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BIORTHOGONAL NONSTATIONARY AND DAUBECHIES WAVELETS: A COMPARATIVE STUDY

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ABSTRACT. In this article, we present a constructive technic of a biorthogonal nonstationary wavelets (Bior-NSW) family which are proposed by [12], and we perform a comparative study between the classical Daubechies biorthogonal wavelets family and the constructed nonstationary wavelets, which prooved the dominance of the nonstationary wavelets in approximation of different kinds signals.

1. INTRODUCTION

Wavelet analysis is a very important famous area in mathematics, it has become a powerful tool in signal processing compared to Fourrier analysis, thanks to the location of information property in time domain. In this context, several researchers have invested in this area such as [3, 4, 7, 9, 10]. Fourier transform only provides the spectral components of signals not their temporal locations, however wavelets can provide a time-frequency representation of signals [5, 8, 11]. Most compression algorithms [2] integrate biorthogonal wavelets with the aim to approximate the signal in the first step have proved their effectiveness. Nonstationary biorthogonal wavelets is a generalization of the Daubechies biorthogonal wavelets family introduced recently in [12]. The main difference with the construction of the

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standard biorthogonal wavelets is, that the multiresolution spaces are generated by scale dependent functions. This property offers a good approximation of $L^2(\mathbb{R})$ signals class using this kind of wavelets basis. In this paper, we present a construction method of a Bior-NSW family based on [12] and we carry out a comparative study concerning the approximation of signals between our proposed Bior-NSW and the standard Daubechies biorthogonal wavelets. Our study consists in approximating the different kinds of Matlab signals classes such as doppler, bumps, blocks and heavy sine,..., etc in the biorthogonal wavelets basis.

The rest of paper is organized as follows. Section 2 gives some notions on nonstationary multiresolution analysis and wavelets. Section 3 present the constructed method of Bior-NSW. Section 4 present a exprimental and comparative study between the obtain results. Finally, a conclusion is drawn in section 5.

2. Preliminaries

In this section we present a basic properties and notions concerning biorthogonal nonstationary multiresolutions and nonstationary wavelets.

2.1. Biorthogonal nonstationary multiresolutions analysis.

Definition 2.1. *Let* $(\varphi_j(t))_{j\le j_0}$ *and* $(\tilde{\varphi}_j(t))_{j\le j_0}$ *be a sequences of functions in* $L^2(\mathbb{R})$, *such that* $j_0\in\mathbb{Z}$. *The couple* $\left((V_j)_{j\le j_0},\ \left(\tilde{V}_j\right)_{j\le j_0}\right)$ *such as:*

$$\begin{cases} V_j=\left\{f(t)=\sum\limits_{k\in\mathbb{Z}} a_k\varphi_j\left(2^{-j}t-k\right),\ a\in l^2(\mathbb{Z})\right\}; \\ \tilde{V}_j=\left\{g(t)=\sum\limits_{k\in\mathbb{Z}} b_k\tilde{\varphi}_j\left(2^{-j}t-k\right),\ b\in l^2(\mathbb{Z})\right\}, \end{cases}$$

defines a biorthogonal nonstationary multiresolution if and only if:

(1) *For all* $j\le j_0$, $\varphi_j(t)$ *and* $\tilde{\varphi}_j(t)$ *generate a Riesz basis, i.e:* $\exists\, C_{1j},\ C_{2j}\in\mathbb{R}_+^*\setminus\forall$ $(c_k)_{k\in\mathbb{Z}}\in l^2(\mathbb{Z})$:

$$C_{1j}\left\|c\right\|_{l^2(\mathbb{Z})}\le\left\|\sum_{k\in\mathbb{Z}} c_k\varphi_j(.-k)\right\|_{L^2(\mathbb{R})}\le C_{2j}\left\|c\right\|_{l^2(\mathbb{Z})},$$

where, C_{1j}, C_{2j} *are called lower and upper Riesz bounds respectively, it is the same in the case of* $\tilde{\varphi}_j(t)$.

(2) *For all* $j \leq j_0,\ V_{j+1} \subset V_j,\ \tilde{V}_{j+1} \subset \tilde{V}_j$,
(3) $\overline{\bigcup_{j \leq j_0} V_j} = \overline{\bigcup_{j \leq j_0} \tilde{V}_j} = L^2(\mathbb{R})$,
(4) $\bigcap_{j \leq j_0} V_j = \bigcap_{j \leq j_0} \tilde{V}_j = \{0\}$.

In the stationary case $\forall j \leq j_0 : \varphi_j(t) = \varphi(t)$ and $\tilde{\varphi}_j(t) = \tilde{\varphi}(t)$.

2.2. **Nonstationary biorthogonal wavelets.** In this subsection, we denote by

(1) The functions $\varphi_{j,k}$ and $\tilde{\varphi}_{j,k},\ j \leq j_0$ generate spaces V_j and $\tilde{V}_j$ respectively, that is to say

$$\begin{cases} \varphi_{j,k}(t) = \sqrt{2}^j \varphi_j(2^j t - k); \\ \tilde{\varphi}_{j,k}(t) = \sqrt{2}^j \tilde{\varphi}_j(2^j t - k), \end{cases}$$

where,

$$\begin{cases} V_j = span\{\varphi_{j,k}(t)\}_{j,k \in \mathbb{Z}}; \\ \tilde{V}_j = span\{\tilde{\varphi}_{j,k}(t)\}_{j,k \in \mathbb{Z}}, \end{cases}$$

and, $\varphi_{j,k}(t),\ \tilde{\varphi}_{j,k}(t)$ called the scaling functions

(2) From the embeding condition (2) of definition, we obtain

$$\begin{cases} \varphi_{j+1}(t) = \sum_{k \in \mathbb{Z}} h_k^{[j]} \varphi_j(2t - k); \\ \tilde{\varphi}_{j+1}(t) = \sum_{k \in \mathbb{Z}} \tilde{h}_k^{[j]} \tilde{\varphi}_j(2t - k). \end{cases} \tag{1}$$

The relation (1) called scaling equations, and $(h_k^{[j]})_{j,k \in \mathbb{Z}}, (\tilde{h}_k^{[j]})_{j,k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ are the scaling filters.

Definition 2.2. *(Biorthogonal nonstationary wavelets) The orthogonal complements of V_{j+1} in V_j and $\tilde{V}_{j+1}$ in $\tilde{V}_j$ are W_{j+1} and $\tilde{W}_{j+1}$ respectively such that*

$$\begin{cases} V_j = V_{j+1} \oplus W_{j+1} \\ \tilde{V}_j = \tilde{V}_{j+1} \oplus \tilde{W}_{j+1} \end{cases} \quad \textit{and} \quad \begin{cases} W_{j+1} \perp \tilde{V}_{j+1}; \\ \tilde{W}_{j+1} \perp V_{j+1}. \end{cases}$$

The spaces W_{j+1} and $\tilde{W}_{j+1}$ are define by

$$\begin{cases} W_j = span\{\psi_{j,k}(t)\}_{j,k \in \mathbb{Z}}; \\ \tilde{W}_j = span\left\{\tilde{\psi}_{j,k}(t)\right\}_{j,k \in \mathbb{Z}}. \end{cases}$$

According to the embeding $W_{j+1} \subset V_j$ we obtain the following relation

$$\begin{cases} \psi_{j+1}(t) = \sum_{k\in\mathbb{Z}} g_k^{[j]} \varphi_j (2t-k); \\ \tilde{\psi}_{j+1}(t) = \sum_{k\in\mathbb{Z}} \tilde{g}_k^{[j]} \tilde{\varphi}_j (2t-k), \end{cases}$$

from the biorthogonal condition of the wavelets basis we define the sequences $(g_k^{[j]})_{j,k\in\mathbb{Z}}$, $(\tilde{g}_k^{[j]})_{j,k\in\mathbb{Z}}$ as follow

$$\begin{cases} g_k^{[j]} = (-1)^k \tilde{h}_{1-k}^{[j]}; \\ \tilde{g}_k^{[j]} = (-1)^k h_{1-k}^{[j]}. \end{cases} \tag{2}$$

Here, the functions ψ_j, $\tilde{\psi}_j$ are called wavelets functions and the sequences $(g_k^{[j]})_{j,k\in\mathbb{Z}}$, $(\tilde{g}_k^{[j]})_{j,k\in\mathbb{Z}} \in l^2(\mathbb{Z})$ are the wavelets filters.

3. Construction method

Generally, in the $z-$domaine the biorthogonality condition equivalent to:

$$H_j(z)\tilde{H}_j\left(z^{-1}\right) + H_j(-z)\tilde{H}_j\left(-z^{-1}\right) = 4, \tag{3}$$

as shown in [12], we set:

$$\begin{cases} H_j(z) = R_{2^j\vec{\alpha}}(z)\, Q_j(z); \\ \tilde{H}_j(z) = R_{2^j\vec{\tilde{\alpha}}}(z)\, \tilde{Q}_j(z), \end{cases} \tag{4}$$

with

$$\begin{cases} R_{2^j\vec{\alpha}}(z) = \prod_{k=1}^{N}\left(1 + e^{2^j\alpha_k} z\right); \\ R_{2^j\vec{\tilde{\alpha}}}(z) = \prod_{k=1}^{\tilde{N}}\left(1 + e^{2^j\tilde{\alpha}_k} z\right). \end{cases}$$

Our constractive method consists of the following steps.

Step 1.

By replacing (4) in (3) we obtain

$$C_j(Z)\, D_{0j}(Z) + C_j(-Z)\, D_{0j}(-Z) = 2, \tag{5}$$

where

$$C_j\left(Z\right) = \tilde{c}_j z^{\frac{N-\tilde{N}}{2}} R_{2^j \overrightarrow{\alpha}}\left(z\right) R_{2^j \overrightarrow{\tilde{\alpha}}}\left(z^{-1}\right),$$

such as: N and $\tilde{N}$ are the lengths of the complex vectors $\overrightarrow{\alpha}$ and $\overrightarrow{\tilde{\alpha}}$ respectively, $\tilde{c}_j = \prod_{k=1}^{\tilde{N}} e^{-2^j \alpha_k}$ and $Z = \frac{z+z^{-1}}{2}$.

In this step we find the shortest solution $D_{0j}\left(Z\right)$ of the equation (5), using the interpolation technic, see [1].

Step 2.

Performing the spectral factorization

$$2z^{\frac{N-\tilde{N}}{2}} \tilde{c}_j D_{0j}\left(\frac{z+z^{-1}}{2}\right) = Q_j\left(z\right) \tilde{Q}_j\left(z\right),$$

as proposed in [12], $Q_j\left(z\right) = 2^{1-N}$ then

$$\tilde{Q}_j\left(z\right) = 2^N z^{\frac{\tilde{N}-N}{2}} \tilde{c}_j D_{0j}\left(\frac{z+z^{-1}}{2}\right).$$

So

$$\begin{cases} H_j\left(z\right) = 2^{1-N} R_{2^j \overrightarrow{\alpha}}\left(z\right); \\ \tilde{H}_j\left(z\right) = R_{2^j \overrightarrow{\tilde{\alpha}}}\left(z\right) 2^N z^{\frac{\tilde{N}-N}{2}} \tilde{c}_j D_{0j}\left(\frac{z+z^{-1}}{2}\right). \end{cases}$$

3.1. **The proposed nonstationary biorthogonal wavelets.** Our proposed Bior-NSW is based on the choise of vectors $\overrightarrow{\alpha}$ and $g\ \overrightarrow{\tilde{\alpha}}$ with the following manner $\overrightarrow{\alpha} = \overrightarrow{\tilde{\alpha}} = \left(i\omega,\ -i\omega,\ 0\right),\ \omega \in \mathbb{R}$.

By formulating equation (5) and applying the steps (1) and (2) we obtain:

$$H_j(z) = \frac{\sqrt{2}}{4(\beta+1)} z^{-3} + \frac{\sqrt{2}(2\beta+1)}{4(\beta+1)} z^{-2} + \frac{\sqrt{2}(2\beta+1)}{4(\beta+1)} z^{-1} + \frac{\sqrt{2}}{4(\beta+1)}$$

and

$$\begin{aligned} &\tilde{H}_j\left(z\right) \\ &= \frac{\sqrt{2}(2\beta+1)}{32\beta^3(\beta+1)} z^{-5} - \frac{\sqrt{2}(2\beta+1)^2}{32\beta^3(\beta+1)} z^{-4} - \frac{\sqrt{2}(8\beta^3+4\beta^2-4\beta-1)}{32\beta^3(\beta+1)} z^{-3} \\ &+ \frac{\sqrt{2}(16\beta^4+24\beta^3+8\beta^2-2\beta-1)}{32\beta^3(\beta+1)} z^{-2} + \frac{\sqrt{2}(16\beta^4+24\beta^3+8\beta^2-2\beta-1)}{32\beta^3(\beta+1)} z^{-1} \\ &- \frac{\sqrt{2}(8\beta^3+4\beta^2-4\beta-1)}{32\beta^3(\beta+1)} - \frac{\sqrt{2}(2\beta+1)^2}{32\beta^3(\beta+1)} z + \frac{\sqrt{2}(2\beta+1)}{32\beta^3(\beta+1)} z^2, \end{aligned}$$

where, $\beta = \cos(2^j\omega)$. Thus, the biorthogonal scaling filters are

$$\boxed{h_{-3}^{[j]} = \frac{\sqrt{2}}{4(\beta+1)},\ h_{-2}^{[j]} = \frac{\sqrt{2}(2\beta+1)}{4(\beta+1)},\ h_{-1}^{[j]} = \frac{\sqrt{2}(2\beta+1)}{4(\beta+1)},\ h_{0}^{[j]} = \frac{\sqrt{2}}{4(\beta+1)}}$$

$$\begin{array}{|ll|}\hline \tilde{h}_{-5}^{[j]} = \frac{\sqrt{2}(2\beta+1)}{32\beta^3(\beta+1)} & \tilde{h}_{-1}^{[j]} = \frac{\sqrt{2}\left(16\beta^4+24\beta^3+8\beta^2-2\beta-1\right)}{32\beta^3(\beta+1)} \\ \tilde{h}_{-4}^{[j]} = -\frac{\sqrt{2}(2\beta+1)^2}{32\beta^3(\beta+1)} & \tilde{h}_{0}^{[j]} = -\frac{\sqrt{2}\left(8\beta^3+4\beta^2-4\beta-1\right)}{32\beta^3(\beta+1)} \\ \tilde{h}_{-3}^{[j]} = -\frac{\sqrt{2}\left(8\beta^3+4\beta^2-4\beta-1\right)}{32\beta^3(\beta+1)} & \tilde{h}_{1}^{[j]} = -\frac{\sqrt{2}(2\beta+1)^2}{32\beta^3(\beta+1)} \\ \tilde{h}_{-2}^{[j]} = \frac{\sqrt{2}\left(16\beta^4+24\beta^3+8\beta^2-2\beta-1\right)}{32\beta^3(\beta+1)} & \tilde{h}_{2}^{[j]} = \frac{\sqrt{2}(2\beta+1)}{32\beta^3(\beta+1)} \\ \hline \end{array}$$

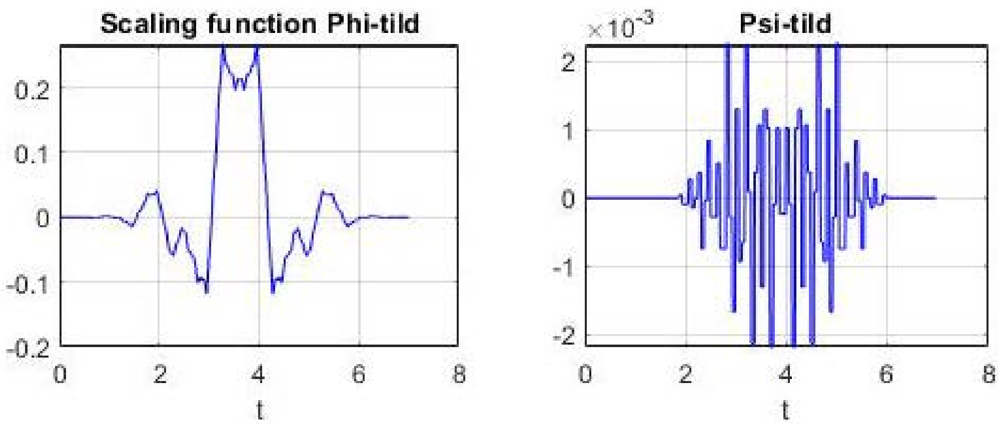
For obtaining the wavelets filters we use the relation (2).

Example 1. *If we choose $\omega = 0.0240$ and $j = 1$ we obtain the following biorthogonal scaling an wevelets filters*

$$\begin{array}{|ll|}\hline h_{-3}^{[1]} = 0.1769 & h_{-1}^{[1]} = 0.5302 \\ h_{-2}^{[1]} = 0.5302 & h_{0}^{[1]} = 0.1769 \\ \hline \end{array} \qquad \begin{array}{|ll|}\hline \tilde{g}_{1}^{[1]} = -0.1769 & \tilde{g}_{3}^{[1]} = -0.5302 \\ \tilde{g}_{2}^{[1]} = 0.5302 & \tilde{g}_{4}^{[1]} = 0.1769 \\ \hline \end{array}$$

$$\begin{array}{|ll|}\hline \tilde{h}_{-5}^{[1]} = 0.0665 & \tilde{h}_{-1}^{[1]} = -0.1994 \\ \tilde{h}_{-4}^{[1]} = -0.1546 & \tilde{h}_{0}^{[1]} = 0.9946 \\ \tilde{h}_{-3}^{[1]} = 0.9946 & \tilde{h}_{1}^{[1]} = -0.1546 \\ \tilde{h}_{-2}^{[1]} = -0.1994 & \tilde{h}_{2}^{[1]} = 0.0665 \\ \hline \end{array} \qquad \begin{array}{|ll|}\hline g_{-1}^{[1]} = -0.0665 & g_{3}^{[1]} = 0.1994 \\ g_{0}^{[1]} = -0.1546 & g_{4}^{[1]} = 0.9946 \\ g_{1}^{[1]} = -0.9946 & g_{5}^{[1]} = 0.1546 \\ g_{2}^{[1]} = -0.1994 & g_{6}^{[1]} = 0.0665 \\ \hline \end{array}$$

Using cascad algorithm [6], we obtain the following biorthogonal scaling and wevelets functions



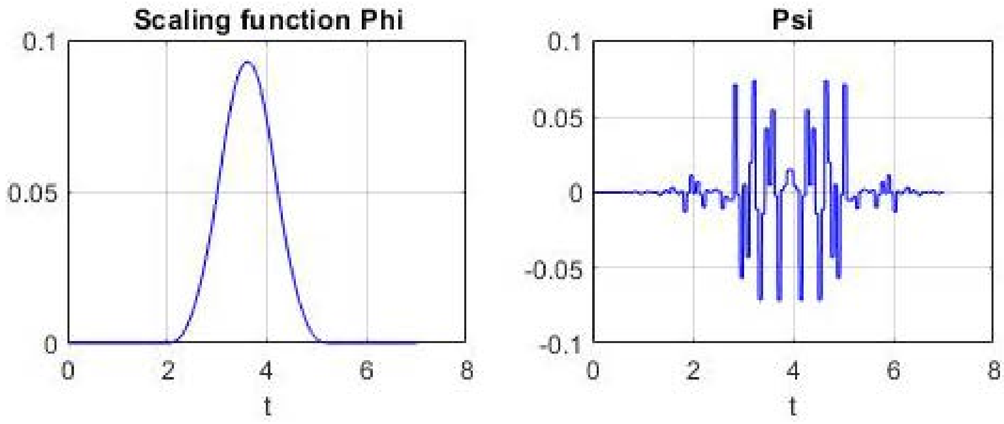


FIGURE 1. Functions $\phi, \tilde{\phi}, \psi, \tilde{\psi}$ for $j = 1, \omega = 0.0240$

4. EXPERIMENTAL AND COMPARATIVE STUDY

In this section, we effect a comparative study for the purpose of choosing a best Bior-NSW to generate an $L^2(\mathbb{R})$ basis for approximating signals and images. Note that, our technique of choosing the best Bior-NSW wavelet is based on the choice of ω through the maximum value of the PSNR quality measure (short for Peak Signal to Noise Ratio), where

$$PSNR = 10\log_{10}\left(\frac{\|x\|_2^2}{\|x - xr\|_2^2}\right),$$

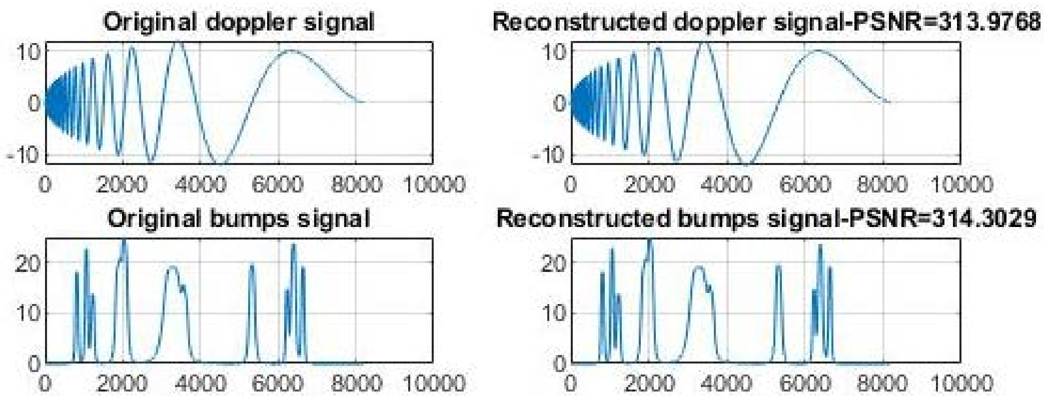
and x, xr are respectively the original and the constructed signal. Our comparative study is composed by two parts: approximation part and thresholding part.

Approximation part:

In this part of our comparative study, we approximate a class of signals in the proposed Bior-NSW basis and in the Daubechies family of biorthogonal wavelets basis, then we compare between the obtain results using the PSNR quality measure. In our study we use the following class of signals: Doppler, blocks, heavy sine, bumps, ECG and cameraman image (CI), then we get the following table of results.

TABLE 1. Approximation results obtained by the proposed Bior-NSW and Daubechies biorthogonal wavelets family

Bior Daubechies	PSNR Doppler	PSNR Blocks	PSNR Heavy Sine	PSNR Bumps	PSNR ECG	PSNR CI
Bior 1.1	304,8595	304,8595	304,8595	304,8595	305.0227	300.8817
Bior 1.3	304,5391	304,5391	304,5391	304,5391	303.9560	3008962
Bior 1.5	304,1961	304,1961	304,1961	304,1961	303.0740	300.8513
Bior 2.2	302,7128	302,7128	302,7128	302,7128	302.2617	298.8938
Bior 2.4	308,2876	308,2876	308,2876	308,2876	307.9377	305.2787
Bior 2.6	307,9099	307,9099	307,9099	307,9099	307.8459	305.1269
Bior 2.8	304,5545	304,5545	304,5545	304,5545	304.0225	301.3887
Bior 3.1	299,7088	299,7088	299,7088	299,7088	299.7691	296.1518
Bior 3.3	304,7780	304,7780	304,7780	304,7780	304,2777	300.7060
Bior 3.5	304,0868	304,0868	304,0868	304,0868	303.6168	300.6334
Bior 3.7	302,8794	302,8794	302,8794	302,8794	303.0174	299.7753
Bior 3.9	302,4278	302,4278	302,4278	302,4278	303.2998	299.7939
Bior 4.4	252,7010	252,7010	252,7010	252,7010	259.7322	238.9062
Bior 5.5	248,8891	248,8891	248,8891	248,8891	257.6999	236.3364
Bior 6.8	267,6460	267,6460	267,6460	267,6460	273.5575	252.8666
Proposed Bior-NSW	**313,6768**	**317,1361**	**314,3498**	**314,3029**	**315.2481**	**312.9809**
Best value of ω	0.0240	1.0560	0.1260	0.0240	0.1400	0.0410



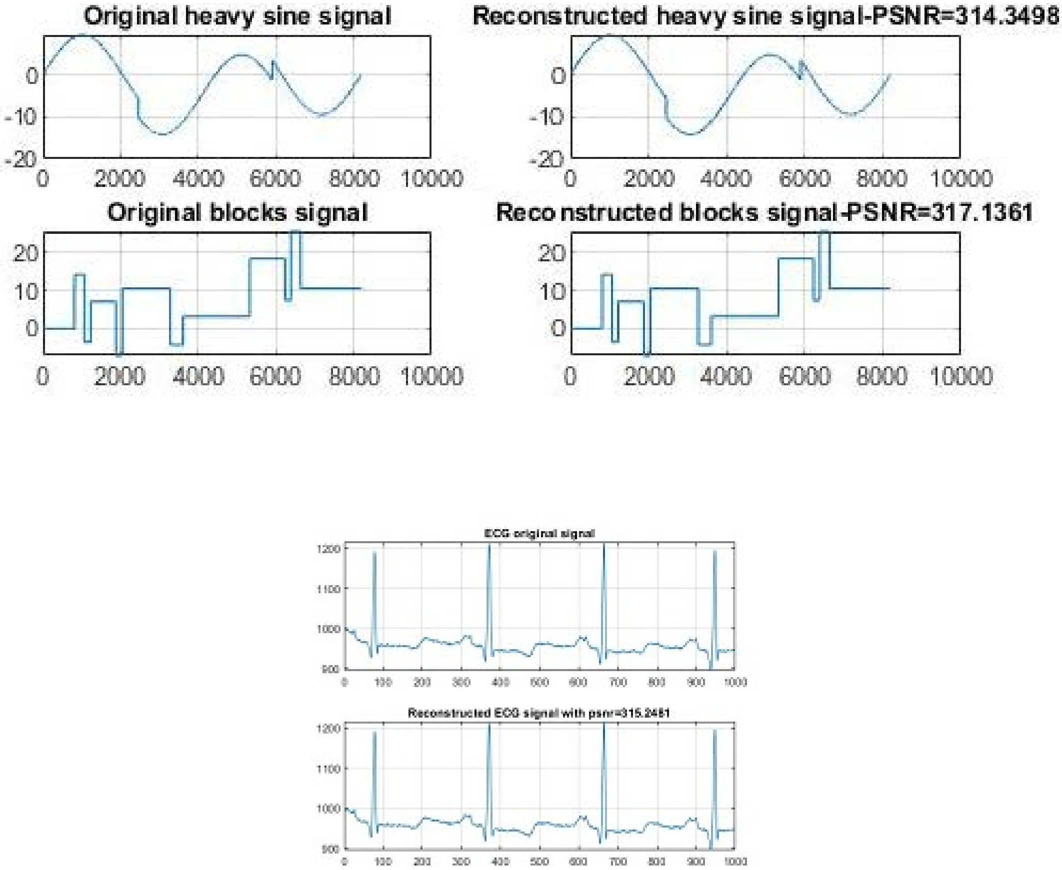


FIGURE 2. Original and reconstructed signals by the proposed Bior-NSW

FIGURE 3. Original and reconstructed cameraman image 256×256 by the proposed Bior-NSW

In general, as it can be observed from these obtained results, our proposed Bior-NSW exhibits a power of appromation compared to the Daubechies biorthogonal wavelets family.

Thresholding part:

Generaly, during the process of data lossy compresion, thresholding is an important operation this means, find a threshold (TH) according to a quality creteria and cancelling the signal coeffients which have a modulus less than th with preserving the quality of the reconstructed signal. In this part of our study we choose the best biorthgonal wavelet concerning Daubechies family which has a best PSNR and our proposed Bior-NSW with the aim of making a comparison study between them in the thresholding operation. In our case, we perform an analysis of the PSNR measure with respect to the threshold (TH) and we obtain the following results.

From the above results, we remark that: the process of thresholding in the wavelets basis which are genarated by our Bior-NSW gives a good results compared to the Daubechies biorthogonal wavelets family.

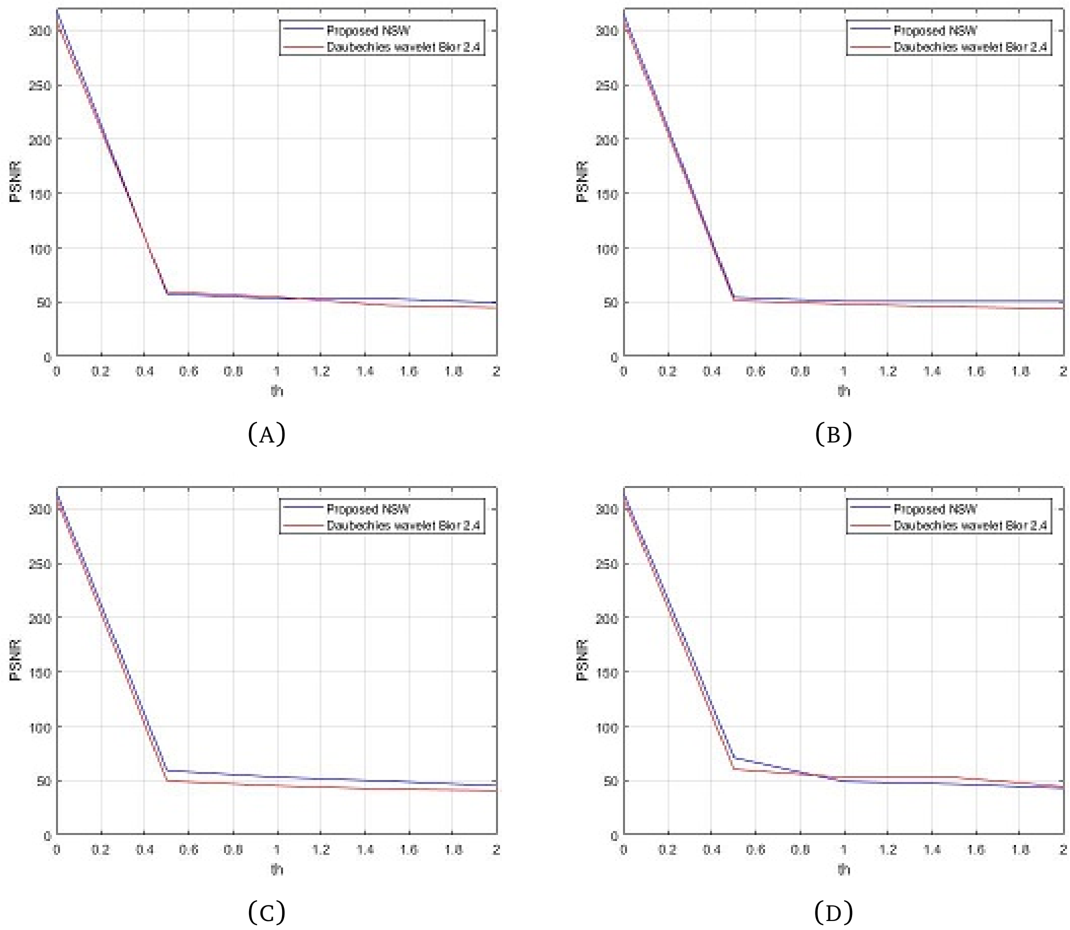


FIGURE 4. Compared Bior2.4 with proposed Bior-NSW during the thresholding process applying in doppler (A), bumps (B), heavy sine (C) and blocks (D)

5. CONCLUSION

In this paper, we have proposed a class of Bior-NSW with the aim to approximating and thresholding signals in the wavelets basis, which are an important operations in the compression process. In fact, our study results show a superior quality of approximation concerning our proposed Bior-NSW compared to Daubechies biorthogonal wavelets.

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