ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.7, 591–600
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.7.3

APPROXIMATE SOLUTIONS OF WEAKLY NONLINEAR DIFFERENTIAL EQUATIONS

S. Meftah and B. Ghanmi

ABSTRACT. In this work, we study the most useful approximation methods for proving solutions to analytic approximation for solving a weak second-order nonlinear differential equation in a power series with small parameters. We prove the second-order periodic approximation solution and also the best third-order approximation of the weak nonlinear differential equation.

1. PRESENTATION

Lately, researchers have developed a study of approximation methods for large-scale differential equation systems, known as the perturbation method has many applications, see for example [1, 2, 6].

We use Lindstedt's method, which gives periodic approximations, see [4,5,7]. In [5], the existence of first-order analytic periodic approximations has been demonstrated.

The aim of this work is to demonstrate second-order analytic periodic approximations.

[1]*corresponding author: safia-meftah@univ-eloued.dz*
2020 *Mathematics Subject Classification.* 76M45, 41A60, 35B10.
Key words and phrases. Perturbations, Asymptotic approximations, Lindstedt method, Periodic solutions.
Submitted: 18.06.2022; *Accepted:* 04.07.2022; *Published:* 05.07.2022.

First, we give a presentation for our search. Second, we introduce necessary initial concepts and definitions. Finally in section 3 we confirm the general second approximation. Next, we prove the third best approximation solution for weak second-order nonlinear differential equations.

2. Preliminaries

We present approximation methods that depend on constructing an extended solution. We find an analytical solutions to the following equation

$$\frac{d^2y}{dt^2}(t,\varepsilon)+y(t,\varepsilon)=\varepsilon F\left(y(t,\varepsilon),\frac{dy}{dt}(t,\varepsilon)\right),\qquad 0<\varepsilon<<1, \tag{2.1}$$

with $y(0,\varepsilon)=A$, $\frac{dy}{dt}(0,\varepsilon)(0)=0$, F is an analytical function of $y(t,\varepsilon)$ and $dy/dt\ (t,\varepsilon)$.

If $\varepsilon=0$ we obtain the following non perturbed problem

$$\frac{d^2y}{dt^2}(t,0)+y(t,0)=0. \tag{2.2}$$

2.1. **Simple approximation method.** Suppose (2.3) a solution of (2.1) with

$$y(t,\varepsilon)=\sum_{m=0}^{n}\varepsilon^m y_m(t,0)+O\left(\varepsilon^{n+1}\right). \tag{2.3}$$

The principle of this method is to substitute (2.3) into equation (2.1), according to ε. But it gives neither analytic approximations nor periodic.

Remark 2.1. *Generally, in the simple approximation method the terms* $y_2(t,0)$ *and* $y_3(t,0)$ *are non periodic.*

This leads us to the Lindstedt-Poincaré method which solve this problemes.

2.2. **Lindstedt-Poincaré method.** Keeping a few terms of (2.3) defines a function non-periodic and infinite as $t\to+\infty$.

Definition 2.1. $t^m cos(pt)$, $t^m sin(nt)$, $m,\ n\in\mathbb{N}^*$, $p\in\mathbb{N}$ *are called secular terms.*

The astronomer Lindstedt defined a new variable $\eta=\vartheta(\varepsilon)t$ with $\vartheta_0=\vartheta(0)=1$, $\vartheta(\varepsilon)\neq 1$, and $y(\eta,\varepsilon)$, $\vartheta(\varepsilon)$ definded as follows

$$y(\eta,\varepsilon)=y_0(\eta,0)+\varepsilon y_1(\eta,0)+\cdots+\varepsilon^n y_n(\eta,0)+\cdots, \tag{2.4}$$

$$\vartheta(\varepsilon) = 1 + \varepsilon\vartheta_1 + \cdots + \varepsilon^n\vartheta_n + \cdots,$$

First, we note

$$\dot{y} \equiv \frac{dy}{d\eta}(\eta,\varepsilon), \qquad \ddot{y} \equiv \frac{d^2y}{d\eta^2}(\eta,\varepsilon),$$

$$F_y(y,\vartheta\dot{y}) \equiv \frac{\partial F(y(\eta,\varepsilon),\dot{y})}{\partial y(\eta,\varepsilon)}, \qquad F_{\dot{y}}(y,\dot{y}) \equiv \frac{\partial F(y(\eta,\varepsilon),\dot{y})}{\partial \dot{y}},$$

the equation (2.1) becomes

$$\vartheta^2\ddot{y} + y = \varepsilon F(y,\vartheta\dot{y}), \qquad 0 < \varepsilon << 1, \tag{2.5}$$

with $y(0,\varepsilon) = A$, $\dot{y}(0,\varepsilon) = 0$. When we substitute expansion (2.4) into equation (2.5) we have

$$\begin{aligned} &\left(1 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + \varepsilon^3\vartheta_3 + \cdots\right)^2 \left(\ddot{y}_0 + \varepsilon\ddot{y}_1 + \varepsilon^2\ddot{y}_2 + \varepsilon^3\ddot{y}_3 + \cdots\right) \\ &+ y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \cdots = \varepsilon F(y_0,\dot{y}_0) \\ &+ \varepsilon^2\frac{\partial F(y_0,\dot{y}_0)}{\partial\varepsilon} + \frac{\varepsilon^3}{2}\frac{\partial^2 F(y_0,\dot{y}_0)}{\partial\varepsilon^2} + \cdots . \end{aligned} \tag{2.6}$$

Matching all powers of ε to zero, we find the following equations

$$\ddot{y}_0 + y_0 = 0, \tag{2.7}$$

$$\ddot{y}_1 + y_1 = -2\vartheta_1\ddot{y}_0 + F(y_0,\dot{y}_0) =: G_1(y_0(\eta,0),\dot{y}_0(\eta,0)) = G_1(\eta), \tag{2.8}$$

$$\begin{aligned} \ddot{y}_2 + y_2 &= -2\vartheta_1\ddot{y}_1 - \left(\vartheta_1^2 + 2\vartheta_2\right)\ddot{y}_0 + F_y(y_0,\dot{y}_0)\,y_1 + F_{\dot{y}}(y_0,\dot{y}_0)(\vartheta_1\dot{y}_0 + \dot{y}_1) \\ &=: G_2(y_0(\eta,0),y_1(\eta,0),\dot{y}_0(\eta,0),\dot{y}_1(\eta,0)) = G_2(\eta), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\ddot{y}_3 + y_3 \\ &=: G_3(y_0(\eta,0),y_1(\eta,0),y_2(\eta,0);\dot{y}_0(\eta,0),\dot{y}_1(\eta,0),\dot{y}_2(\eta,0)) = G_3(\eta), \end{aligned} \tag{2.10}$$

$$\vdots$$

$$\ddot{y}_n + y_n = G_n\left(y_0(\eta,0),\ldots,y_{n-1}(\eta,0);\dot{y}_0(\eta,0),\ldots,\dot{y}_{n-1}(\eta,0)\right) = G_n(\eta); \tag{2.11}$$

note here that G_i, $i = 1,\ldots,n$ is also an analytical function of y_0, y_1, $\ldots$, $y_{i-1}; \dot{y}_0, \dot{y}_1, \ldots, \dot{y}_{i-1}$.

To solve the equation (2.5), we first have to solve the equations (2.8), (2.9), …, (2.11).

3. Our outcomes

3.1. **Second order periodic approximate solution for the nonlinear differential equation.** In the following proposal, we prove the periodic conditions of the solution $y_2(\eta)$. We mention here that the proof of the periodicity of the solution $y_1(\eta)$ is in [5].

Proposition 3.1. *$y_2(\eta)$ is a periodic solution of the equation* (2.9) *if and only if*

$$\begin{cases} w_2 = -\dfrac{w_1^2}{2} + \dfrac{1}{2A\pi}\displaystyle\int_0^{2\pi}(2w_1\cos\eta\ddot{y}_1 - \cos\eta F_y(A\cos\eta, -A\cos\eta)y_1 \\ \qquad - (w_1 A\sin\eta\cos\eta + \dot{y}_1\cos\eta)F_{\dot{y}}(A\cos\eta, -A\sin\eta))d\eta, \\ \displaystyle\int_0^{2\pi}\sin\eta(F_y(A\cos\eta, -A\cos\eta)y_1 + F_{\dot{y}}(A\cos\eta, -A\sin\eta)\dot{y}_1) \\ \qquad - \vartheta_1\displaystyle\int_0^{2\pi}(2\sin\eta\ddot{y}_1 + AF_{\dot{y}}(A\cos\eta, -A\sin\eta)\sin^2\eta)d\eta = 0. \end{cases} \tag{3.1}$$

Proof. The periodic condition gives

$$\int_\eta^{\eta+2\pi}\sin(\eta-\tau)\,G(\tau)d\tau = 0 \Rightarrow \begin{cases} \int_0^{2\pi}\cos\eta\, G(\eta) = 0, \\ \int_0^{2\pi}\sin\eta\, G(\eta) = 0. \end{cases} \tag{3.2}$$

According to the equation (2.9), we have

$$G_2(\eta) = -2\vartheta_1\ddot{y}_1 - (\vartheta_1^2 + 2\vartheta_2)\ddot{y}_0 + F_y(y_0, \ddot{y}_0)y_1 + F_{\dot{y}}(y_0, \dot{y}_0)(\vartheta_1\dot{y}_0, \dot{y}_1),$$

with

$$y_0(\eta) = A\cos\eta, \quad y_1(\eta) = \int_0^\eta \sin(\eta-\tau)\,(2w_1A\cos\tau + F(A\cos\tau, -A\sin\tau))\,d\tau.$$

We rewrite (3.2) as

$$\begin{cases} \int_0^{2\pi}\cos\eta\,[-2w_1\ddot{y}_1 - (w_1^2+2w_2)(-A\cos\eta) + F_y(A\cos\eta, -A\cos\eta)y_1]\,d\eta \\ \quad + \int_0^{2\pi}\cos\eta\,\left[F_{\dot{y}}(A\cos\eta, -A\sin\eta)(-w_1A\sin\eta + \dot{y}_1)\right]\,d\eta = 0, \\ \int_0^{2\pi}\sin\eta\,[-2w_1\ddot{y}_1 - (w_1^2+2w_2)(-A\cos\eta) + F_y(A\cos\eta, -A\cos\eta)y_1]\,d\eta \\ \quad + \int_0^{2\pi}\cos\eta\,\left[F_{\dot{y}}(A\cos\eta, -A\sin\eta)(-w_1A\sin\eta + \dot{y}_1)\right]\,d\eta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} w_2 = -\frac{w_1^2}{2} + \frac{1}{2A\pi}\int_0^{2\pi}(2w_1 \cos\eta\ddot{y}_1 - \cos\eta F_y(A\cos\eta, -A\cos\eta)y_1 \\ \quad +w_1 A F_{\dot{y}}(A\cos\eta, -A\sin\eta)\sin\eta\cos\eta - F_{\dot{y}}(A\cos\eta, -A\sin\eta)\dot{y}_1\cos\eta)d\eta, \\ -2w_1\int_0^{2\pi}\sin\eta\ddot{y}_1 d\eta + \int_0^{2\pi}\sin\eta F_y(A\cos\eta, -A\cos\eta)y_1 d\eta \\ \quad -w_1 A\int_0^{2\pi} F_{\dot{y}}(A\cos\eta, -A\sin\eta)\sin^2\eta d\eta \\ \quad +\int_0^{2\pi} F_{\dot{y}}(A\cos\eta, -A\sin\eta)\dot{y}_1\sin\eta d\eta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} w_2 = -\frac{w_1^2}{2} + \frac{1}{2A\pi}\int_0^{2\pi}(2w_1\cos\eta\ddot{y}_1 - \cos\eta F_y(A\cos\eta, -A\cos\eta)y_1 \\ \quad -(w_1 A\sin\eta\cos\eta + \dot{y}_1\cos\eta)F_{\dot{y}}(A\cos\eta, -A\sin\eta))d\eta, \\ \int_0^{2\pi}\sin\eta(F_y(A\cos\eta, -A\cos\eta)y_1 + F_{\dot{y}}(A\cos\eta, -A\sin\eta)\dot{y}_1) \\ \quad -\vartheta_2\int_0^{2\pi}(2\sin\eta\ddot{y}_1 + AF_{\dot{y}}(A\cos\eta, -A\sin\eta)\sin^2\eta)d\eta = 0. \end{cases}$$

□

3.2. **Comparisons of Approximate Solutions.** We continue our search in [3], and find the best third-order approximation in his general form

$$\frac{d^2\tilde{y}}{dt^2}(t,\varepsilon) + \tilde{y}(t,\varepsilon) = g(\varepsilon)F\left(\tilde{y}(t,\varepsilon), \frac{d\tilde{y}}{dt}(t,\varepsilon)\right), \qquad 0 < \varepsilon << 1, \tag{3.3}$$

with $\tilde{y}(0,\varepsilon) = A$, $\frac{d\tilde{y}}{dt}(0,\varepsilon) = 0$, and F is an analytic function of $\tilde{y}(t,\varepsilon)$ and $d\tilde{y}/dt\,(t,\varepsilon)$.

We introduce the variable $\tilde{\eta} = \tilde{\vartheta}t$, with $\tilde{y}$ and $\tilde{\vartheta}$ are defined with

$$\tilde{y}(\tilde{\eta},\varepsilon) = \tilde{y}_0(\tilde{\eta},0) + \varepsilon\tilde{y}_1(\tilde{\eta},0) + \varepsilon^2\tilde{y}_2(\tilde{\eta},0) + \varepsilon^3\tilde{y}_3(\tilde{\eta},0)\dots, \tag{3.4}$$

$$\tilde{\vartheta}(\varepsilon) = 1 + \varepsilon\tilde{\vartheta}_1 + \varepsilon^2\tilde{\vartheta}_2 + \varepsilon^3\tilde{\vartheta}_3 + \dots + \varepsilon^n\tilde{\vartheta}_n + \dots \tag{3.5}$$

To find the third order uniformly approximate periodic solution (3.3), we must give the general formula of $\tilde{y}_0(\tilde{\eta},0)$, $\tilde{y}_1(\tilde{\eta},0)$, $\tilde{y}_2(\tilde{\eta},0)$ and $\tilde{y}_3(\tilde{\eta},0)$.

Remark 3.1. *In [3], we proved* $\tilde{y}_0(\tilde{\eta},0)$, $\tilde{y}_1(\tilde{\eta},0)$, $\tilde{y}_2(\tilde{\eta},0)$. *Here, we prove* $\tilde{y}_3(\tilde{\eta},0)$.

Proposition 3.2. *The terms* $\tilde{y}_0(\tilde{\eta},0)$, $\tilde{y}_1(\tilde{\eta},0)$, $\tilde{y}_2(\tilde{\eta},0)$ *and* $\tilde{y}_3(\tilde{\eta},0)$ *are respectively solutions of* (3.6),(3.7), (3.8) *and* (3.9) *such that*

$$\ddot{\tilde{y}}_0 + \tilde{y}_0 = 0, \tag{3.6}$$

$$\ddot{\tilde{y}}_1 + \tilde{y}_1 = \tilde{G}_1(\tilde{\eta}), \tag{3.7}$$

$$\ddot{\tilde{y}}_2 + \tilde{y}_2 = \tilde{G}_2(\tilde{\eta}), \tag{3.8}$$

$$\ddot{\tilde{y}}_3 + \tilde{y}_3 = \tilde{G}_3(\tilde{\eta}), \tag{3.9}$$

with

$$\tilde{G}_1(\tilde{\eta}) = -2\tilde{\vartheta}_1\ddot{\tilde{y}}_0 + c_1 F\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right),$$

$$\begin{aligned}\tilde{G}_2(\tilde{\eta}) &= -2\tilde{\vartheta}_1\ddot{\tilde{y}}_0 + c_1 F\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right) - 2\tilde{\vartheta}_1\ddot{\tilde{y}}_1 - \left(\tilde{\vartheta}_1^2 + 2\tilde{\vartheta}_2\right)\ddot{\tilde{y}}_0 \\ &+ c_1\left(F_{\tilde{y}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right)\tilde{y}_1 + F_{\dot{\tilde{y}}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right)\left(\tilde{\vartheta}_1\dot{\tilde{y}}_0 + \dot{\tilde{y}}_1\right)\right) \\ &+ c_2 F\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right),\end{aligned}$$

$$\begin{aligned}\tilde{G}_3(\tilde{\eta}) &= -2\tilde{\vartheta}_1\ddot{\tilde{y}}_2 - \left(\tilde{\vartheta}_1{}^2 + 2\tilde{\vartheta}_2\right)\ddot{\tilde{y}}_1 - \left(2\tilde{\vartheta}_3 + 2\tilde{\vartheta}_1\tilde{\vartheta}_2\right)\ddot{\tilde{y}}_0 \\ &+ c_1\left[\frac{\tilde{y}_1{}^2}{2}F_{\tilde{y}\tilde{y}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right) + \left(\tilde{\vartheta}_1\dot{\tilde{y}}_0 + \dot{\tilde{y}}_1\right)\tilde{y}_1 F_{\tilde{y}\dot{\tilde{y}}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right)\right. \\ &+ \tilde{y}_2 F_{\tilde{y}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right) + \left(\tilde{\vartheta}_2\dot{\tilde{y}}_0 + \tilde{\vartheta}_1\dot{\tilde{y}}_1 + \dot{\tilde{y}}_2\right)F_{\dot{\tilde{y}}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right) \\ &+ \left.\frac{1}{2}\left(\tilde{\vartheta}_1\dot{\tilde{y}}_0 + \dot{\tilde{y}}_1\right)^2 F_{\dot{\tilde{y}}\dot{\tilde{y}}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right)\right] + c_2\left[F_{\tilde{y}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right)\tilde{y}_1\right. \\ &+ \left.F_{\dot{\tilde{y}}}\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right)\left(\tilde{\vartheta}_1\dot{\tilde{y}}_0 + \dot{\tilde{y}}_1\right)\right] + c_3 F\left(\tilde{y}_0(\tilde{\eta},0), \dot{\tilde{y}}_0(\tilde{\eta},0)\right).\end{aligned}$$

Proof. The equation (3.3) will be written as

$$\begin{aligned}\frac{d^2\tilde{y}(\tilde{\eta},\varepsilon)}{dt^2} + \tilde{y}(\tilde{\eta}) &= \sum_{k\geq 1}\varepsilon^k c_k F\left(\tilde{y}(\tilde{\eta},\varepsilon), \frac{d\tilde{y}(\tilde{\eta},\varepsilon)}{dt}\right) \\ &= \varepsilon\left(\sum_{k\geq 0}\varepsilon^k c_{k+1} F\left(\tilde{y}(\tilde{\eta},\varepsilon), \frac{d\tilde{y}(\tilde{\eta},\varepsilon)}{dt}\right)\right) = \varepsilon K\left(\tilde{y}(\tilde{\eta},\varepsilon), \frac{d\tilde{y}(\tilde{\eta},\varepsilon)}{dt}\right),\end{aligned}$$

with K is an analytic function of y and dy/dt.

Replacing the equation (3.4) into the equation (3.3), we have

$$\begin{aligned}&\left(1 + \varepsilon\tilde{\vartheta}_1 + \varepsilon^2\tilde{\vartheta}_2 + \varepsilon^3\tilde{\vartheta}_3 + \cdots\right)^2\left(\ddot{\tilde{y}}_0 + \varepsilon\ddot{\tilde{y}}_1 + \varepsilon^2\ddot{\tilde{y}}_2 + \varepsilon^3\ddot{\tilde{y}}_3 + \cdots\right) + \tilde{y}_0 + \varepsilon\tilde{y}_1 + \varepsilon^2\tilde{y}_2 \\ &+ \varepsilon^3\tilde{y}_3 + \cdots = \varepsilon K\left(\tilde{y}_0, \dot{\tilde{y}}_0\right) + \varepsilon^2\frac{\partial K\left(\tilde{y}_0, \dot{\tilde{y}}_0\right)}{\partial\varepsilon} + \frac{\varepsilon^3}{2}\frac{\partial^2 K\left(\tilde{y}_0, \dot{\tilde{y}}_0\right)}{\partial\varepsilon^2} + \cdots\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon c_1 F(\tilde{y}_0, \dot{\tilde{y}}_0) + \varepsilon^2 \left[c_1 \frac{\partial F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \varepsilon} + c_2 F(\tilde{y}_0, \dot{\tilde{y}}_0) \right] \\
&\quad + \varepsilon^3 \left[c_3 F(\tilde{y}_0, \dot{\tilde{y}}_0) + c_2 \frac{\partial F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \varepsilon} + \frac{c_1}{2} \frac{\partial^2 F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \varepsilon^2} \right] + \cdots , \\
&= \varepsilon c_1 F\left(\tilde{y}_0, \dot{\tilde{y}}_0\right) + \varepsilon^2 \left[c_1 \left(\frac{\partial F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \tilde{y}} . \frac{\partial \tilde{y}}{\partial \varepsilon} + \frac{\partial F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \dot{\tilde{y}}} . \frac{\partial \vartheta \dot{\tilde{y}}}{\partial \varepsilon} \right) + c_2 F\left(\tilde{y}_0, \dot{\tilde{y}}_0\right) \right] \\
&\quad + \varepsilon^3 \left[c_3 F(\tilde{y}_0, \dot{\tilde{y}}_0) + c_2 \left(\frac{\partial F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \tilde{y}} . \frac{\partial \tilde{y}}{\partial \varepsilon} + \frac{\partial F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \dot{\tilde{y}}} . \frac{\partial \vartheta \dot{\tilde{y}}}{\partial \varepsilon} \right) + \frac{c_1}{2} \frac{\partial^2 F(\tilde{y}_0, \dot{\tilde{y}}_0)}{\partial \varepsilon^2} \right] \\
&= \varepsilon c_1 F(\tilde{y}_0, \dot{\tilde{y}}_0) + \varepsilon^2 \left[c_1 \left(F_{\tilde{y}}(\tilde{y}_0, \dot{\tilde{y}}_0) \tilde{y}_1 + F_{\dot{\tilde{y}}}(\tilde{y}_0, \dot{\tilde{y}}_0)(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1) \right) + c_2 F(\tilde{y}_0, \dot{\tilde{y}}_0) \right] \\
&\quad + \varepsilon^3 \left[c_3 F(\tilde{y}_0, \dot{\tilde{y}}_0) + c_2 \left(F_{\tilde{y}}\left(\tilde{y}_0, \dot{\tilde{y}}_0\right) \tilde{y}_1 + F_{\dot{\tilde{y}}}\left(\tilde{y}_0, \dot{\tilde{y}}_0\right) \left(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \right) \right) \right. \\
&\quad + c_1 \left(\frac{\tilde{y}_1{}^2}{2} F_{\tilde{y}\tilde{y}}(\tilde{y}_0, \dot{\tilde{y}}_0) + (\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1) \tilde{y}_1 F_{\tilde{y}\dot{\tilde{y}}}(\tilde{y}_0, \dot{\tilde{y}}_0) + \tilde{y}_2 F_{\tilde{y}}(\tilde{y}_0, \dot{\tilde{y}}_0) + \left(\tilde{\vartheta}_2 \dot{\tilde{y}}_2 + \tilde{\vartheta}_1 \dot{\tilde{y}}_1 \right.\right. \\
&\quad \left.\left.\left. + \dot{\tilde{y}}_2 \right) F_{\dot{\tilde{y}}}(\tilde{y}_0, \dot{\tilde{y}}_0) \right) \frac{1}{2} (\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1)^2 F_{\dot{\tilde{y}}\dot{\tilde{y}}}(\tilde{y}_0, \dot{\tilde{y}}_0) \right] .
\end{aligned}
$$

By identification we get equations (3.6), (3.7), (3.8), (3.9). □

3.3. **Closest approximation.** We build an approach of (2.2), which gives an closer approximation to (3.3) more than an approximation of (2.1).

Lemma 3.1. *If* $g(\varepsilon) = \sum_{k \geq 1} \varepsilon^k c_k$*, such that* c_k *are constants, we have*

1) $\tilde{y}_0(\tilde{\eta}, 0) = y_0(\tilde{\eta}, 0)$.
2) $\tilde{y}_1(\tilde{\eta}, 0) = c_1 y_1(\tilde{\eta}, 0)$.
3) $\tilde{y}_2(\tilde{\eta}, 0) = c_1^2 y_2(\tilde{\eta}, 0) + c_2 y_1(\tilde{\eta}, 0)$.
4) $\tilde{y}_3(\tilde{\eta}, 0) = c_1^3 y_3(\tilde{\eta}, 0) + 2c_1 c_2 y_2(\tilde{\eta}, 0) + c_3 y_1(\tilde{\eta}, 0)$.

Remark 3.2. *The proof of 1, 2 and 3 is in Lemma 3.1 in [3], here we will prove 4.*

Proof. 4) We have

$$
\begin{cases}
\int_0^{2\pi} cos\tilde{\eta} \tilde{G}_3(\tilde{\eta})\, d\tilde{\eta} = 0, \\
\int_0^{2\pi} sin\tilde{\eta}\, \tilde{G}_3(\eta)\, d\tilde{\eta} = 0.
\end{cases}
$$

$$\Rightarrow \begin{cases} \int_0^{2\pi} \cos\tilde{\eta} \left[-2\tilde{\vartheta}_1 \ddot{\tilde{y}}_2 - \left(\tilde{\vartheta}_1^{\ 2} + 2\tilde{\vartheta}_2 \right) \ddot{\tilde{y}}_1 - \left(2\tilde{\vartheta}_3 + 2\tilde{\vartheta}_1\tilde{\vartheta}_2 \right) \ddot{\tilde{y}}_0 \right] d\tilde{\eta} \\ \quad + c_2 \int_0^{2\pi} \cos\tilde{\eta} \left[F_{\tilde{y}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \tilde{y}_1 + F_{\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \left(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \right) \right] d\tilde{\eta} \\ \quad + c_1 \int_0^{2\pi} \cos\tilde{\eta} \left[\frac{\tilde{y}_1{}^2}{2} F_{\tilde{y}\tilde{y}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) + \left(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \right) \tilde{y}_1 F_{\tilde{y}\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) + \tilde{y}_2 F_{\tilde{y}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \right] d\tilde{\eta} \\ \quad + c_1 \int_0^{2\pi} \cos\tilde{\eta} \left[\left(\tilde{\vartheta}_2 \dot{\tilde{y}}_0 + \tilde{\vartheta}_1 \dot{\tilde{y}}_1 + \dot{\tilde{y}}_2 \right) F_{\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) + \frac{1}{2} \left(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \right)^2 F_{\dot{\tilde{y}}\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \right] d\tilde{\eta} \\ \quad + c_3 F \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) d\tilde{\eta} = 0. \\ \int_0^{2\pi} \sin\tilde{\eta} \left[-2\tilde{\vartheta}_1 \ddot{\tilde{y}}_2 - \left(\tilde{\vartheta}_1^{\ 2} + 2\tilde{\vartheta}_2 \right) \ddot{\tilde{y}}_1 - \left(2\tilde{\vartheta}_3 + 2\tilde{\vartheta}_1\tilde{\vartheta}_2 \right) \ddot{\tilde{y}}_0 \right] d\tilde{\eta} \\ \quad + c_2 \int_0^{2\pi} \sin\tilde{\eta} \left[F_{\tilde{y}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \tilde{y}_1 + F_{\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \left(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \right) \right] d\tilde{\eta} \\ \quad + c_1 \int_0^{2\pi} \sin\tilde{\eta} \left[\frac{\tilde{y}_1{}^2}{2} F_{\tilde{y}\tilde{y}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) + \left(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \right) \tilde{y}_1 F_{\tilde{y}\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) + \tilde{y}_2 F_{\tilde{y}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \right] d\tilde{\eta} \\ \quad + c_1 \int_0^{2\pi} \sin\tilde{\eta} \left[\left(\tilde{\vartheta}_2 \dot{\tilde{y}}_0 + \tilde{\vartheta}_1 \dot{\tilde{y}}_1 + \dot{\tilde{y}}_2 \right) F_{\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) + \frac{1}{2} \left(\tilde{\vartheta}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \right)^2 F_{\dot{\tilde{y}}\dot{\tilde{y}}} \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) \right] d\tilde{\eta} \\ \quad + c_3 F \left(\tilde{y}_0, \dot{\tilde{y}}_0 \right) d\tilde{\eta} = 0. \end{cases}$$

$$\Rightarrow \begin{cases} -2A\pi c_1^3 \vartheta_3 - 4A\pi c_1 c_2 \vartheta_2 - 2A\pi c_3 \vartheta_1 = -2A\pi\tilde{\vartheta}_3, \\ -2\vartheta_3 c_1^3 A \int_0^{2\pi} \sin\tilde{\eta}\cos\tilde{\eta} d\tilde{\eta} + 2\vartheta_2 c_1 c_2 A \int_0^{2\pi} \sin\tilde{\eta}\cos\tilde{\eta} d\tilde{\eta} + c_3 \int_0^{2\pi} \sin\tilde{\eta} F(y_0, \dot{y_0}) d\tilde{\eta} \\ \quad = -2\tilde{\vartheta}_3 A \int_0^{2\pi} \sin\tilde{\eta}\cos\tilde{\eta} d\tilde{\eta}. \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{\vartheta}_3 = c_1^3 \vartheta_3 + 2c_1 c_2 \vartheta_2 + c_3 \vartheta_1, \\ \int_0^{2\pi} \sin\tilde{\eta} F(y_0, \dot{y_0}) d\tilde{\eta} = 0. \end{cases}$$

On the other hand, according to (3.3) the solution $\tilde{y}_3(\tilde{\eta}, 0)$ of equation (3.9) is given by

$$\tilde{y}_3(\tilde{\eta}, 0) = \int_0^{2\pi} sin(\tilde{\eta} - \tilde{\tau}) \tilde{G}_3(\tilde{\eta})$$

$$\begin{aligned} = \int_0^{2\pi} & sin(\tilde{\eta} - \tilde{\tau}) [-2\tilde{w}_1 \ddot{\tilde{y}}_2 - (\tilde{w}_1^2 + 2\tilde{w}_2) \ddot{\tilde{y}}_2 - (2\tilde{w}_3 + 2\tilde{w}_1\tilde{w}_2) \ddot{\tilde{y}}_0 \\ & + c_2 [F_{\tilde{y}} (y_0 (\eta, 0), \dot{y}_0 (\eta, 0))) \tilde{y}_1 + F_{\dot{\tilde{y}}} (y_0 (\eta, 0), \dot{y}_0 (\eta, 0))) (\tilde{w}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1)] \\ & + c_1 [\frac{\tilde{y}_1^2}{2} F_{\tilde{y}\tilde{y}} (y_0 (\eta, 0), \dot{y}_0 (\eta, 0))) + \tilde{y}_1 (\tilde{w}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1) F_{\tilde{y}\dot{\tilde{y}}} (y_0 (\eta, 0), \dot{y}_0 (\eta, 0))) \\ & + \tilde{y}_2 F_{\tilde{y}} (y_0 (\eta, 0), \dot{y}_0 (\eta, 0))) + (\tilde{w}_2 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1 \tilde{w}_1 + \dot{\tilde{y}}_2) F_{\dot{\tilde{y}}} (y_0 (\eta, 0), \dot{y}_0 (\eta, 0))) \\ & + \frac{1}{2} (\tilde{w}_1 \dot{\tilde{y}}_0 + \dot{\tilde{y}}_1)^2 F_{\dot{\tilde{y}}\dot{\tilde{y}}} (y_0 (\eta, 0), \dot{y}_0 (\eta, 0)))] + c_3 F (y_0 (\eta, 0), \dot{y}_0 (\eta, 0)))], \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \tilde{y}_3(\tilde{\eta},0) = &\int_0^{2\pi} sin(\tilde{\eta}-\tilde{\tau})[-2c_1w_1(\ddot{y}_2) - (c_1^2w_1^2 + 2(c_1^2w_2 + c_2w_1))c_1\ddot{y}_1 \\
&- (2\left(c_1^3w_3 + c_1c_2 2w_2 + c_3w_1\right) + (2c_1w_1(c_1^2w_2 + c_2w_1))\ddot{y}_0 \\
&+ c_2[F_y((y_0(\eta,0), \dot{y}_0(\eta,0))\, c_1y_1 + F_{\dot{y}}((y_0(\eta,0), \dot{y}_0(\eta,0)))(c_1w_1\dot{y}_0 \\
&+ c_1\dot{y}_1)] + c_1[(c_1w_1\dot{y}_0 + c_1\dot{y}_1)c_1y_1F_{y\dot{y}}((y_0(\eta,0), \dot{y}_0(\eta,0))) \\
&+ (c_1^2y_2 + c_2y_1)F_y((y_0(\eta,0), \dot{y}_0(\eta,0))) \\
&+ ((c_1^2w_2 + c_2w_1)\dot{y}_0 + c_1w_1\dot{y}_0 + c_1\dot{y}_2)F_{\dot{y}}((y_0(\eta,0), \dot{y}_0(\eta,0))) \\
&+ \frac{1}{2}(c_1w_1\dot{y}_0 + c_1\dot{y}_1)^2F_{\dot{y}\dot{y}}((y_0(\eta,0), \dot{y}_0(\eta,0)))] + c_3F((y_0(\eta,0), \dot{y}_0(\eta,0)))],
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \tilde{y}_3(\tilde{\eta},0) = c_1^3 &\int_0^{\tilde{\eta}} sin(\tilde{\eta}-\tilde{\tau})[-2\vartheta_1\ddot{y}_2 - \left(\vartheta_1^2 + 2\vartheta_2\right)\ddot{y}_1 - (2\vartheta_3 + 2\vartheta_1\vartheta_2)\ddot{y}_0 \\
&+ y_2F_y(y_0,\dot{y}_0) + \frac{{y_1}^2}{2}F_{yy}(y_0,\dot{y}_0) + y_1(\vartheta_1\dot{y}_0 + \dot{y}_1)F_{y\dot{y}}(y_0,\dot{y}_0) \\
&+ (\vartheta_2\dot{y}_0 + \vartheta_1\dot{y}_1 + \dot{y}_2)F_{\dot{y}}(y_0,\dot{y}_0) + \frac{1}{2}(\vartheta_1\dot{y}_0 + \dot{y}_1)^2F_{\dot{y}\dot{y}}(y_0,\dot{y}_0).]d\tilde{\tau} \\
&+ 2c_1c_2\int_0^{\tilde{\eta}} sin(\tilde{\eta}-\tilde{\tau})[-2\vartheta_1\ddot{y}_1 - \left(\vartheta_1^2 + 2\vartheta_2\right)\ddot{y}_0 + F_y(y_0,\ddot{y}_0)\,y_1 \\
&+ F_{\dot{y}}(y_0,\dot{y}_0)(\vartheta_1\dot{y}_0 + \dot{y}_1)]d\tilde{\tau} + c_3\int_0^{\tilde{\eta}} sin(\tilde{\eta}-\tilde{\tau})[-2\vartheta_1\ddot{y}_0 \\
&+ F((y_0(\eta,0), \dot{y}_0(\eta,0)))]d\tilde{\tau},
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \tilde{y}_3(\tilde{\eta},0) &= c_1^3\int_0^{\tilde{\eta}} sin(\tilde{\eta}-\tilde{\tau})G_3(\tilde{\tau})d\tilde{\tau} + 2c_1c_2\int_0^{\tilde{\eta}} sin(\tilde{\eta}-\tilde{\tau})G_2(\tilde{\tau})d\tilde{\tau} \\
&\quad + c_3\int_0^{\tilde{\eta}} sin(\tilde{\eta}-\tilde{\tau})G_1(\tilde{\tau})d\tilde{\tau} \\
\Rightarrow \quad \tilde{y}_3(\tilde{\eta},0) &= c_1^3y_3(\tilde{\eta},0) + 2c_1c_2y_2(\tilde{\eta},0) + c_3y_3(\tilde{\eta},0).
\end{aligned}$$

□

REFERENCES

[1] M. BAGAYOGO, Y. PARE, Y. MINOUNGOU: *An Approached Solution of Wave Equation with Cubic Damping by Homotopy Perturbation Method (HPM), Regular Pertubation Method (RPM)*

and Adomian Decomposition Method (ADM), Journal of Mathematics Research, **10**(2) (2018), 166-181.

[2] T.LEJDEL ALI, S.MEFTAH, L.NISSE: *Analysis of approximate solutions of weak nonlinear differential equations in their general form with generalized order*, Advances in Mathematics: Scientific Journal **11**(3) (2022), 211–228.

[3] S. MEFTAH: *A New Approach to Approximate Solutions for Nonlinear Differential Equation*, International Journal of Mathematics and Mathematical Sciences, **2018** (2018), Article ID 5129502, 8 pages.

[4] R-E. MICKENS: *An Introduction to nonlinear oscillations*, Cambridge University, 1981.

[5] R.E. MICKENS: *Oscillations in planar dynamics systems*, World Scientifics, Singapore, 1996.

[6] A.H. NAYFEH: *Introduction to perturbation techniques*, Wiley, New York, 1981.

[7] F. VERHULST: *Weak nonlinear differential equations and Dynamical Systems*, 2nd edition, Springer, Berlin, 1989.

[8] F. VERHULST: *Methods and Applications of Singular Perturbations*, 2nd edition, Springer, Berlin, 2005.

DEPARTMENT OF MATHEMATICS
FACULTY OF EXACT SCIENCES, OPERATOR THEORY, EDP AND APPLICATIONS LABORATORY
UNIVERSITY OF ECHAHID HAMMA LAKHDAR EL-OUED
ALGERIA.
Email address: safia-meftah@univ-eloued.dz

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES OF GAFSA
UNIVERSITY CAMPUS SIDI AHMED ZARROUG-2112 GAFSA
TUNISIA.
Email address: bghanmi@gmail.com