BIORTHOGONAL NONSTATIONARY AND DAUBECHIES WAVELETS: A COMPARATIVE STUDY

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ABSTRACT. In this article, we present a constructive technic of a biorthogonal non-stationary wavelets (Bior-NSW) family which are proposed by [12], and we perform a comparative study between the classical Daubechies biorthogonal wavelets family and the constructed nonstationary wavelets, which proved the dominance of the nonstationary wavelets in approximation of different kinds signals.

1. INTRODUCTION

Wavelet analysis is a very important famous area in mathematics, it has become a powerful tool in signal processing compared to Fourrier analysis, thanks to the location of information property in time domain. In this context, several researchers have invested in this area such as [3,4,7,9,10]. Fourier transform only provides the spectral components of signals not their temporal locations, however wavelets can provide a time-frequency representation of signals [5,8,11]. Most compression algorithms [2] integrate biorthogonal wavelets with the aim to approximate the signal in the first step have proved their effectiveness. Nonstationary biorthogonal wavelets is a generalization of the Daubechies biorthogonal wavelets family introduced recently in [12]. The main difference with the construction of the

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standard biorthogonal wavelets is, that the multiresolution spaces are generated by scale dependent functions. This property offers a good approximation of \( L^2(\mathbb{R}) \) signals class using this kind of wavelets basis. In this paper, we present a construction method of a Bior-NSW family based on [12] and we carry out a comparative study concerning the approximation of signals between our proposed Bior-NSW and the standard Daubechies biorthogonal wavelets. Our study consists in approximating the different kinds of Matlab signals classes such as doppler, bumps, blocks and heavy sine,..., etc in the biorthogonal wavelets basis.

The rest of paper is organized as follows. Section 2 gives some notions on nonstationary multiresolution analysis and wavelets. Section 3 present the constructed method of Bior-NSW. Section 4 present a exprimental and comparative study between the obtain results. Finally, a conclusion is drawn in section 5.

2. Preliminaries

In this section we present a basic properties and notions concerning biorthogonal nonstationary multiresolutions and nonstationary wavelets.

2.1. Biorthogonal nonstationary multiresolutions analysis.

**Definition 2.1.** Let \((\varphi_j(t))_{j \leq j_0}\) and \((\tilde{\varphi}_j(t))_{j \leq j_0}\) be a sequences of functions in \( L^2(\mathbb{R}) \), such that \( j_0 \in \mathbb{Z} \). The couple \((V_j)_{j \leq j_0}, (\tilde{V}_j)_{j \leq j_0}\) such as:

\[
\begin{align*}
V_j &= \left\{ f(t) = \sum_{k \in \mathbb{Z}} a_k \varphi_j(2^{-j}t - k), \ a \in l^2(\mathbb{Z}) \right\}; \\
\tilde{V}_j &= \left\{ g(t) = \sum_{k \in \mathbb{Z}} b_k \tilde{\varphi}_j(2^{-j}t - k), \ b \in l^2(\mathbb{Z}) \right\},
\end{align*}
\]

defines a biorthogonal nonstationary multiresolution if and only if:

1. For all \( j \leq j_0 \), \( \varphi_j(t) \) and \( \tilde{\varphi}_j(t) \) generate a Riesz basis, i.e: \( \exists C_{1j}, C_{2j} \in \mathbb{R}_+^* \) \( \forall (c_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) : \)

\[
C_{1j} \|c\|_{l^2(\mathbb{Z})} \leq \left\| \sum_{k \in \mathbb{Z}} c_k \varphi_j(\cdot - k) \right\|_{L^2(\mathbb{R})} \leq C_{2j} \|c\|_{l^2(\mathbb{Z})},
\]

where, \( C_{1j}, C_{2j} \) are called lower and upper Riesz bounds respectively, it is the same in the case of \( \tilde{\varphi}_j(t) \).
(2) For all \( j \leq j_0 \), \( V_{j+1} \subset V_j \), \( \tilde{V}_{j+1} \subset \tilde{V}_j \).

(3) \( \bigcup_{j \leq j_0} V_j = \bigcup_{j \leq j_0} \tilde{V}_j = L^2(\mathbb{R}) \).

(4) \( \bigcap_{j \leq j_0} V_j = \bigcap_{j \leq j_0} \tilde{V}_j = \{0\} \).

In the stationary case \( \forall j \leq j_0 : \varphi_j(t) = \varphi(t) \) and \( \tilde{\varphi}_j(t) = \tilde{\varphi}(t) \).

### 2.2. Nonstationary biorthogonal wavelets

In this subsection, we denote by

(1) The functions \( \varphi_{j,k} \) and \( \tilde{\varphi}_{j,k} \), \( j \leq j_0 \) generate spaces \( V_j \) and \( \tilde{V}_j \) respectively, that is to say

\[
\begin{align*}
\varphi_{j,k}(t) &= \sqrt{2^j} \varphi_j(2^jt - k); \\
\tilde{\varphi}_{j,k}(t) &= \sqrt{2^j} \tilde{\varphi}_j(2^jt - k),
\end{align*}
\]

where,

\[
\begin{align*}
V_j &= \text{span} \{ \varphi_{j,k}(t) \}_{j,k \in \mathbb{Z}}; \\
\tilde{V}_j &= \text{span} \{ \tilde{\varphi}_{j,k}(t) \}_{j,k \in \mathbb{Z}},
\end{align*}
\]

and, \( \varphi_{j,k}(t) \), \( \tilde{\varphi}_{j,k}(t) \) called the scaling functions.

(2) From the embedding condition (2) of definition, we obtain

\[
\begin{align*}
\varphi_{j+1}(t) &= \sum_{k \in \mathbb{Z}} h_k^{[j]} \varphi_j(2t - k); \\
\tilde{\varphi}_{j+1}(t) &= \sum_{k \in \mathbb{Z}} \tilde{h}_k^{[j]} \tilde{\varphi}_j(2t - k).
\end{align*}
\]

The relation (1) called scaling equations, and \( (h_k^{[j]})_{j,k \in \mathbb{Z}}, (\tilde{h}_k^{[j]})_{j,k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \) are the scaling filters.

**Definition 2.2. (Biorthogonal nonstationary wavelets)** The orthogonal complements of \( V_{j+1} \) in \( V_j \) and \( \tilde{V}_{j+1} \) in \( \tilde{V}_j \) are \( W_{j+1} \) and \( \tilde{W}_{j+1} \) respectively such that

\[
\begin{align*}
V_j &= V_{j+1} \oplus W_{j+1} \\
\tilde{V}_j &= \tilde{V}_{j+1} \oplus \tilde{W}_{j+1}
\end{align*}
\]

and

\[
\begin{align*}
W_{j+1} \perp \tilde{V}_{j+1}; \\
\tilde{W}_{j+1} \perp V_{j+1}.
\end{align*}
\]

The spaces \( W_{j+1} \) and \( \tilde{W}_{j+1} \) are defined by

\[
\begin{align*}
W_j &= \text{span} \{ \psi_{j,k}(t) \}_{j,k \in \mathbb{Z}}; \\
\tilde{W}_j &= \text{span} \{ \tilde{\psi}_{j,k}(t) \}_{j,k \in \mathbb{Z}}.
\end{align*}
\]
According to the embedding $W_{j+1} \subset V_j$ we obtain the following relation
\[
\begin{cases}
\psi_{j+1}(t) = \sum_{k \in \mathbb{Z}} g_k^{[j]} \varphi_j(2t - k); \\
\tilde{\psi}_{j+1}(t) = \sum_{k \in \mathbb{Z}} \tilde{g}_k^{[j]} \tilde{\varphi}_j(2t - k),
\end{cases}
\]

from the biorthogonal condition of the wavelets basis we define the sequences $(g_k^{[j]})_{j,k \in \mathbb{Z}}$, $(\tilde{g}_k^{[j]})_{j,k \in \mathbb{Z}}$ as follow
\[
\begin{cases}
g_k^{[j]} = (-1)^k \tilde{h}_1^{[j]}; \\
\tilde{g}_k^{[j]} = (-1)^k h_1^{[j]}.
\end{cases}
\]

Here, the functions $\psi_j$, $\tilde{\psi}_j$ are called wavelets functions and the sequences $(g_k^{[j]})_{j,k \in \mathbb{Z}}$, $(\tilde{g}_k^{[j]})_{j,k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ are the wavelets filters.

3. Construction method

Generally, in the $z$-domaine the biorthogonality condition equivalent to:

\[
H_j(z) \tilde{H}_j(z^{-1}) + H_j(-z) \tilde{H}_j(-z^{-1}) = 4,
\]

as shown in [12], we set:

\[
\begin{cases}
H_j(z) = R_{2^j \alpha} (z) Q_j(z); \\
\tilde{H}_j(z) = R_{2^j \tilde{\alpha}} (z) \tilde{Q}_j(z),
\end{cases}
\]

with
\[
\begin{cases}
R_{2^j \alpha} (z) = \prod_{k=1}^{\tilde{N}} \left( 1 + e^{2i\alpha_k z} \right); \\
R_{2^j \tilde{\alpha}} (z) = \prod_{k=1}^{N} \left( 1 + e^{2i\tilde{\alpha}_k z} \right) .
\end{cases}
\]

Our constractive method consists of the following steps.

Step 1.

By replacing (4) in (3) we obtain

\[
C_j(Z) D_{0j}(Z) + C_j(-Z) D_{0j}(-Z) = 2,
\]
where

\[ C_j(Z) = \tilde{c}_j z^{\frac{N-N}{2}} R_{2^j \tilde{a}}(z) R_{2^j \tilde{a}}(z^{-1}), \]

such as: \( N \) and \( \tilde{N} \) are the lengths of the complex vectors \( \tilde{\alpha} \) and \( \tilde{\alpha} \) respectively, \( \tilde{c}_j = \prod_{k=1}^{\tilde{N}} e^{-2^i \alpha_k} \) and \( Z = \frac{z + z^{-1}}{2} \).

In this step we find the shortest solution \( D_{0j}(Z) \) of the equation (5), using the interpolation technic, see [1].

**Step 2.**

Performing the spectral factorization

\[ 2z^\frac{N-N}{2} \tilde{c}_j D_{0j} \left( \frac{z + z^{-1}}{2} \right) = Q_j(z) \tilde{Q}_j(z), \]

as proposed in [12], \( Q_j(z) = 2^{1-N} \) then

\[ \tilde{Q}_j(z) = 2^N z^\frac{N-N}{2} \tilde{c}_j D_{0j} \left( \frac{z + z^{-1}}{2} \right). \]

So

\[
\begin{align*}
H_j(z) &= 2^{1-N} R_{2^j \tilde{\alpha}}(z); \\
\tilde{H}_j(z) &= R_{2^j \tilde{\alpha}}(z) 2^N z^\frac{N-N}{2} \tilde{c}_j D_{0j} \left( \frac{z + z^{-1}}{2} \right).
\end{align*}
\]

3.1. **The proposed nonstationary biorthogonal wavelets.** Our proposed Bior-NSW is based on the choice of vectors \( \tilde{\alpha} \) and \( \alpha \) with the following manner \( \tilde{\alpha} = \alpha = (i\omega, -i\omega, 0), \omega \in \mathbb{R} \).

By formulating equation (5) and applying the steps (1) and (2) we obtain:

\[ H_j(z) = \frac{\sqrt{2}}{4(\beta + 1)} z^{-3} + \frac{\sqrt{2}(2\beta + 1)}{4(\beta + 1)} z^{-2} + \frac{\sqrt{2}(2\beta + 1)}{4(\beta + 1)} z^{-1} + \frac{\sqrt{2}}{4(\beta + 1)} \]

and

\[ \tilde{H}_j(z) = \frac{\sqrt{2}(2\beta + 1)}{32\beta^3(\beta + 1)} z^{-5} - \frac{\sqrt{2}(2\beta + 1)^2}{32\beta^3(\beta + 1)} z^{-4} - \frac{\sqrt{2}(8\beta^3 + 4\beta^2 - 4\beta - 1)}{32\beta^3(\beta + 1)} z^{-3} \]

\[ + \frac{\sqrt{2}(16\beta^4 + 24\beta^3 + 8\beta^2 - 2\beta - 1)}{32\beta^3(\beta + 1)} z^{-2} + \frac{\sqrt{2}(16\beta^4 + 24\beta^3 + 8\beta^2 - 2\beta - 1)}{32\beta^3(\beta + 1)} z^{-1} \]

\[ - \frac{\sqrt{2}(8\beta^3 + 4\beta^2 - 4\beta - 1)}{32\beta^3(\beta + 1)} - \frac{\sqrt{2}(2\beta + 1)^2}{32\beta^3(\beta + 1)} z^{-2} + \frac{\sqrt{2}(2\beta + 1)}{32\beta^3(\beta + 1)} z^2; \]
where, $\beta = \cos(2^j \omega)$. Thus, the biorthogonal scaling filters are

$$h_{-3}^{[j]} = \frac{\sqrt{2}}{4(\beta+1)}, \quad h_{-2}^{[j]} = \frac{\sqrt{2}(2\beta+1)}{4(\beta+1)} - \frac{\sqrt{2}}{4(\beta+1)}, \quad h_{-1}^{[j]} = \frac{\sqrt{2}(2\beta+1)}{4(\beta+1)}, \quad h_0^{[j]} = \frac{\sqrt{2}}{4(\beta+1)}$$

$$\tilde{h}_{-5}^{[j]} = \frac{\sqrt{2}(2\beta+1)}{32\beta^3(\beta+1)}$$

$$\tilde{h}_{-4}^{[j]} = -\frac{\sqrt{2}(2\beta+1)^2}{32\beta^3(\beta+1)}$$

$$\tilde{h}_{-3}^{[j]} = -\frac{\sqrt{2}(8\beta^3+4\beta^2-4\beta-1)}{32\beta^3(\beta+1)}$$

$$\tilde{h}_{-2}^{[j]} = \frac{\sqrt{2}(16\beta^4+24\beta^3+8\beta^2-2\beta-1)}{32\beta^3(\beta+1)}$$

For obtaining the wavelets filters we use the relation (2).

**Example 1.** If we choose $\omega = 0.0240$ and $j = 1$ we obtain the following biorthogonal scaling and wavelets filters

$$h_{-3}^{[1]} = 0.1769, \quad h_{-1}^{[1]} = 0.5302$$

$$h_{-2}^{[1]} = 0.5302, \quad h_0^{[1]} = 0.1769$$

$$\tilde{h}_{-5}^{[1]} = 0.0665, \quad \tilde{h}_{-1}^{[1]} = -0.1994$$

$$\tilde{h}_{-4}^{[1]} = -0.1546, \quad \tilde{h}_{-1}^{[1]} = 0.9946$$

$$\tilde{h}_{-3}^{[1]} = 0.9946, \quad \tilde{h}_{-1}^{[1]} = -0.1546$$

$$\tilde{h}_{-2}^{[1]} = -0.1994, \quad \tilde{h}_{-2}^{[1]} = 0.0665$$

$$\tilde{g}_{-1}^{[1]} = -0.1769, \quad \tilde{g}_{-1}^{[1]} = -0.5302$$

$$\tilde{g}_{-2}^{[1]} = 0.5302, \quad \tilde{g}_{-2}^{[1]} = 0.1769$$

$$\tilde{g}_{-1}^{[1]} = -0.0665, \quad \tilde{g}_{-1}^{[1]} = 0.1994$$

$$\tilde{g}_{-2}^{[1]} = -0.1546, \quad \tilde{g}_{-2}^{[1]} = 0.9946$$

$$\tilde{g}_{-3}^{[1]} = -0.9946, \quad \tilde{g}_{-3}^{[1]} = 0.1546$$

$$\tilde{g}_{-2}^{[1]} = -0.1994, \quad \tilde{g}_{-2}^{[1]} = 0.0665$$

Using cascad algorithm [6], we obtain the following biorthogonal scaling and wavelets functions
4. Experimental and comparative study

In this section, we effect a comparative study for the purpose of choosing a best Bior-NSW to generate an $L^2(\mathbb{R})$ basis for approximating signals and images. Note that, our technique of choosing the best Bior-NSW wavelet is based on the choice of $\omega$ through the maximum value of the PSNR quality measure (short for Peak Signal to Noise Ratio), where

$$PSNR = 10 \log_{10} \left( \frac{\|x\|_2^2}{\|x - xr\|_2^2} \right),$$

and $x, xr$ are respectively the original and the constructed signal. Our comparative study is composed by two parts: approximation part and thresholding part.

**Approximation part:**

In this part of our comparative study, we approximate a class of signals in the proposed Bior-NSW basis and in the Daubechies family of biorthogonal wavelets basis, then we compare between the obtain results using the PSNR quality measure. In our study we use the following class of signals: Doppler, blocks, heavy sine, bumps, ECG and cameraman image (CI), then we get the following table of results.
Table 1. Approximation results obtained by the proposed Bior-NSW and Daubechies biorthogonal wavelets family

<table>
<thead>
<tr>
<th>Bior Daubechies</th>
<th>PSNR Doppler</th>
<th>PSNR Blocks</th>
<th>PSNR Heavy Sine</th>
<th>PSNR Bumps</th>
<th>PSNR ECG</th>
<th>PSNR CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bior 1.1</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>305.0227</td>
<td>300.8817</td>
<td></td>
</tr>
<tr>
<td>Bior 1.3</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.9560</td>
<td>300.8962</td>
<td></td>
</tr>
<tr>
<td>Bior 1.5</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.0740</td>
<td>300.8513</td>
<td></td>
</tr>
<tr>
<td>Bior 2.2</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>302.2617</td>
<td>298.8938</td>
<td></td>
</tr>
<tr>
<td>Bior 2.4</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>301.3887</td>
<td>301.3887</td>
<td></td>
</tr>
<tr>
<td>Bior 2.6</td>
<td>307,9099</td>
<td>307,9099</td>
<td>307,9099</td>
<td>307.8549</td>
<td>305.1269</td>
<td></td>
</tr>
<tr>
<td>Bior 2.8</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304.0225</td>
<td>301.3887</td>
<td></td>
</tr>
<tr>
<td>Bior 3.1</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.6168</td>
<td>300.6334</td>
<td></td>
</tr>
<tr>
<td>Bior 3.3</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.0174</td>
<td>299.7753</td>
<td></td>
</tr>
<tr>
<td>Bior 3.5</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.2998</td>
<td>299.7939</td>
<td></td>
</tr>
<tr>
<td>Bior 3.7</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.0174</td>
<td>299.7753</td>
<td></td>
</tr>
<tr>
<td>Bior 3.9</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.2998</td>
<td>299.7939</td>
<td></td>
</tr>
<tr>
<td>Bior 4.4</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.2998</td>
<td>299.7939</td>
<td></td>
</tr>
<tr>
<td>Bior 5.5</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.2998</td>
<td>299.7939</td>
<td></td>
</tr>
<tr>
<td>Bior 6.8</td>
<td>304,8595</td>
<td>304,8595</td>
<td>304,8595</td>
<td>303.2998</td>
<td>299.7939</td>
<td></td>
</tr>
<tr>
<td>Proposed Bior-NSW</td>
<td>313,6768</td>
<td>317,1361</td>
<td>314,3498</td>
<td>314,3029</td>
<td>315.2481</td>
<td></td>
</tr>
<tr>
<td>Best value of ω</td>
<td>0.0240</td>
<td>1.0560</td>
<td>0.1260</td>
<td>0.0240</td>
<td>0.1400</td>
<td>0.0410</td>
</tr>
</tbody>
</table>
Figure 2. Original and reconstructed signals by the proposed BiorNSW
In general, as it can be observed from these obtained results, our proposed Bior-NSW exhibits a power of approximation compared to the Daubechies biorthogonal wavelets family.

**Thresholding part:**

Generally, during the process of data lossy compression, thresholding is an important operation this means, find a threshold (TH) according to a quality criteria and cancelling the signal coefficients which have a modulus less than th with preserving the quality of the reconstructed signal. In this part of our study we choose the best biorthogonal wavelet concerning Daubechies family which has a best PSNR and our proposed Bior-NSW with the aim of making a comparison study between them in the thresholding operation. In our case, we perform an analysis of the PSNR measure with respect to the threshold (TH) and we obtain the following results.

From the above results, we remark that: the process of thresholding in the wavelets basis which are generated by our Bior-NSW gives a good results compared to the Daubechies biorthogonal wavelets family.
Figure 4. Compared Bior2.4 with proposed Bior-NSW during the thresholding process applying in doppler (A), bumps (B), heavy sine (C) and blocks (D)

5. Conclusion

In this paper, we have proposed a class of Bior-NSW with the aim to approximating and thresholding signals in the wavelets basis, which are an important operations in the compression process. In fact, our study results show a superior quality of approximation concerning our proposed Bior-NSW compared to Daubechies biorthogonal wavelets.
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