widetilde{\beta}} \overrightarrow{\widetilde{N}}_{\widetilde{\beta}} = 0.$$

Conversely, let the equation (4.1) be provided. From (2.7)

$$\frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\beta}}}{d\widetilde{s}^2}+\widetilde{\kappa}_{\widetilde{\beta}}^2\overrightarrow{\widetilde{N}}_{\widetilde{\beta}}=-\widetilde{\kappa}_{\widetilde{\beta}}'\overrightarrow{\widetilde{T}}_{\widetilde{\beta}}-\widetilde{\tau}_{\widetilde{\beta}}^2\overrightarrow{\widetilde{N}}_{\widetilde{\beta}}+\widetilde{\tau}_{\widetilde{\beta}}'\overrightarrow{\widetilde{B}}_{\widetilde{\beta}}. \tag{4.4}$$

Hence we get $\widetilde{\kappa}_{\widetilde{\beta}}=$ nowhere pure dual constant and $\widetilde{\tau}_{\widetilde{\beta}}=0$. Also $\widetilde{\psi}$ is a dual slant helix. □

Theorem 4.2. *The dual curve $\widetilde{\psi}:I\to\mathbb{D}^3$ is a dual slant helix iff*

$$\frac{d^3\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^3}-3\frac{\widetilde{\kappa}_{\widetilde{\alpha}}'}{\widetilde{\kappa}_{\widetilde{\alpha}}}\frac{d^2\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^2}-\left[\frac{\widetilde{\kappa}_{\widetilde{\alpha}}''}{\widetilde{\kappa}_{\widetilde{\alpha}}}-3\left(\frac{\widetilde{\kappa}_{\widetilde{\alpha}}'}{\widetilde{\kappa}_{\widetilde{\alpha}}}\right)^2-\widetilde{\lambda}_1\widetilde{\kappa}_{\widetilde{\alpha}}^2\right]\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}}=0, \tag{4.5}$$

where $\widetilde{\lambda}_1\neq 1$ is dual constant ($\widetilde{\lambda}_1=1+\frac{1}{\widetilde{c}_1^2}$ and $\widetilde{c}_1$ is nowhere pure dual number).

Proof. We assume that $\widetilde{\psi}$ is a dual slant helix. Also the dual tangent indicatrix $\widetilde{\alpha}$ of $\widetilde{\psi}$ is a dual general helix. From (2.3), we have $\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}}=\widetilde{\kappa}_{\widetilde{\alpha}}\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}$. By differentiating this equation, we get

$$\frac{d^3\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^3}=-2\widetilde{\kappa}_{\widetilde{\alpha}}\widetilde{\kappa}_{\widetilde{\alpha}}'\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}-\widetilde{\kappa}_{\widetilde{\alpha}}^2\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}}+\widetilde{\kappa}_{\widetilde{\alpha}}''\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}+\widetilde{\kappa}_{\widetilde{\alpha}}'\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}}+2\widetilde{\kappa}_{\widetilde{\alpha}}'\widetilde{\tau}_{\widetilde{\alpha}}\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}+\widetilde{\kappa}_{\widetilde{\alpha}}\widetilde{\tau}_{\widetilde{\alpha}}\frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}}=0. \tag{4.6}$$

By using (2.3), we get (4.5).

Conversely, we suppose that (4.5) is valid. From (2.3), we obtain

$$\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}=\frac{1}{\widetilde{\tau}_{\widetilde{\alpha}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}}+\frac{\widetilde{\kappa}_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}. \tag{4.7}$$

Differentiating the last equality, we have

$$\begin{aligned}\frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}}&=\frac{1}{\widetilde{\kappa}_{\widetilde{\alpha}}\widetilde{\tau}_{\widetilde{\alpha}}}\left\{\frac{d^3\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^3}-3\frac{\widetilde{\kappa}_{\widetilde{\alpha}}'}{\widetilde{\kappa}_{\widetilde{\alpha}}}\frac{d^2\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^2}-\left[\frac{\widetilde{\kappa}_{\widetilde{\alpha}}''}{\widetilde{\kappa}_{\widetilde{\alpha}}}-3\left(\frac{\widetilde{\kappa}_{\widetilde{\alpha}}'}{\widetilde{\kappa}_{\widetilde{\alpha}}}\right)^2-\widetilde{\kappa}_{\widetilde{\alpha}}^2-\widetilde{\tau}_{\widetilde{\alpha}}^2\right]\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}}\right\}\\&+\frac{1}{\widetilde{\kappa}_{\widetilde{\alpha}}^2}\left(\frac{\widetilde{\kappa}_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\right)'\frac{d^2\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^2}-\left(\frac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}+\frac{\widetilde{\kappa}_{\widetilde{\alpha}}'}{\widetilde{\kappa}_{\widetilde{\alpha}}^3}\left(\frac{\widetilde{\kappa}_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\right)'\right)\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}}+\left(\frac{\widetilde{\kappa}_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\right)'\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}.\end{aligned} \tag{4.8}$$

By using (2.3) and (4.5), we get

$$\left(\frac{\widetilde{\kappa}_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\right)'=0 \quad \text{and} \quad \frac{\widetilde{\kappa}_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}=\sqrt{\frac{1}{\widetilde{\lambda}_1-1}}=\widetilde{c}_1 \quad \text{(nowhere pure dual constant).}$$

Thus from (3.1), we obtain $\widetilde{\sigma}_{\widetilde{\beta}} = \frac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}$ =dual constant, hence $\widetilde{\psi}$ is a dual slant helix. □

Theorem 4.3. *The dual curve $\widetilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff*

$$\frac{d^3\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^3} - 3\frac{\widetilde{\tau}'_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\frac{d^2\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}^2} - \left[\frac{\widetilde{\tau}''_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}} - 3\left(\frac{\widetilde{\tau}'_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\right)^2 - \widetilde{\mu}_1\widetilde{\tau}^2_{\widetilde{\alpha}}\right]\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}} = 0, \tag{4.9}$$

where $\widetilde{\mu}_1 \neq 1$ is a dual constant ($\widetilde{\mu}_1 = 1 + \widetilde{c}_1^2$ and $\widetilde{c}_1$ is nowhere pure dual number).

Theorem 4.4. *The dual curve $\widetilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff*

$$\frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}^2} - \frac{\widetilde{\kappa}'_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}} + \widetilde{\lambda}_1\widetilde{\kappa}^2_{\widetilde{\alpha}}\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}} = 0, \tag{4.10}$$

where $\widetilde{\lambda}_1 \neq 1$ is dual constant ($\widetilde{\lambda}_1 = 1 + \frac{1}{\widetilde{c}_1^2}$ and $\widetilde{c}_1$ is nowhere pure dual number).

Proof. Let $\widetilde{\psi}$ be a dual slant helix. Hence the dual tangent indicatrix $\widetilde{\alpha}$ of $\widetilde{\psi}$ is a dual general helix. We differentiate $\frac{d\overrightarrow{\widetilde{\beta}}}{d\widetilde{s}} = -\widetilde{\kappa}_{\widetilde{\alpha}}\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}} + \widetilde{\tau}_{\widetilde{\alpha}}\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}$, we get

$$\frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}^2} = -\widetilde{\kappa}'_{\widetilde{\alpha}}\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}} + \widetilde{\tau}'_{\widetilde{\alpha}}\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}} - (\widetilde{\kappa}^2_{\widetilde{\alpha}} + \widetilde{\tau}^2_{\widetilde{\alpha}})\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}. \tag{4.11}$$

From the equation (2.3) and (4.11), we obtain (4.10).

Conversely, let the equation (4.10) be provided. According to (2.3), we obtain

$$\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}} = -\frac{1}{\widetilde{\kappa}_{\widetilde{\alpha}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}} + \frac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}. \tag{4.12}$$

Differentiating the last equality, we have

$$\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\alpha}}}{d\widetilde{s}} = -\frac{1}{\widetilde{\kappa}_{\widetilde{\alpha}}}\left[\frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}^2} - \frac{\widetilde{\kappa}'_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}} + (\widetilde{\kappa}^2_{\widetilde{\alpha}} + \widetilde{\tau}^2_{\widetilde{\alpha}})\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}\right] + \widetilde{\kappa}_{\widetilde{\alpha}}\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}} + \left(\frac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}\right)'\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}. \tag{4.13}$$

By using (2.3) and (4.10), we get

$$\left(\frac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}\right)' = 0 \quad \text{and} \quad \frac{\widetilde{\kappa}_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}} = \sqrt{\frac{1}{\widetilde{\lambda}_1 - 1}} = \widetilde{c}_1 \quad \text{(nowhere pure dual constant).}$$

Thus from (3.1), we obtain $\widetilde{\sigma}_{\widetilde{\beta}} = \frac{\widetilde{\tau}_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}$ =dual constant. Also $\widetilde{\psi}$ is a dual slant helix. □

Theorem 4.5. *The dual curve* $\widetilde{\psi} : I \to \mathbb{D}^3$ *is a dual slant helix iff*

$$\frac{d^2 \overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}^2} - \frac{\widetilde{\tau}'_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}} \frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\alpha}}}{d\widetilde{s}} + \widetilde{\mu}_1 \widetilde{\tau}^2_{\widetilde{\alpha}} \overrightarrow{\widetilde{N}}_{\widetilde{\alpha}} = 0, \tag{4.14}$$

where $\widetilde{\mu}_1 \neq 1$ *is dual constant* ($\widetilde{\mu}_1 = 1 + \widetilde{c}_1^2$ *and* $\widetilde{c}_1$ *is nowhere pure dual number*).

Theorem 4.6. *The dual curve* $\widetilde{\psi} : I \to \mathbb{D}^3$ *is a dual slant helix iff*

$$\frac{d^3 \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}^3} - 3\frac{\widetilde{\kappa}'_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}} \frac{d^2 \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}^2} - \left[\frac{\widetilde{\kappa}''_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}} - 3\left(\frac{\widetilde{\kappa}'_{\widetilde{\alpha}}}{\widetilde{\kappa}_{\widetilde{\alpha}}}\right)^2 - \widetilde{\lambda}_1 \widetilde{\kappa}^2_{\widetilde{\alpha}} \right] \frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}} = 0, \tag{4.15}$$

where $\widetilde{\lambda}_1 \neq 1$ *is dual constant* ($\widetilde{\lambda}_1 = 1 + \frac{1}{\widetilde{c}_1^2}$ *and* $\widetilde{c}_1$ *is nowhere pure dual number*).

Theorem 4.7. *The dual curve* $\widetilde{\psi} : I \to \mathbb{D}^3$ *is a dual slant helix iff*

$$\frac{d^3 \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}^3} - 3\frac{\widetilde{\tau}'_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}} \frac{d^2 \overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}^2} - \left[\frac{\widetilde{\tau}''_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}} - 3\left(\frac{\widetilde{\tau}'_{\widetilde{\alpha}}}{\widetilde{\tau}_{\widetilde{\alpha}}}\right)^2 - \widetilde{\mu}_1 \widetilde{\tau}^2_{\widetilde{\alpha}} \right] \frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\alpha}}}{d\widetilde{s}} = 0, \tag{4.16}$$

where $\widetilde{\mu}_1 \neq 1$ *is a dual constant* ($\widetilde{\mu}_1 = 1 + \widetilde{c}_1^2$ *and* $\widetilde{c}_1$ *is nowhere pure dual number*).

Theorem 4.8. *The dual curve* $\widetilde{\psi} : I \to \mathbb{D}^3$ *is a dual slant helix iff*

$$\frac{d^3 \overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^3} - 3\frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}} \frac{d^2 \overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^2} - \left[\frac{\widetilde{\kappa}''_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}} - 3\left(\frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\right)^2 - \widetilde{\lambda}_2 \widetilde{\kappa}^2_{\widetilde{\gamma}} \right] \frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}} = 0, \tag{4.17}$$

where $\widetilde{\lambda}_2 \neq 1$ *is dual constant* ($\widetilde{\lambda}_2 = 1 + \frac{1}{\widetilde{c}_2^2}$ *and* $\widetilde{c}_2$ *is nowhere pure dual number*).

Proof. We suppose that $\widetilde{\psi}$ is a dual slant helix. Also the dual binormal indicatrix $\widetilde{\gamma}$ of $\widetilde{\psi}$ is a dual general helix. If we differentiate following equation $\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}} = \widetilde{\kappa}_{\widetilde{\gamma}} \overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}$, then we get

$$\frac{d^3 \overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^3} = -2\widetilde{\kappa}_{\widetilde{\gamma}}\widetilde{\kappa}'_{\widetilde{\gamma}} \overrightarrow{\widetilde{T}}_{\widetilde{\gamma}} - \widetilde{\kappa}^2_{\widetilde{\gamma}} \frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}} + \widetilde{\kappa}''_{\widetilde{\gamma}} \overrightarrow{\widetilde{N}}_{\widetilde{\gamma}} + \widetilde{\kappa}'_{\widetilde{\gamma}} \frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}} + 2\widetilde{\kappa}'_{\widetilde{\gamma}}\widetilde{\tau}_{\widetilde{\gamma}} \overrightarrow{\widetilde{B}}_{\widetilde{\gamma}} + \widetilde{\kappa}_{\widetilde{\gamma}}\widetilde{\tau}_{\widetilde{\gamma}} \frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}} = 0. \tag{4.18}$$

By using (2.7), we get (4.17).

Conversely let us assume that (4.17) holds. From (2.7), we have

$$\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}} = \frac{1}{\widetilde{\tau}_{\widetilde{\gamma}}} \frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}} + \frac{\widetilde{\kappa}_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}} \overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}. \tag{4.19}$$

Differentiating the last equality, we have

$$\frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}}=\frac{1}{\widetilde{\kappa}_{\widetilde{\gamma}}\widetilde{\tau}_{\widetilde{\gamma}}}\left\{\frac{d^3\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^3}-3\frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\frac{d^2\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^2}-\left[\frac{\widetilde{\kappa}''_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}-3\left(\frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\right)^2-\widetilde{\kappa}^2_{\widetilde{\gamma}}-\widetilde{\tau}^2_{\widetilde{\gamma}}\right]\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}}\right\}$$
$$+\frac{1}{\widetilde{\kappa}^2_{\widetilde{\gamma}}}\left(\frac{\widetilde{\kappa}_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\right)'\frac{d^2\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^2}-\left(\frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}+\frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}^3_{\widetilde{\gamma}}}\left(\frac{\widetilde{\kappa}_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\right)'\right)\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}}+\left(\frac{\widetilde{\kappa}_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\right)'\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}. \tag{4.20}$$

By using (2.7) and (4.17), we get

$$\left(\frac{\widetilde{\kappa}_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\right)'=0 \quad \text{and} \quad \frac{\widetilde{\kappa}_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}=\sqrt{\frac{1}{\widetilde{\lambda}_2-1}}=\widetilde{c}_2 \quad \text{(nowhere pure dual constant).}$$

Thus from (3.1), we obtain $\widetilde{\sigma}_{\widetilde{\beta}}=-\frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}=$ dual constant. Also $\widetilde{\psi}$ is a dual slant helix. □

Theorem 4.9. *The dual curve $\widetilde{\psi}: I\to\mathbb{D}^3$ is a dual slant helix iff*

$$\frac{d^3\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^3}-3\frac{\widetilde{\tau}'_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\frac{d^2\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}^2}-\left[\frac{\widetilde{\tau}''_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}-3\left(\frac{\widetilde{\tau}'_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\right)^2-\widetilde{\mu}_2\widetilde{\tau}^2_{\widetilde{\gamma}}\right]\frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}}=0, \tag{4.21}$$

where $\widetilde{\mu}_2\neq 1$ is a dual constant ($\widetilde{\mu}_2=1+\widetilde{c}_2^2$ and $\widetilde{c}_2$ is nowhere pure dual number).

Theorem 4.10. *The dual curve $\widetilde{\psi}: I\to\mathbb{D}^3$ is a dual slant helix iff*

$$\frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}^2}-\frac{\widetilde{\tau}'_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}}+\widetilde{\mu}_2\widetilde{\tau}^2_{\widetilde{\gamma}}\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}=0, \tag{4.22}$$

where $\widetilde{\mu}_2\neq 1$ is dual constant ($\widetilde{\mu}_2=1+\widetilde{c}_2^2$ and $\widetilde{c}_2$ is nowhere pure dual number).

Proof. Let $\widetilde{\psi}$ be a dual slant helix. Thus the dual binormal indicatrix $\widetilde{\gamma}$ of $\widetilde{\psi}$ is a dual general helix. If we differentiate following the equation $\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}}=-\widetilde{\kappa}_{\widetilde{\gamma}}\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}+\widetilde{\tau}_{\widetilde{\gamma}}\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}$, then we get

$$\frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}^2}=-\widetilde{\kappa}'_{\widetilde{\gamma}}\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}+\widetilde{\tau}'_{\widetilde{\gamma}}\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}-(\widetilde{\kappa}^2_{\widetilde{\gamma}}+\widetilde{\tau}^2_{\widetilde{\gamma}})\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}. \tag{4.23}$$

From (2.7) and (4.23), we get (4.22).

Conversely, let the equation (4.22) be provided. According to (2.7), we obtain

$$\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}=-\frac{1}{\widetilde{\kappa}_{\widetilde{\gamma}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}}+\frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}. \tag{4.24}$$

Differentiating the last equality, we have

$$(4.25)\qquad \frac{d\overrightarrow{\widetilde{T}}_{\widetilde{\gamma}}}{d\widetilde{s}} = -\frac{1}{\widetilde{\kappa}_{\widetilde{b}}}\left[\frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}^2} - \frac{\widetilde{\tau}'_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}} + (\widetilde{\kappa}^2_{\widetilde{\gamma}} + \widetilde{\tau}^2_{\widetilde{\gamma}})\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}\right] + \widetilde{\kappa}_{\widetilde{\gamma}}\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}} + \left(\frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\right)'\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}.$$

By using (2.7) and (4.22), we get

$$\left(\frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\right)' = 0 \quad \text{and} \quad \frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}} = \sqrt{\widetilde{\mu}_2 - 1} = \widetilde{c}_2 \quad \text{(nowhere pure dual constant).}$$

Thus from (3.1), we obtain $\widetilde{\sigma}_{\widetilde{\beta}} = -\frac{\widetilde{\tau}_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}} =$ dual constant. Also $\widetilde{\psi}$ is a dual slant helix. □

Theorem 4.11. *The dual curve $\widetilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff*

$$(4.26)\qquad \frac{d^2\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}^2} - \frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\frac{d\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}}}{d\widetilde{s}} + \widetilde{\lambda}_2\widetilde{\kappa}^2_{\widetilde{\gamma}}\overrightarrow{\widetilde{N}}_{\widetilde{\gamma}} = 0,$$

where $\widetilde{\lambda}_2 \neq 1$ is dual constant ($\widetilde{\lambda}_2 = 1 + \frac{1}{\widetilde{c}_2^2}$ and $\widetilde{c}_2$ is nowhere pure dual number).

Theorem 4.12. *The dual curve $\widetilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff*

$$(4.27)\qquad \frac{d^3\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}^3} - 3\frac{\widetilde{\tau}'_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\frac{d^2\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}^2} - \left[\frac{\widetilde{\tau}''_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}} - 3\left(\frac{\widetilde{\tau}'_{\widetilde{\gamma}}}{\widetilde{\tau}_{\widetilde{\gamma}}}\right)^2 - \widetilde{\mu}_2\widetilde{\tau}^2_{\widetilde{\gamma}}\right]\frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}} = 0,$$

where $\widetilde{\mu}_2 \neq 1$ is a dual constant ($\widetilde{\mu}_2 = 1 + \widetilde{c}_2^2$ and $\widetilde{c}_2$ is nowhere pure dual number).

Theorem 4.13. *The dual curve $\widetilde{\psi} : I \to \mathbb{D}^3$ is a dual slant helix iff*

$$(4.28)\qquad \frac{d^3\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}^3} - 3\frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\frac{d^2\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}^2} - \left[\frac{\widetilde{\kappa}''_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}} - 3\left(\frac{\widetilde{\kappa}'_{\widetilde{\gamma}}}{\widetilde{\kappa}_{\widetilde{\gamma}}}\right)^2 - \widetilde{\lambda}_2\widetilde{\kappa}^2_{\widetilde{\gamma}}\right]\frac{d\overrightarrow{\widetilde{B}}_{\widetilde{\gamma}}}{d\widetilde{s}} = 0,$$

where $\widetilde{\lambda}_2 \neq 1$ is dual constant ($\widetilde{\lambda}_2 = 1 + \frac{1}{\widetilde{c}_2^2}$ and $\widetilde{c}_2$ is nowhere pure dual number).

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