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SMALL PSEUDO QUASI PRINCIPALLY INJECTIVE ACTS

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ABSTRACT. In act theory, Pseudo injective acts and their generalizations are essential. As a result, the purpose of this work is to give a generalization of pseudo quasi principally injective acts specifically small pseudo quasi principally injective acts. Besides, the concept of a small pseudo injective was presented, which can be employed later. If each S-monomorphism from a small M-cyclic sub-act of M_S to N_S is extended to S-homomorphism from M_S to N_S, An S-act N_S is called a small pseudo-M-principally-injective (for simply small pseudo-MP-injective). Also, if an S-act M_S is a small pseudo-MP-injective act, then it is called small pseudo quasi principally injective. This form of generalization is given several new characterizations and properties. Following that conditions are shown under which sub acts inherit the property of small pseudo quasi principally injectivity. Furthermore, the connection between the classes of small pseudo quasi principally injective S-acts with classes of injectivity is addressed, and as a result, requirements to coincide these classes are shown. Our work's conclusions have been explained.

1. INTRODUCTION

In [26] small injective rings are studied and then in [27-29] several generalizations are introduced to that notion. One of these was small pseudo QP-injective

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Modules. This motivated us to extend this notion in acts. Besides, in [1], the author introduced a generalization of quasi principally injective S-acts in [2] which was pseudo quasi principally injective S-acts (for simply, PQP-injective)and obtained some results. More generally, during this work, we still to find another weak form of pseudo quasi principally injectivity acts and pseudo injective S-acts called small pseudo QP-injective acts and small pseudo injective acts to review the behavior of quasi injective through the property, "an S-act M_S is quasi injective if, and only if, it's invariant in injective envelope of itself", which is satisfying in small pseudo injective S-act for each S-monomorphism from injective envelope of S-act M_S to itself. Thus, by using this property and taking cog-reversible S-acts, we obtain coinciding between small quasi injective acts with small pseudo injective acts in Theorem 2.1. It is common for an S-act to often be found in other terms as S-systems, S-sets, S-polygons, S-automata [19].

For more details about semigroups, S-acts and injective acts, we refer the reader to the references [1], [3-25]. Where injective act are often defined as follows: an S-act AS is named injective if for every monomorphism $\alpha : C_S \to B_S$ and every S-homomorphism $\beta : C_S \to A_S$, there exists an S-homomorphism $\sigma : B_S \to A_S$ such $\sigma\alpha = \beta$[20]. In ([25], Definition 3.1.15), the author define cog-reversible where an S-act M_S is named cog-reversible if each congruenceρ on M_S with $\rho \neq I_M$ is large on M_S. A subact N of an S-act M_S, is a non-empty subset of M such that $xs \in N$ for all $x \in N$ and $s \in S$ [20]. In [7], the author and Ahmed define small subact where a subact N of a right S-act M_S is named small (or superfluous) in M_S if for each subact H of M_S, N$\bigcup$H=M_S implies H=M_S. An S-act M_S is named simple if it contains no subacts other than M_S itself and M_S is called Θ-simple if it contains no subacts other than M_S and one element subact Θ_S[19]. In ([25], Definition 1.1.11 the author define M-cyclic subact as follows, a subactN of S-act M_S is named M-cyclic, if there's an S-epimorphism from M_S onto N. This is equivalent to say, there's a congruence Ï on M_S such that$N \cong M_S/\rho$. An S-act B_S is named retracted of S-act A_S if, and only if, there exists a subact W of A_S and an epimorphism $f : A_S \to W$ such that$B_S \cong W$ and $f(w) = w$ for each $w \in W$ [19, P.84]. Let M_S, N_S be right S-acts. An S-act E is named injective if for each S-monomorphism $f : M_S \to N_S$ and each S-homomorphism $g : M_S \to E$, there's an S-homomorphism $h : N_S \to E$ such $hf = g$[30]. A right S-act N_S is named

M-injective if for every S-monomorphismf from S-act B_S into S-actM_S and each S-homomorphism $g : B_S \to N_S$, there's an S-homomorphism $h : M_S \to N_S$ such $hf = g$. Thus, N_S is injective if, and only if, $N_S$is M-injective for all S-act M_S [31]. The concept of injectivity was generalized to quasi injective S-act by Lopez [21], such an S-act N_S is quasi injective if, and only if, N_S is N-injective. More generally, Yan introduced pseudo injective as a generalization of quasi injective S-act. An S-act M_S is named pseudo-injective if each S-monomorphism of a subact of M_S into M_S extends to an S-endomorphism of M_S [31]. An S-act N_S is named M-principally injective if for each S-homomorphism from M-cyclic subact of M_S into N_S are extended to S-homomorphism from M_S into N_S (for short N_S is MP-injective) [2]. An S-act M_S is named quasi-principally injective if it's MP-injective, that's every S-homomorphism from M-cyclic subact of M_S to M_S are extended to S-endomorphism of M_S (M_S is QP-injective) [2]. An S-act N_S is called pseudo M-principally-injective, if for each S-monomorphism from M-cyclic subact of M_S to N_S are extended to S-homomorphism from M_S to N_S (if this is the case, we write N_S is pseudo MP-injective). An S-act M_S is called pseudo quasi principally injective if it's pseudo MP-injective act (if this is the case, we write M_S is pseudo QP-injective).

Throughout this paper, S is going to be a monoid, and every act is unitary.

2. Results

In the next sections (two and three), we'll give interesting theorems, propositions, concepts, and lots of more results concerning small pseudo injective S-acts, and small pseudo quasi principally injective acts.

2.1. **Small Pseudo Injective S-acts.**

Definition 2.1. *Let M_S, N_S be S-acts. N_S is a small M-pseudo injective if for each small subact A of M_S, each S-monomorphism $f : A \to N_S$ are extended to an S-homomorphism $f : M_S \to N_S$. An S-act N_S is called small pseudo injective if it's a small N-pseudo injective.*

Remark 2.1.

1. *Retract subact of small pseudo injective S-act is small pseudo injective.*

2. *Let M_S, N_S, W_S be S-acts. If N_S is a small M-pseudo injective and$M_S \cong W_S$, then it's easy to prove that N_S is a small W-pseudo injective act. Also, every isomorphic S-act to small M-pseudo injective act is a small M-pseudo injective act.*

Proof. Proof of part 1. Let M_S be a small pseudo injective S-act and N be a retract subact of M_S. Let A be a small subact of M_S and $f : A \to N$ be S-monomorphism. Define $\alpha : A \to M_S$ by $\alpha = j_N \circ f$, wherej_N be the injection map of N into M_S, thenα is S-monomorphism. Since M_S is small pseudo injective, so there exists S-homomorphism $\beta : M_S \to M_S$ such that $\beta \circ i_A = \alpha$, wherei_A be the inclusion map of A into M_S. Now, let π_N be the projection map of M_S onto N. Then, define $\sigma(= \pi_N \beta) : M_S \to N$. Thus, for every $a \in A$ we've $\sigma \circ i_A(a) = (\pi_N \circ \beta \circ i_A)(a) = \pi_N(\alpha(a)) = \pi_N(j_N \circ f(a)) = \pi_N(f(a)) = f(a)$. Therefore, σ is an extension of f. Thus, N is a small pseudo injective act. □

Proposition 2.1. *Let M_S and N_S be S-acts. Then:*

1. *If N_S is a small M-pseudo injective, then any S-monomorphism $f : N_S \to M_S$ splits.*
2. *N_S is injective S-act if, and only if, N_S is a small M-pseudo injective for all M_S.*

Proof.

1. It's clear that N_S is isomorphic to $f(N)$, so $f(N)$ is a small M-pseudo injectivity.
2. By (1), if N_S is small M-pseudo injective for all M_S, then every S-monomorphism $f : N_S \to M_S$ splits for all S-acts M_S. Hence N_S is injective.

□

Proposition 2.2. *Every small M-pseudo injective S-act is a small A-pseudo injective for any subact A of M_S.*

Proof. Let H be a small subact of A, then H is a small subact in M_S by Lemma 2.2.4 in [7]. Let f be S-monomorphism fromH into N. Then, since N is a small M-pseudo injective, so there exists S-homomorphism $g : M_S \to N$ which extends f. Consider figure 1, where $i_1(i_2)$ be the inclusion map of H(A) in A(M_S). Then, we've $g \circ i_2 \circ i_1 = f$. Now, put $g^/(= g_{/A}) : A \to N_S$ be S-homomorphism which extends f also. Hence, N_S is a small A-pseudo injective. □

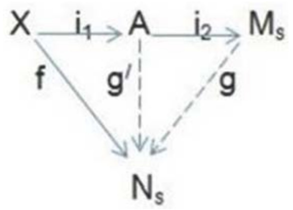

FIGURE 1. Explains that N is a small A-pseudo injective act.

Lemma 2.1. *Every retracts subact of small M-pseudo injective S-act is a small M-pseudo injective.*

Proof. Assume that N_S is a small M-pseudo injective S-act, and A be a retract subact of N_S, so there a subactW of N_S and S-epimorphism$\alpha : N_S \to W$such $A \cong W$ and$\alpha_{'W} = i_W$. This suggests$\alpha(w) = w$, $\forall w \in W$. Thus,we've S-epimorphism$\alpha : N_S \to A$ such that$\alpha(a) = a$, $\forall a \in A$. Let H be a small subact of M_S and$f : H \to A$be S-monomorphism. Define $g : H \to N_S$ by$g(x) = (f(x), 0)$, $\forall h \in H$. This meansg is S-monomorphism (in fact, if $g(h_1) = g(h_2)$, this implies $(f(h_1), 0) = (f(h_2), 0)$, so f $(h_1) = f\,(h_2)$). Sincef is S-monomorphism, so$h_1 = h_2$, thus g is S-monomorphism. Since N_S is a small M-pseudo injective, so there exists S-homomorphism $g^/ : M_S \to N_S$ such $g^/ \circ i_H = g$. Letj and π be the injection and projection map of A into N_S(and N_S onto A). Now, define $K(= \pi \circ g^/) : M_S \to A$ be S-homomorphism such$K \circ i_H = \pi \circ g^/ \circ i_H = \pi \circ g = f$, so $K \circ i_H = f$. This suggests h extends f and A is M-pseudo injective. □

Before subsequent proposition, we'd like the subsequent lemma.

Lemma 2.2. *[15] Let M_S and N_S be S-acts andÏ• $\in$ Hom(M_S, N_S). If A_S is intersection large in N_S, then$\phi^{-1}(A_S)$ is intersection large in M_S. (In particular, if N is intersection large in M, then for each $m \in M_S$, $[N, m] = \{s \in S : ms \in N\}$ is intersection large right ideal in S_S).*

Proposition 2.3. *If an S-act N_S is a small M-pseudo injective with $\psi_M = I_M$, then $\alpha(M) \subseteq N_S$ for every S-monomorphism$\alpha : E(M_S) \to E(N_S)$. In particular, if H_S is a small pseudo injective with$\psi_M = I_M$, then $\alpha(H_S) \subseteq H_S$ for every S-monomorphism $\alpha \in End(E(H_S))$.*

Proof. Let N_S be a small M-pseudo injective andα be S-monomorphism from E(M) into E(N). Define $X = \{m \in M_S | \alpha(m) \in N_S\}$. Since N_S is a small M-pseudo injective, so $\alpha\prime_X$ are often extended to $\beta : M_S \to N_S$. Since E(N) is E(M)-injective, so E(N) is small M-injective by Proposition 2.2. This means, there exists S-homomorphism $h : M_S \to E(N)$ which extend$\alpha\prime_X$. The proof is complete, when $\beta(M) = h(M)$. Assume that $\beta(m_0)' = h(m_0)$ for some $m_0 \in M_S$. Since N is important in E(N) and $\Theta \neq h(m_0) \in E(N)$, so there exists $s \in S$, such $\Theta \neq h(m_0)s \in N$. Thus, $h(m_0 s) \in N$implies that $m_0 s \in X$. On the opposite hand, $(m_0)s = \beta(m_0 s) \in N$. Note that, since N_S is $\bigcap$-large in E(N), so $[N, h(m_0)]$ is $\bigcap$-large right ideal in S_S by Lemma 2.2. Thus, for$h(m_0)\psi_M\beta(m_0)$, and since $\psi_M = I_M$,we've $h(m_0) = \beta(m_0)$ and this is often a contradiction. Hence, $h(M) = \beta(M) \subseteq N_S$. Since$h(M) = \alpha(M)$, then this suggests that $\alpha(M) = \beta(M) \subseteq N_S$. □

Before subsequent theorem, we'd like the subsequent definition:

Definition 2.2. *Let M_S, N_S be S-acts. N_S is a small M-injective if for each a small subact A of M_S, each S-homomorphism $f : A \to N_S$ are often extended to an S-homomorphism $f : M_S \to N_S$. An S-act N_S is named small quasi injective if it's small N- injective.*

Theorem 2.1. *Let M_S be a cog-reversible nonsingular S-act with$\updownarrow_M(s) = \Theta$ for every $s \in S$. Then M_S is small pseudo injective act if, and only if, M_S is small quasi injective.*

Proof. Let A be a small sub act of an S-act M_S and f be a nonzero S-homomorphism from A into M_S. If f is S-monomorphism, then there's nothing to prove. So assume f isn't S-monomorphism.

Since E(M) is injective, then E(M) is an M (respectively E(M))-injective). Thus, there's S-homomorphism $h : M_S \to E(M)$such$h \circ \omega_A = \omega_M \circ f$, where ω_A (respectively ω_M)is the inclusion mapping of A (respectively M_S)into M_S (respectively E(M)). Again there's an S-homomorphism $g : E(M) \to E(M)$such$g \circ \omega_M = h$. Then, either$ker(h) = I_M$or$ker(h) \neq I_M$. If $ker(h) = I_M$, then h is S-monomorphism. Largeness of M_S in E(M) implies thatg is S-monomorphism, so $g(M_S) \subseteq M_S$by Proposition 2.3. Thus, $h(M_S) \subseteq M_S$which is extension of f, since $h(A) = h \circ \omega_A(A) = \omega_M \circ f(A) = f(A)$. If $ker(h) \neq I_M$, then ker(h) is large on M_S, so Theorem 2.15 in[1] implies that $\frac{M_S}{ker(h)}$is singular. But$\frac{M_S}{ker(h)} \cong h(M) \subseteq M_S$ so, $\frac{M_S}{ker(h)}$ is nonsingular. These two cases imply that$ker(h) = M \times M$. This suggests that h (and hence f) is a zero map. □

Recall that an S-acts A_S and B_S are called mutually small (pseudo) injective if A_S is a small B-(pseudo) injective and B_S is a small A-(pseudo) injective.

Proposition 2.4. *Let A_S and B_S be mutually small pseudo injective S-acts, with $\psi_A = i_A and \psi_B = i_B$. $E(A_S) \cong E(B_S)$, then every S-isomorphism $\alpha : E(A_S) \to E(B_S)$ reduces to an S-isomorphism$\alpha^{/} : A_S \to B_S$. Especially, $A_S \cong B_S$, consequently, A_S and B_S are pseudo injective S-acts.*

Proof. Let $f : E(A_S) \to E(B_S)$ be an S-isomorphism. Since $\psi_A = i_A$, so by Proposition 2.3 $f(A_S) \subseteq B_S$, similarly, since $f^{-1} : E(B_S) \to E(A_S)$ be an S-isomorphism and $\psi_B = i_B$, so by Proposition 2.3 $f^{-1}(B_S) \subseteq A$. Thus, $B_S = (ff^{-1})(B_S) = f(f^{-1}(B_S)) \subseteq f(A_S) \subseteq B_S$. Hence $f(A_S) = B_S$. Therefore, $f_{'A} : A_S \to B_S$ is an S-isomorphism, so $A_S \cong B_S$. Moreover, as A_S is a small B-pseudo injective and $B_S \cong A_S$, we've A_S is a small A-pseudo injective. This suggests A_S is a small pseudo injective. □

For more properties of a small pseudo injective S-acts, we have:

Theorem 2.2. *Let M_1 and M_2 be S-acts. If $M_1 \bigoplus M_2$ is a small pseudo injective, then M_1 and M_2 are mutually a small injective.*

Proof. Let A be a small subact of M_2 and f $: A \to M_1$ be an S-homomorphism. Define $\alpha : A \to M_1 \bigoplus M_2$ by $\alpha(a) = (f(a), a)$, $\forall a \in A$, then α is S-monomorphism. By Proposition 2.2, $M_1 \bigoplus M_2$ is a small M_2-pseudo injective, so there exists S-homomorphism $\beta : M_2 \to M_1 \bigoplus M_2$such$\beta \circ i_A = \alpha$. Now, let$j_1$ and π_1 be the injection and projection map of M_1 into $M_1 \bigoplus M_2$ and $M_1 \bigoplus M_2$ onto M_1.

Then, define $\sigma(= \pi_1\beta) : M_2 \to M_1$be S-homomophism extends f, this suggests $\sigma i = \pi_1\beta i_A = \pi_1\alpha = \pi_1 j_1 f = I_{M_1} f = f$, which means $\sigma i = f$. Figure (2) explains that: □

Corollary 2.1. *If $\bigoplus_{i\in I} M_i$ is a small pseudo injective, thenM_j is a small M_K-injective for all distinct $j, k \in I$.*

Before subsequent corollary, we'd like the subsequent proposition:

Proposition 2.5. *Let M_S be an S-act and$\{N_i | i \in I\}$ be a family of S-acts. Then $\prod_{i\in I} N_i$ is small M-injective if, and only if, N_i is a small M-injective for each$i \in I$.*

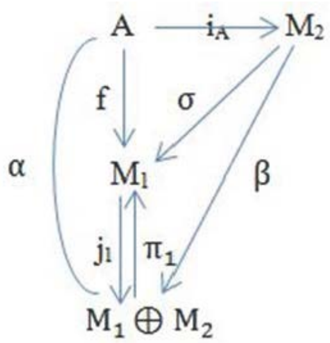

FIGURE 2. Clarifies that M_1 is a small M_2-injective act

Proof. Assume that$N_S = \prod_{i \in I} N_i$ is a small M-injective act. Let A be a small subact of M_S and f be homomorphism$g : M_S \to N_S$such $g \circ i_A = j \circ f$, where i_A is the inclusion map of A into M_S and j is the injection map of N_i into N_S. Define $h : M_S \to N_i$ such$h = \pi_i \circ g$, where π_i is the projection map of N_S into N_i, then$h \circ i_A = \pi_i \circ g \circ i_A = \pi_i \circ j \circ f = f$. That's for all $a \in A$, $h(a) = h\left(i_A(a)\right) = \pi_i\left(g(a)\right) = \pi_i\left(g\left(i_A(a)\right)\right) = \pi_i\left(j\left(f(a)\right)\right) = (\pi_i \circ j)(f(a)) = f(a)$. Conversely, assume that N_i is a small M-injective for every $i \in I$ andf is S-homomorphism from a small subact A of M_S into N_S. Since N_i is a small M-injective act, then there exists S-homomorphism $\beta_i : M_S \to N_i$,such that$\beta_i \circ i_A = \pi_i \circ f$,whereπ_i is the natural projection of N_S intoN_i.So there exists S-homomorphism $\beta : M_S \to N_S$such$\beta_i = \pi_i \circ \beta$. We claim that $\beta \circ i_A = f$. For this since$\beta_i \circ i_A = \pi_i \circ \beta \circ i_A$, then $\pi_i \circ f = \pi_i \circ \beta \circ i_A$, so we obtain $f = \beta \circ i$. Therefore, N_S is a small M-injective act. □

Corollary 2.2. *For any integer $n \geq 2$. Let M_S be a cog-reversible nonsingular S-act with $\updownarrow_M(s) = \Theta$for every $s \in S$. Then M_S^n is a small pseudo injective act if, and only if, M_S is a small quasi injective act.*

Proof. If M_S^n is a small pseudo injective act, then by Proposition 2.2 each M_i is a small M-pseudo injective act. So by Theorem 2.1, each M_i is a small quasi injective act. Then, by Proposition 2.5, M_S is a small quasi injective act.Conversely, if M_S is a small quasi injective act, then by Proposition 2.5, M_S^n is a small quasi injective act and especially is a small pseudo injective. □

Proposition 2.6. *Let $M_S = \bigoplus_{i \in I} M_i$ be a direct sum of a cog-reversible nonsingular S-acts M_i. An S-act M_S is a small quasi injective if, and only if, it's small pseudo injective act.*

Proof. Let M_S be a small pseudo injective act. Then, by Corollary 2.1, each M_j is a small M_i-injective, for all distinct$i, j \in I$. Now, by Lemma 2.1 each M_j is a small M_i-pseudo injective act, so by Theorem 2.1, each M_j is a small quasi injective act. Therefore, by Proposition 2.5 M_S is a small quasi injective act. The opposite direction is a clear. □

Recall that an S-act M_S satisfy C_2**-condition**. If a subact N may be a retract of M_S and $H \cong N$, where H may be sub act of M_S, then H may be a retract of M_S.

Theorem 2.3. *Every a small pseudo injective act satisfies C_2-condition.*

Proof. Let M_S be a small pseudo injective act and A be a small retract subact of M_S with $A \cong B$. Let f be an S-isomorphism from subact B of M_S into A, then f is S-monomorphism from B into M_S. Since M_S is a small pseudo injective act and A be a small retract of M_S, so A is a small M-pseudo injective act by Lemma 2.1. Since$A \cong B$, so by the remarks(2.2)(2), B is a small M-pseudo injective act. Then, by Proposition 2.1(1), f is split. Hence, B is retracting sub act of M_S then M_S satisfies C_2-condition. □

3. Small Pseudo Quasi Principally Injective Acts

Definition 3.1. *An S-act N_S is named **small pseudo M-principally-injective**, if for each S-monomorphism from small M-cyclic sub act of M_S to N_S are often extended to S-homomorphism from M_S to N_S (if this is often the case, we write N_S is **small pseudo MP-injective**). An S-act M_S is named **small pseudo quasi principally injective** if it's small pseudo MP-injective act(if this is the case, we write M_S is **small pseudo QP-injective**).*

Remark 3.1. *Let $S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F may be a field, $M_S = S_S$ and $N_S = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then N_S is small pseudo M-principally injective act.*

Proof. It's easy to point out that A = $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is that the nonzero small M-cyclic subact of M_S. Let α:A→N_S be S-homomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in$ A, there exists x_{11} , $x_{12} \in$ F such$\alpha \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$. Then

$$\alpha \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \alpha \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] = \alpha \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}.$$

It implies that x_{11}=0. Define $\overline{\alpha} : M_s \to N_s$ by $\overline{\alpha} \left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = \begin{pmatrix} \mathrm{xx}_{12} & \mathrm{yx}_{12} \\ 0 & 0 \end{pmatrix}$ for x,y,z∈F. It's clear that $\overline{\alpha}$ is an S-homomorphism. Then, this means that $\overline{\alpha}$ is an extension of α. Thus, N_S is a small pseudo M-principally injective act. □

Remark 3.2. *Retract subact of a small pseudo MP-injective S-act is a small pseudo MP-injective.*

Proof. Let M_S be a small pseudo MP-injective S-act and N be a retract subact of M_S. Let A be a small subact of M_S and$f : A \to N$ be S-monomorphism. Define $\alpha : A \to M_S$ by$\alpha = j_N \circ f$,where j_N be the injection map of N into M_S, then α is S-monomorphism.Since M_S is a small pseudo MP-injective, so there exists S-homomorphism $\beta : M_S \to M_S$ such that $\beta \circ i_A = \alpha$, where i_A be the inclusion map of A into M_S. Now, let π_N be the projection map of M_S onto N. Then, define $\sigma(= \pi_N\beta) : M_S \to N$. Thus, for every $a \in A$, we've$\sigma \circ i_A(a) = (\pi_N \circ \beta \circ i_A)(a) = \pi_N(\alpha(a)) = \pi_N(j_N \circ f(a)) = \pi_N(f(a)) = f(a)$. Therefore, an S-homomorphism σ is extended f. Thus, N is a small pseudo QP-injective act. □

Remark 3.3. *Let A_S, B_S and M_S be the right S-acts and A_S is a small pseudo MP-injective, if B_S isomorphic to A_S, then B_S is additionally, small pseudo MP-injective.*

Remark 3.4. *Let N_S, M_S and A_S be S-acts. If A_S is a small pseudo MP-injective act and$N_S \cong M_S$, then A_S is a small pseudo NP-injective S-act.*

The next proposition represents the generalization of Lemma 2.3 in [29]:

Proposition 3.1. *Let M_S be S-act. M_S is a small pseudo QP-injective act if, and only if, M_S is a small pseudo NP-injective for each M-cyclic subact N of M_S. Especially, if B may be a retract of N, and then M_S is a small pseudo BP-injective act.*

Proof. Let N be M-cyclic subact of S-act M_S. Assume that A is a small N-cyclic subact of N, so A is a small in M_S by Lemma 2.2.4 in [7]. Let f be S-monomorphism from A into M_S and $i_1(i_2)$ be the inclusion map of A(N) into N(M_S).

Since M_S is a small pseudo QP-injective, so there exists S-homomorphism $g : M_S \to M_S$ such that$g \circ i_2 \circ i_2 = f$, this suggests$g$ is an extension of f. Define an S-homomorphism $g_1(= g \circ i_2) : N \to M_S$, then $g_1 \circ i_1 = g \circ i_2 \circ i_2 = f$. Thus, g_1 is an extension of f and M_S is a small pseudo NP-injective act. Conversely, by taking M_S is M-cyclic subact of M_S. □

Corollary 3.1. *Let M_S be S-act and N_S be a small pseudo MP-injective act, then N may be a retract subact of M_S if, and only if, N is M-cyclic subact of M_S.*

Proof. As every retract subact of an S-act M_S is M-cyclic subact of M_S [2]. Conversely, by taking f is the identity map of N within the proof of Proposition 3.1. □

Before subsequent proposition, we'd like to give the following concept:

Let N_S and M_S be two S-acts. Recall that N_S is M-projective or projective relative to M_S, where M_S be S-act, if, for each S-act C_S, every S-homomorphism f from S-act N_S into S-act C_S are lifted to every S-epimorphism g from M_S into C_S, that's there exists S-homomorphism h from N_S into M_S such gh = f[22]. An S-act N_S is named projective if it is projective relative to each right S-act. Also, an S-act N_S is named quasi-projective if N_S is N-**projective**[22]. Note that if N_S is M-projective, then every S-epimorphism from S-act M_S into N_S is split. Also, retract of M-projective S-act is M-projective[9].

Proposition 3.2. *Let M_S be a small pseudo QP-injective S-act,and $\alpha \in T = End(M)$. The following statements are equivalent:*

1. $\alpha(M)$ *is a retract subact of M_S,*
2. $\alpha(M)$ *is a small pseudo MP-injective. Additionally, if M_S is a small quasi projective S-act, then,(1)and(2)are equivalent to:*
3. $\alpha(M)$ *is M-projective.*

Proof. (1→2) Follows from Remark 3.2.

(2→1) As $\alpha(M)$ is M-cyclic subact of M_S, so by Corollary 3.1, $\alpha(M)$ may be a retract subact of M_S.

(2→3) By(2)and Corollary 3.1, we've $\alpha(M)$ may be a retract subact of M_S. Since M_S is quasi projective S-act, so$\alpha(M)$ is M-projective.

(3→2) Assume that$\alpha(M)$is M-projective. Let A be a small M-cyclic subact of M_S and σ be S-monomorphism from A into$\alpha(M)$. Since $\alpha(M)$ is M-cyclic, so there exists S-epimorphism$\beta : M_S \to \alpha(M)$. Since$\alpha(M)$ is M-projective, soβ split.This suggests there is S-homomorphism k from$\alpha(M)$ intoM_S, such$\beta \circ k = I_{\alpha(M)}$. Then, define $f = k \circ \sigma$. Since f is S-monomorphism (whence$\beta \circ k = I_{\alpha(M)}$)and M_S is small pseudo QP-injective act, so there exists S-homomorphis $h : M_S \to M_S$ such $h \circ i = f$. Since M_S is quasi projective, so $\beta \circ h = g$, where g is an S-homomorphism from M_S into $\alpha(M)$.

Thus, we've $g \circ i = \beta \circ h \circ i = \beta \circ f = \beta \circ k \circ \sigma = I_{\alpha(M)} \circ \sigma$. This suggests $\alpha(M)$ is small pseudo MP-injective act. □

Corollary 3.2. *Let M_S be a small pseudo QP-injective S-act and quasi projective. Then, the subsequent statements hold for M-cyclic subactN of M_S:*

1. *N is a retract subact of M_S.*
2. *N is small pseudo MP-injective. In additiona, if M_S is quasi projective S-act, then (1)and(2)are equivalent to:*
3. *N is M-projective.*

The following proposition explains under which conditions on small pseudo QP-injective to be Small QP-injective act:

Proposition 3.3. *Let M_S be a cog-reversible nonsingular S-act with $\updownarrow_M(s) = \Theta$ for every $s \in S$. If M_S is a small pseudo QP-injective act, then M_S is small QP-injective.*

Proof. Let N be a small M-cyclic subact of S-act M_S and f be S-homomorphism from N into M_S. If f is one-to-one, then, there's nothing to prove. If f isn't one-to-one, then, by using the proof of Theorem 2.1, we get the specified. This suggests that M_S is small QP-injective S-act. □

Proposition 3.4. *Let M_S be a principal self-generator S-act. Then, every a small pseudo QP-injective S-acts is a small pseudo injective.*

Proof. Let N be a small subact of S-act M_S and f be S-monomorphism from N into M_S. Since M_S is principal self-generator, so there exists some $\alpha : M_S \to N$ such

$m = \alpha(m_1)$, $\forall m \in M_S$. This suggestsα is S-epimorphism, thus, N is M-cyclic subact of M_S. As M_S is a small pseudo QP-injective, so f are often extend to S-endomorphism g of M_S such$g \circ i = f$, where i be the inclusion map of N into M_S. Therefore, M_S is a small pseudo injective S-act. □

Theorem 3.1. *Let M_1 and M_2 be two S-acts. If $M_1 \oplus M_2$ is a small pseudo QP–injective act, then M_i is a small M_j-principally injective for i, j = {1, 2}.*

Proof. Let $M_1 \oplus M_2$ be a small pseudo QP –injective act. Let A be a small M_2-cyclic subact of M_2 and f be S-homomorphism from A into M_1. Let j_1 and π_1 be the injection and projection map of M_1 into$M_1 \oplus M_2$ and $M_1 \oplus M_2$ onto M_1 respectively. Define $\alpha : A \to M_1 \oplus M_2$ by $\alpha(a) = (f(a), a)$, $\forall a \in A$. It's clear that α is S-monomorphism. Since $M_1 \oplus M_2$ is a small pseudo QP-injective, so by Proposition 3.1, $M_1 \oplus M_2$ is small pseudo M_2P-injective.Hence, there exists S-homomorphismg from M_2 into$M_1 \oplus M_2$such that$g \circ i = \alpha$, where i be the inclusion map of A into M_2. Now, put $h = \pi_1 \circ g$ from M_2 into M_1. Thus, $\forall a \in A$ we've $h(a) = \pi_1 \circ g(a) = \pi_1 \circ \alpha(a) = \pi_1(\alpha(a)) = \pi_1(f(a), a) = f(a)$. This suggests M_1 is a small M_2P-injective. □

Corollary 3.3. *Let $\{M_i\}_{i \in I}$ be a family of S-acts. If $\bigoplus_{i \in I} M_i$ is a small pseudo M_KP-injective, then M_j is M_KP-injective act for all distinct $j, k \in I$.*

Proposition 3.5. *For any integer $n \geq 2, M_S^n$ is a small pseudo QP-injective act if, and only if, M_S is a small QP-injective.*

Proof. If M_S^n is a small pseudo QP-injective act, then by Theorem 3.1, we've M_S is a small MP-injective. This suggests that M_S is a small QP–injective act. Conversely, assume that M_S is a small QP-injective act, this suggests that M_S is a small MP-injective act. By Proposition 2.5 in [1] M_S^n is a small QP–injective act and hence it's a small pseudo QP-injective act, M_S^n is a small pseudo QP-injective act. □

Proposition 3.6. *Let an S-act M_S be a small pseudo QP-injective act and T=End(M).If ImÎ± is an essential(large) a small sub act of M_S, where $\alpha \in T$, then any S-monomorphism from$\alpha(M)$ into M_S is often extended to an S-monomorphism in T.*

Proof. Let $f : \alpha(M) \to M_S$ be S-monomorphism and $\alpha(M)$ is a small subact of M_S. Since M_S is a small pseudo QP-injective act, so there exists S-homomorphism

$g : M_S \to M_S$ such $f = gi$, where$i : \alpha(M) \to M_S$ is that the inclusion map. Then, $f\alpha = gi\alpha = g\alpha$. Now, let $g(\alpha(m_1)) = g(\alpha(m_2))$, where $m_1, m_2 \in M_S$, then $f(\alpha(m_1)) = f(\alpha(m_2))$. Since f is monomorphism, so $\alpha(m_1) = \alpha(m_2)$ and on the opposite hand$\alpha(M)$ is an essential subact of M_S, so g is monomorphism. □

The following theorems and lemma provide a characterization of a small pseudo QP-injective S-acts:

Theorem 3.2. *Let M_S be an S-act and$\alpha(M)$ is a small subact of M_S. Then, M_S is a small pseudo QP-injective act if, and only if,$ker(\alpha) = ker(\beta)$, impliesTÎ± = TÎ² for all$\alpha, \beta \in T = End(M)$.*

Proof. $\to$)Let $\alpha, \beta \in T$with $ker(\alpha) = ker(\beta)$. Define $\phi : \alpha(M) \to M_S$by $\phi(\alpha(m)) = \beta(m)$for each $m \in M_S$. Let $\alpha(m_1), \alpha(m_2) \in \alpha(M)$such$\alpha(m_1) = \alpha(m_2$. Then, $(m_1, m_2) \in ker(\alpha) = ker(\beta)$, so$\beta(m_1) = \beta(m_2)$. Hence, $\phi(\alpha(m_1)) = \phi(\alpha(m_2))$ and ϕ is well-defined, the reverse steps give thatϕ is S-monomorphism. For each $m \in M_S$ and $s \in S$, we have $\phi(\alpha(ms)) = \beta(ms) = \beta(m)s = \phi(\alpha(m))s$. This shows that$\phi$ is an S-homomorphism. Since M_S is a small pseudo QP-injective act andÎ±(M)is a small M-cyclic subact of M_S, so there exists S-homomorphism $\psi : M_S \to M_S$such that $\psi i = \phi$, where i is the inclusion map of $\alpha(m)$into M_S. Thus, $\beta = \phi\alpha = \psi i\alpha = \psi\alpha \in T\alpha$. Then, $T\beta \subseteq T\alpha$. similarly, $T\alpha \subseteq T\beta$, therefore, $T\alpha = T\beta$.

$\leftarrow$) Let$\alpha \in T$and $f : \alpha(M) \to M_S$ be S-monomorphism from a small M-cyclic $\alpha(M)$of M_S into S-act M_S. Then, $kerf = keri$, where i is that the inclusion map from $\alpha(M)$intoM_S. Since$f(\alpha(M)) \cong \alpha(M)$, and similarly$i(\alpha(M)) \cong \alpha(M)$, so this suggests $f, i \in T$. Then by assumption, $Tf = Ti$, so we've $f \in Ti$. Thus, $f = hi$,for some $h \in T$. This shows that M_S is a small pseudo QP-injective act. □

Lemma 3.1. *Let M_S be a small pseudo QP-injective act. If $ker(\alpha) = ker(\beta)$, where $\alpha, \beta \in T = End(M)$, with $\alpha(M)$ is a small in M_S. Then $T\beta \subseteq T\alpha$.*

Proof. Let $ker(\alpha) = ker(\beta)$, where $\alpha, \beta \in T$ with$\alpha(M)$ is a small in M_S. Define $f : \alpha(M) \to M_S$ by$f(\alpha(m)) = \beta(m)$ for every $m \in M_S$ .It is obvious that f is an S-monomorphism. For this let$f(\alpha(m_1)) = f(\alpha(m_2))$, this suggests that $\beta(m_1) = \beta(m_1)$ and since$ker(\alpha) = ker(\beta)$, so $\alpha(m_1) = \alpha(m_1)$ and f is monomorphism. Since M_S is a small pseudo QP-injective act, so there exists$\overline{f}$ is extension f. Then, $\beta = f\alpha = \overline{f}\alpha \in T\alpha$. Thereby, $T\beta \subseteq T\alpha$. □

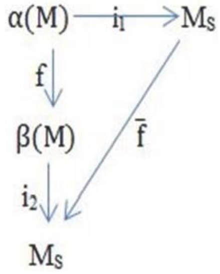

FIGURE 3. Illustrates that M_S is a small pseudo QP-injective act

Theorem 3.3. *Let M_S be a small pseudo QP-injective act and T=End(M)with $\alpha, \beta \in T$ and $\alpha(M)$ is a small sub act of M_S. Then:*

1. *If $\alpha(M)$ embeds in $\beta(M)$, then $T\alpha$ is an image of $T\beta$.*
2. *If $\alpha(M) \cong \beta(M)$, then $T\alpha \cong T\beta$.*

Proof. 1. Let $f : \alpha(M) \to \beta(M)$ be S-monomorphism and $\alpha(M)$ is a small subact of M_S. Let i_1 (respectively i_2) be the inclusion maps of $\alpha(M)$ (respectively $\beta(M)$) into M_S. Since M_S is a small pseudo QP-injective act, therefore, the S-homomorphism $i_2 \circ f$are often extended to S-homomorphism $\overline{f} : M_S \to M_S$ such that$\overline{f} \circ i_1 = i_2 \circ f$ and figure 3, below explain that.

Define $\sigma : T\beta \to T\alpha$ by $\sigma(\lambda\beta) = \lambda\overline{f}\alpha$, $\lambda \in T$. If $\lambda_1\beta = \lambda_2\beta$, for m $\in M_S$. $\overline{f}\alpha(m) = \left(\overline{f} \circ i\right)(\alpha(m)) = (i_2 \circ f)(\alpha(m)) = f(\alpha(m))$and hence$\lambda\overline{f}\alpha(m) = \lambda f(\alpha(m))$, so σ is well-defined. It's clear thatσ is T-homomorphism, in fact, let$\sigma\beta \in T\beta$and $g \in T$, then $\sigma\left(g\left(\text{Î»Î²}\right)\right) = \sigma\left(\left(\text{gÎ»}\right)\beta\right) = g\lambda\overline{f}\alpha =$ g $\left(\lambda\overline{f}\alpha\right) = g\sigma(\lambda\beta)$. We claim that $ker(\overline{f}\alpha) \subseteq ker\alpha$. Let $(x_1, x_2) \in ker(\overline{f}\alpha)$ which means $\overline{f}\alpha(x_1) = \overline{f}\alpha(x_2)$. This suggests $f\alpha(x_1) = f\alpha(x_2)$, since f is monomorphism, so $\alpha(x_1) = \alpha(x_2)$. Thus, $(x_1, x_2) \in ker\alpha$. By Theorem 3.2, we've $T\alpha \subseteq T\overline{f}\alpha$ so there exists $\lambda \in T$ such $\alpha = \lambda\overline{f}\alpha$, then $\alpha = \lambda\overline{f}\alpha = \sigma(\lambda\beta) \in \sigma(T\beta)$. This suggests$T\alpha = \sigma(T\beta)$. Then$\sigma$ is T-epimorphism.

2. As in (1), let $f : \alpha(M) \to \beta(M)$ be monomorphism and $\alpha(M)$ be a small subact of M_S and by assumption f is S-epimorphism. Since M_S is a small pseudo QP-injective act, so $i_2 \circ f$ is often extended to $\overline{f} : M_S \to M_S$ such $\overline{f} \circ i_1 = i_2 \circ f$, where i_1, i_2 are the inclusion map of $\alpha(M)$ into M_S and $\beta(M)$ into M_S respectively. Define $\sigma : T\beta \to T\alpha$ by $\sigma(\lambda\beta) = \lambda\overline{f}\alpha$, for $\lambda \in T$. As in part(1), σ is well-defined, then $\lambda_1\overline{f}\alpha = \lambda_2\overline{f}\alpha$. Since $\overline{f}\alpha(M) = \overline{f} \circ i_1(\alpha(M)) = i_2 f(\alpha(M)) = f\alpha(M) = \beta(M)$,

then $\lambda\overline{f}\alpha(M) = \lambda\beta(M)$, hence $\lambda_1\beta(M) = \lambda_1\overline{f}\alpha(M) = \lambda_2\overline{f}\alpha(M) = \lambda_2\beta(M)$, then $\lambda_1\beta = \lambda_2\beta$. Hence, σ is T-monomorphism and by (1)σ is T-epimorphism, so, we obtained the specified. □

Lemma 3.2. *Let M_S be a small pseudo QP -injective act and T = End(M). If$\alpha(M)$ is a small and simple S-act, $\alpha \in T$, then $T\alpha$ may be a simple T-act.*

Proof. Let $\Theta \neq f\alpha \in T\alpha$. Then $f : \alpha(M) \to f\alpha(M)$ is an S-isomorphism by hypothesis where$\alpha(M)$is a small, so let$\sigma : f\alpha(M) \to \alpha(M)$ be the inverse. If $\overline{\sigma} \in T$ extends σ, then for $m \in M_S$, we've $\alpha(m) = \sigma(f\alpha(m)) = \overline{\sigma}(f\alpha(m)) \in Tf\alpha$ and hence Tα = Tfα. □

Theorem 3.4. *Let M_S be S-act. If every M-cyclic a small subact of M_S is projective, then every factor act of a small PM-principally injective act is a small PM-principally injective.*

Proof. Let Abe a small PM-principally injective act, ρ a congruence on A and $\alpha(M)$ be a small subact of M_S and let $f : \alpha(M) \to \frac{A}{\rho}$ be an S-monomorphism. Hence, by assumption, there exists an S- homomorphism$\overline{f} : \alpha(M) \to A$ such that $f = \pi\overline{f}$ where $\pi : A \to \frac{A}{\rho}$ is the natural S-epimorphism. To point out that $\overline{f}$ is a monomorphism, let $\overline{f}\left(\alpha\left(m_1\right)\right) = \overline{f}\left(\alpha\left(m_2\right)\right)$. Then, $\pi\overline{f}\left(\alpha\left(m_1\right)\right) = \pi\overline{f}\left(\alpha\left(m_2\right)\right)$and this suggests that $f\left(\alpha\left(m_1\right)\right) = f\left(\alpha\left(m_2\right)\right)$. But f is a monomorphism, so, we obtain that $\alpha\left(m_1\right) = \alpha\left(m_2\right)$ and this means that$\overline{f}$ is a monomorphism. Since A is small PM-principally injective act, so there exists an S-homomorphism $\beta : M_S \to A$ which is an extension of $\overline{f}$ to M_S. Then, $\pi\beta$ is an extension of f. □

The next proposition illustrates the connection of a small pseudo QP-injective act with other classes of injective:

Proposition 3.7. *Let M_S be principal and principal self-generator. Then M_S is a small pseudo QP -injective act if, and only if, M_S is a small pseudo PQ-injective act.*

Proof. $\to$) Let N be a small cyclic subact of M_S and f be S-homomorphism from N into M_S.SinceM_S is principal self-generator, so there exists some$\alpha : M_S \to mS$, such $m = \alpha(m_1)$, $\forall m \in M_S$. This suggestsα is S-epimorphism, thus, N is small M-cyclic subact of M_S. Since M_S is a small QP-injective act, so f are often extended to S-homomorphism $g : M_S \to M_S$, such $g \circ i = f$, where i be the inclusion map of N into M_S, therefore, M_S is a small PQ-injective act.

$\leftarrow$) Let N be a small M-cyclic subact of an S-act M_S, so there exists an S-epimorphism $\alpha : M_S \rightarrow N$. Since M_S is principal, so N is principal. Let f be S-homomorphism from N into M_S. Since M_S is a small PQ-injective act, so f is extended to S-homomorphism g from M_S into M_S such$g \circ i = f$, where i be the inclusion map of N into M_S. Thus, M_S is a small QP-injective act. □

4. Conclusions

In this article, we presented new notions which are a small pseudo injective acts and a small pseudo QP-injective acts. Then, we deduced several new characterizations, and properties as shown above within the propositions, and theorems. Besides, we found that a subact must be retracted to inherit the property of small pseudo QP-injectivity. Also, we deduced the connection between the classes of a small pseudo QP-injective acts with the classes of injectivity and then conditions for equivalent these classes. More precisely, we found that M_S must be a cog-reversible nonsingular S-act with$\updownarrow_M(s) = \Theta$ for every $s \in S$ to be the subsequent classes coinciding, class of a small pseudo QP-injective act with the classes of a small QP-injective. Furthermore, every S-act M_S must be a principal self-generator, implying that any small pseudo QP-injective S-act is a small pseudo injective.

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