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HIGHER ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITION AT RESONANCE

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ABSTRACT. In this work, we prove the existence of solution for the following higher-order boundary value problem at resonance $\omega^{(n)}(t) = f(t,\omega(t),\ldots, \omega^{(n-2)}(t))$ $n \geq 3$, $t \in (0,1)$, $\omega(0) = \omega'(0) = \ldots = \omega^{(n-3)}(0) = \omega^{(n-1)}(0) = 0, \omega(1) = \frac{n-1}{\eta^{n-1}} \int_0^\eta \omega(t)dt; \eta \in (0,1)$, we have relied on Mawhin's coincidence degree theory to get existence results.

1. INTRODUCTION

Boundary value problems (BVP) at resonance have been studied in many papers for ordinary differential equations, see for example [1–14, 19–27] and the references therein. In this literature, we show some contributions of researchers to the finding of the existence of the solution for boundary value problems at resonance. Assia Guezane-Lakoud et al [15] studed some existence results for third-order differential equation

$$\omega'''(t) = f(t,\omega(t),\omega'(t)), \qquad t \in (0,1), \tag{1.1}$$

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subject to the following nonlocal condition

$$\omega(0) = \omega^{''}(0) = 0, \omega(1) = \frac{2}{\eta^{\eta}} \int_0^{\eta} \omega(t)dt; \qquad \eta \in (0,1). \tag{1.2}$$

Assia Frioui et al [16] studed the existence of solutions of the higher-order ordinary differential equation

$$\omega^{(n)}(t) = f(t, \omega(t)), \qquad t \in (0, \infty), \tag{1.3}$$

with the integral boundary value conditions

$$\omega^{(i)}(0) = 0,\ i = 0, 1 \ldots, n-2,\ \omega^{(n-1)}(\infty) = \frac{n!}{\eta^{n}} \int_0^{\eta} \omega(t)dt; \qquad \eta > 0,\ n \geq 3. \tag{1.4}$$

In [17], the focus of this paper is to provide sufficient conditions that ensure the existence of solutions for the following nonlinear third-order boundary value problem

$$\omega^{'''}(t) = f(t, \omega(t), \omega'(t)), \qquad t \in (0, T), \tag{1.5}$$

with the condition

$$\omega(0) = \omega^{''}(0) = 0, \omega(T) = \frac{2T}{\eta^{2}} \int_0^{\eta} \omega(t)dt; \eta \in (0,1). \tag{1.6}$$

In [18], the existence of at least one solution for the following third-order integral and m-point boundary value problem on the half-line at resonance

$$(\rho(t)\omega'(t))^{''} = f(t, \omega(t), \omega'(t), u^{''}), \qquad t \in [0, \infty), \tag{1.7}$$

with

$$\omega(0) = \sum_{j=1}^{m} \alpha_j \int_0^{\eta_j} \omega(t)dt, \qquad \omega'(0) = 0, \qquad \lim_{t \to \infty} (\rho(t)\omega'(t))' = 0. \tag{1.8}$$

In this paper, we discuss existence results for higher-order differential equation, these results are determined by applying Mawhin's coincidence degree theory. Our assumed problem will more complicated and general than the problems considered before and aforementioned above, we study the existence of solutions for the higher-order differential equation given by

$$\omega^{(n)}(t) = f(t, \omega(t), \omega'(t), \ldots, \omega^{(n-2)}(t)), \qquad n \geq 3,\ t \in (0,1), \tag{1.9}$$

with the following nonlocal condition

$$\omega(0) = \omega'(0) = \ldots = \omega^{(n-3)}(0) = \omega^{(n-1)}(0) = 0, \quad \omega(1) = \frac{n-1}{\eta^{n-1}} \int_0^{\eta} \omega(t)dt, \tag{1.10}$$

$\eta \in (0,1)$, where $f : [0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}$ is caratheodary function, and $\eta \in (0,1)$, we say that the BVP 1.9 is a resonance problem if the linear equation $Lx = \omega^{(n)}$, with the PVC 1.10 has nontrivial solution i.e., $\dim \ker L \geq 1$.

2. PRELIMINAIRES

For the convenience of the reader to understand the coincidence degree theory, we briefly recall some definitions [13–15].

Definition 2.1. *Let X, Y be reaI Banach spaces, the linear operator $L : domL \subset X \to Y$ is said to be a Fredholm map of index zero provided that $\ker L$, the kernel of L, is of the same finite dimension as the Y/ImL, where ImL is the image of L.*

Let L be a Fredholm map of index zero, and $P : X \to X$, $Q : Y \to Y$ be continuous projectors, such that $Im\, P = \ker L$, $\ker Q = Im\, L$, $X = \ker L \oplus \ker P$, and $Y = Im\, L \oplus Im\, Q$.

We denote the inverse of the map $L|_{dom\, L \cap \ker P} : dom\, L \cap \ker P \to Im\, L$ by Kp, i.e.,

$$Kp = (L|_{dom\, L \cap \ker P})^{-1} : ImL \to dom\, L \cap \ker P.$$

Definition 2.2. *Let L be a Fredholm map of index zero and Ω be an open bounded subset of Y,such that $domL \cap \Omega \neq \emptyset$, the map $N : X \to Y$ is said to be $L - compact$ on $\overline{\Omega}$, if the map $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.*

For more details, see [14, 15].

Theorem 2.1. *Let L be a Fredholm operator of index zero and let N be $L-$compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*

(i) $L\omega \neq \lambda N\omega$, *for every* $(\omega, \lambda) \in \big[(domL \backslash KerL) \cap \partial\Omega\big] \times (0,1)$.

(ii) $N\omega \notin ImL$, *for every* $\omega \in KerL \cap \partial\Omega$.

(iii) $deg(JQN|_{\ker L}, \ker L \cap \partial\Omega, 0) \neq 0$, *where $J : ImQ \to \ker L$ is a linear isomorphism, $Q : Y \to Y$ is a projection as above with $ImL = \ker Q$.*

Then, the equation $L\omega = N\omega$ has at least one solution in $domL \cap \overline{\Omega}$.

In the following, we shall use the classical spaces $C[0,1], C^1[0,1], \ldots, C^{n-1}[0,1]$ and $L^1[0,1]$.

For $\omega \in C^{n-1}[0,1]$, we use the norm

$$\|\omega\|_\infty = \max_{\omega\in[0,1]} |\omega(t)|,$$

$$\|\omega\| = \max_{\omega\in[0,1]} \left\{ \|\omega\|_\infty, \|\omega'\|_\infty, \ldots, \|\omega^{(n-2)}\|_\infty \right\},$$

and denote the norm in $L^1[0,1]$ by $\|\omega\|_1$. We will use the Sobolev space $W^{n,1}(0,1)$, which may be defined by

$$W^{n,1}(0,1) = \Big\{\omega : [0,1] \to \mathbb{R} : \omega, \ldots, \omega^{(n-1)} \text{ are absolutely continuous on } [0,1] \,\text{with}\, \omega^{(n)} \in L^1[0,1]\Big\}.$$

Let $X = C^{n-1}[0,1]$, $Y = L^1[0,1]$, L is the linear operator from $domL \subset X \to Y$ with

$$domL = \Big\{\omega \in W^{n,1}(0,1) : \omega \text{ verify the condition 1.10}\Big\}$$

and

$$L\omega = \omega^{(n)}, \qquad \omega \in domL.$$

We define $N : X \to Y$ by setting

$$N\omega = f(t, \omega(t), \omega'(t), .., \omega^{(n-2)}(t)), t \in (0,1).$$

Then, BVP 1.9, can be written as $L\omega = N\omega$.

In order to apply Theorem 2.1, in the following Lemma 2.1, we shall show that L is a Fredholm operator of index zero and construct a linear continuous projector operator Q satisfying condition (iii) in Theorem 2.1.

Lemma 2.1. *We have*

(i) $\ker L = \big\{\omega \in domL : \omega = ct^{n-2}, c \in \mathbb{R}, t \in (0,1]\big\}$.

(ii) $ImL = \big\{y \in Y : \int_0^1 (1-s)^{n-1} y(s)ds - \frac{(n-1)}{n\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s)ds = 0\big\}$.

(iii) $ImL : domL \subset X \to Y$ *is a Fredholm operator of index zero, and the linear continuous projector operator* $Q : Y \to Y$ *can be defined as* $Qy = k(Ry)t$ *such that*

$$Ry = \int_0^1 (1-s)^{n-1} y(s)ds - \frac{(n-1)}{n\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s)ds.$$

(iv) *The linear operator* $Kp : ImL \to domL \cap \ker P$ *can be written as*

$$K_p y = \frac{1}{(n-1)} \int_0^\eta (\eta - s)^{n-1} y(s) ds.$$

(v) $\|K_p y\| < \|y\|_1$, *for all* $y \in ImL$.

Proof.

(i)

$$\begin{aligned}\ker L = \Big\{ & \omega \in domL : \omega(t) = \omega(0) + \omega'(0)t + \dots \\ & + \frac{\omega^{(n-1)}(0)}{(n-1)} \omega^{(n-1)}(0) t^{n-1}, t \in (0,1] \Big\} \\ = & \Big\{ \omega \in domL : \omega(t) = \frac{\omega^{(n-2)}(0)}{(n-2)} t^{n-2}, t \in (0,1] \Big\} \\ = & \{ \omega \in domL : x = ct^{n-2}, c \in \mathbb{R}, t \in (0,1] \}\end{aligned}$$

(ii) The problem

$$\omega^{(n)} = y \tag{2.1}$$

has a solution $\omega(t)$ satisfied

$$\begin{aligned}&\omega(0) = \omega'(0) = \dots = \omega^{(n-3)}(0) = \omega^{(n-1)}(0) = 0, \\ &\omega(1) = \frac{n-1}{\eta^{n-1}} \int_0^\eta \omega(t) dt; \eta \in (0,1),\end{aligned} \tag{2.2}$$

if and only if

$$\int_0^1 (1-s)^{n-1} y(s) ds - \frac{(n-1)}{n\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s) ds = 0. \tag{2.3}$$

Then, from 2.1, we have

$$\omega(t) = \omega(0) + \omega'(0)t + \dots + \frac{\omega^{(n-3)}(0)}{(n-3)!} t^{n-3} + \frac{\omega^{(n-2)}(0)}{(n-2)!} t^{n-2}$$

$$+ \frac{\omega^{(n-1)}(0)}{(n-1)} \omega^{(n-1)}(0) t^{n-1} + \frac{1}{(n-1)!} \int_0^\eta (\eta - s)^{n-1} y(s) ds$$

$$= \frac{\omega^{(n-2)}(0)}{(n-2)} \omega^{(n-2)}(0) t^{n-2} + \frac{1}{(n-1)!} \int_0^\eta (\eta - s)^{n-1} y(s) ds. \tag{2.4}$$

From 2.2, we have

$$\omega(1) = \frac{(n-1)!}{\eta^{n-1}} \int_0^\eta \omega(t)dt$$

$$= \frac{(n-1)}{\eta^{n-1}} \int_0^\eta \left[\frac{\omega^{(n-1)}(0)}{(n-2)} \omega^{(n-2)}(0) t^{n-2} + \frac{1}{(n-1)} \int_0^t (t-s)^{n-1} y(s) ds \right] dt$$

$$= \frac{(n-1)}{\eta^{n-1}} \left[\int_0^\eta \frac{\omega^{(n-2)}(0)}{(n-2)} \omega^{(n-2)}(0) t^{n-2} dt + \frac{1}{(n-1)} \int_0^\eta \int_0^t (t-s)^{n-1} y(s) ds dt \right]$$

$$= \frac{(n-1)}{\eta^{n-1}} \left[\frac{\omega^{(n-2)}(0)}{(n-2)(n-1)} \eta^{(n-1)} + \frac{1}{(n-1)} \int_0^\eta \int_0^t (t-s)^{n-1} y(s) ds dt \right]$$

(2.5)

$$= \frac{\omega^{(n-1)}(0)}{(n-2)} \omega^{(n-2)}(0) + \frac{(n-1)}{n!\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s) ds.$$

From 2.4 and 2.5, we obtain

$$\frac{(n-1)}{n!\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s) ds = \frac{1}{(n-1)!} \int_0^1 (1-s)^{n-1} y(s) ds$$

$$\int_0^1 (1-s)^{n-1} y(s) ds - \frac{(n-1)}{n\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s) ds = 0.$$

We consider the condition 2.3 verified, from 2.4 find

$$\omega(t) = ct^{(n-2)} + \frac{1}{(n-1)!} \int_0^1 (t-s)^{n-1} y(s) ds,$$

where c is an arbitrary constant, then $\omega(t)$ is a solution of 2.1, Hence,

$$ImL = \left\{ y \in Y : \int_0^1 (1-s)^{n-1} y(s) ds - \frac{(n-1)}{n\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s) ds = 0 \right\}$$

is valid.

(iii) For $y \in Y$, we take the projector Qy as $Q : Y \to Y$ can be defined as $Qy = k.(Ry).t$ such that

$$Ry = \int_0^1 (1-s)^{n-1} y(s) ds - \frac{(n-1)}{n\eta^{n-1}} \int_0^\eta (\eta - s)^n y(s) ds$$

is clear that $\dim ImQ = 1$, we have

$$
\begin{aligned}
Q^2y &= Q(Qy) \\
&= k\left[\int_0^1 (1-s)^{n-1}(kRy)\,sds - \frac{(n-1)}{n\eta^{n-1}}\int_0^\eta (\eta-s)^n (kRy)\,sds\right]t \\
&= (kRy)\,t = Qy
\end{aligned}
$$

which implies that the operator Q is projector. Futher $ImL = \ker Q$. Let

$$y = (y - Qy) + Qy$$

$$y - Qy \in \ker Q = ImL$$

$$Qy \in ImQ$$

and

$$Q^2y = Qy,$$

that

$$ImQ \cap \ker L = \{0\},$$

than we have

$$Y = ImL \oplus \ker Q$$

since

$$\dim\ker L = 1 = \dim ImQ = co\dim ImL = 1,$$

L is a Fredholm operator of index zero,

(iv) Taking $P : X \longrightarrow Y$ as follows

$$P\omega(t) = \omega^{(n-2)}(0)t.$$

Then, the generalized inverse $Kp : ImL \to domL \cap KerP$ of L can be written as

$$Kpy = \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-1}y(s)ds.$$

Obviously $ImP = \ker L$ and

$$P^2\omega = P(P\omega) = P(\omega^{(n-2)}(0)t) = \omega^{(n-2)}(0)t = P\omega.$$

It follows from $\omega = (\omega - P\omega) + P\omega$ that

$$X = \ker P + \ker L, \qquad \ker P \cap \ker L = \{0\}.$$

Then $X = \ker P \oplus \ker L$. From the definitions of Kp and P it is easy to se that generalized invers of L is Kp. in fact for $y \in ImL$, we have

$$(LKp)y(t) = [(Kpy)\, t]^{(n)} = y(t).$$

For $\omega \in domL \cap \ker P$, we know

$$\begin{aligned}(KpL)\omega(t) = (Kp)\, \omega\, (t)^{(n)} &= \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\omega^{(n)}(s) \\ &= \omega(t) - \omega(0) - \ldots - \frac{\omega^{(n-1)}(0)}{(n-1)!}.\end{aligned}$$

In view of $\omega \in domL \cap \ker P$

$$\omega(0) = \ldots\ldots = \omega^{(n-3)}(0) = \omega^{(n-1)}(0) = 0$$

and $Px = 0$ thus $(KpL)\omega(t) = \omega(t)$. This shows that

$$Kp = (L|_{domL \cap Kp})^{-1}.$$

(v) We have

$$\|K_p y\|_\infty \leq \int_0^1 (1-s)^{n-1} |y(s)|\, ds \leq \int_0^1 |y(s)|\, ds = \|y\|_1,$$

and from

$$\left[K_p y\right]^{(n-2)} = \frac{(n-1)\times(n-2)\times\ldots\times 2}{(n-1)!}\int_0^t (t-s)y(s)ds = \int_0^t (t-s)y(s)ds$$

$$\|[K_p y]^{(n-2)}\|_\infty \leq \int_0^1 (1-s)|y(s)|ds \leq \int_0^1 |y(s)|ds = \|y\|_1.$$

Then $\|K_p y\| < \|y\|_1$, for all $y \in ImL$. This completes the proof of lemma □

3. Main Results

Theorem 3.1. *Let f :* $[0,1]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function, assume that:*

(H1) *There exist functions* $a_1(t), a_2(t), \ldots, a_{n-1}(t), b(t), \in L^1[0,1]$*, such that, for all* $(\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n, t \in [0,1]$*, satisfying one of the following inequalities:*

$$|f(t,\omega_1,\omega_2,\ldots,\omega_{n-1})| \le \sum_{i=1}^{n-1} a_i(t)|\omega_i| + b(t). \tag{3.1}$$

(H2) *There exists a constant $M>0$, such that, for $\omega \in domL$, if $|\omega^{(n-2)}(t)| > M$, for all $t\in[0,1]$, then,*

$$\begin{aligned}&\int_0^1 (1-s)^{n-1} f(s,\omega(s),\omega^{(1)}(s),\ldots,\omega^{(n-2)}(s))ds\\ &-\frac{(n-1)}{n\eta^{n-1}}\int_0^\eta (\eta-s)^n f(s,\omega(s),\omega^{(1)}(s),\ldots,\omega^{(n-2)}(s))ds \ne 0.\end{aligned}$$

(H3) *There exists a constant $M^*>0$, such that for any $\omega(t) = ct^{n-2} \in \ker L$ with $|c|>M^*$, either*

$$\begin{aligned}&c\Bigg[\int_0^1 (1-s)^{n-1} f(s,cs^{n-2},(n-2)cs^{n-3},\ldots,(n-2)!c)ds\\ &-\frac{(n-1)}{n\eta^{n-1}}\int_0^\eta (\eta-s)^n f(s,cs^{n-2},(n-2)cs^{n-3},\ldots,(n-2)!c)ds\Bigg] < 0,\end{aligned}$$

or else

$$\begin{aligned}&c\Bigg[\int_0^1 (1-s)^{n-1} f(s,cs^{n-2},(n-2)cs^{n-3},\ldots,(n-2)!c)ds\\ &-\frac{(n-1)}{n\eta^{n-1}}\int_0^\eta (\eta-s)^n f(s,cs^{n-2},(n-2)cs^{n-3},\ldots,(n-2)!c)ds\Bigg] > 0.\end{aligned}$$

Then BVP 1.9 with condition 1.10 has at least one solution in $C^{n-1}[0,1]$, provided

$$\sum_{i=1}^{n-1} \|a_i(t)\| \le \frac{1}{2}.$$

Proof. We need to construct the set Ω satisfying all the conditions in Theorem 2.1, which is separated into the following four steps.

STEP 1. First we show that the following set

$$\Omega_1 = \{\omega \in domL\backslash \ker L : L\omega = \lambda N\omega,\ \text{for some } \lambda \in (0,1]\}$$

is bounded. In fact, Suppose that $\omega \in \Omega_1$, and $L\omega = \lambda N\omega$, thus, $\lambda \ne 0, QN\omega = 0$ so it yields

$$\int_0^1 (1-s)^{n-1} f(s,\omega(s),\omega'(s),\dots,\omega^{(n-2)})ds$$
$$-\frac{(n-1)}{n\eta^{n-1}}\int_0^\eta (\eta-s)^n f(s,\omega(s),\omega'(s),\dots,\omega^{(n-2)})ds = 0,$$

thus, from (H2), there exists $t_0 \in [0,1]$, such that $|\omega^{(n-2)}(t_0)| \leq M$.

In view of

$$\omega^{(n-2)}(0) = \omega^{(n-2)}(t_0) - \int_0^{t_0} \omega^{(n-1)}(t)dt$$

and

$$\omega^{(n-1)}(t) = \omega^{(n-1)}(0) + \int_0^t \omega^{(n)}(s)ds,$$

then, we have

$$\begin{aligned} |\omega^{(n-2)}(0)| \leq & |\omega^{(n-2)}(t_0)| + \int_0^1 \Big(\int_0^1 \omega^{(n)}(s)ds \Big) dt \\ & = M + \|\omega^{(n)}\|_1 \\ & = M + \|L\omega\|_1 \\ & \leq M + \|N\omega\|_1, \end{aligned} \tag{3.2}$$

$$\|P\omega\| = |\omega^{(n-2)}(0)| \leq M + \|N\omega\|_1. \tag{3.3}$$

Again for $\omega \in \Omega_1$, $\omega \in domL \setminus \ker L$, then $(I-P)\omega \in domL \cap KerP$, $LPx = 0$, thus from Lemma 2.1, we know

$$\|(I-P)\omega\| = \|KpL(I-P)\omega\| \leq \|L(I-P)\omega\|_1 = \|L\omega\|_1 \leq \|N\omega\|_1. \tag{3.4}$$

From (3.3)(3.4), we have

$$\|\omega\| \leq \|P\omega\| + \|(I-P)\omega\| \leq 2\|N\omega\|_1 + M. \tag{3.5}$$

If (3.1) holds, then from (3.5), we obtain

$$\|\omega\| \leq 2\left[\sum_{i=1}^{n-1} \|a_i\|_1 \|\omega^{(i-1)}\|_\infty + \|b\|_1 + \frac{M}{2}\right]. \tag{3.6}$$

From $\|\omega\|_\infty \leq \|\omega\|$, and (3.6) we have

$$(3.7)\quad \|\omega\|_\infty \leq \frac{2}{1-2\|a_1\|_1}\left[\|a_2\|_1\|\omega^{(1)}\|_\infty + \ldots + \|a_{n-1}\|_1\|\omega^{(n-2)}\|_\infty + \|b\|_1 + \frac{M}{2}\right].$$

From $\|\omega'\|_\infty \leq \|\omega\|$, (3.6) and (3.7) one has

$$\|\omega'\|_\infty\left[1 - \frac{2\|a_2\|_1}{1-2\|a_1\|_1}\right] \leq \frac{2\left[\|a_3\|_1\|\omega^{(2)}\|_\infty + \ldots + \|a_{n-1}\|_1\|\omega^{(n-2)}\|_\infty + \|b\|_1 + \frac{M}{2}\right]}{1-2\left\|a_1\right\|_1},$$

i.e.,

$$(3.8)\qquad \|\omega'\|_\infty \leq \frac{2\left[\|a_3\|_1\|\omega^{(2)}\|_\infty + \ldots + \|a_{n-1}\|_1\|\omega^{(n-2)}\|_\infty + \|b\|_1 + \frac{M}{2}\right]}{1-2\left\|a_1\right\|_1 - 2\left\|a_2\right\|_1}.$$

Similarly, we can find

$$(3.9)\qquad \|\omega^{(2)}\|_\infty \leq \frac{2\left[\|a_4\|_1\|\omega^{(3)}\|_\infty + \ldots + \|a_{n-1}\|_1\|\omega^{(n-2)}\|_\infty + \|b\|_1 + \frac{M}{2}\right]}{1-2\left\|a_1\right\|_1 - 2\left\|a_2\right\|_1 - 2\left\|a_3\right\|_1}.$$

$$\vdots$$

$$(3.10)\qquad \left\|\omega^{(n-3)}\right\|_\infty \leq \frac{2\left[\left\|a_{n-1}\right\|_1 \|\omega^{(n-2)}\|_\infty \|b\|_1 + \frac{M}{2}\right]}{1-2\left\|a_1\right\|_1 - 2\left\|a_2\right\|_1 - \ldots - 2\left\|a_{n-2}\right\|_1} = M_1$$

$$(3.11)\qquad \left\|\omega^{(n-2)}\right\|_\infty \leq \frac{2}{1-2\left\|a_1\right\|_1 - 2\left\|a_2\right\|_1 - \ldots - 2\left\|a_{n-1}\right\|_1}\left[\|b\|_1 + \frac{M}{2}\right].$$

From (3.11) there exists $M_1 > 0$, such that

$$(3.12)\qquad \left\|\omega^{(n-2)}\right\|_\infty \leq M_1.$$

Thus, from (3.10) and (3.12), there exist $M_2 > 0$, such that

$$(3.13)\qquad \left\|\omega^{(n-3)}\right\|_\infty \leq M_2.$$

Similarly there exist $M_i > 0 \ \ (i = 1, 2, \ldots, n-1)$

$$(3.14)\qquad \left\|\omega^{(n-i-1)}\right\|_\infty \leq M_i.$$

Hence,

$$\|\omega\| = \max\left\{\|\omega\|_\infty, \|\omega\prime\|_\infty, \ldots, \left\|\omega^{(n-2)}\right\|_\infty\right\} \leq \max\left\{M_1, M_2, \ldots, M_{n-1}\right\}.$$

Again, from (3.1), and (3.12)-(3.13), we have

$$\|\omega^{(n)}\|_1 = \|L\omega\|_1 \leq \|N\omega\| \leq \|a_1\|_1 M_1 + \ldots + \|a_{n-2}\|_1 M_{n-2} + \|b\|_1.$$

So, Ω_1 is bounded.

STEP 2. The set $\Omega_2 = \{\omega \in \ker L : N\omega \in ImL\}$ is bounded. In fact, $\omega \in \Omega_2, \omega \in \ker L = \{\omega \in domL : \omega = ct^{n-2}, c \in \mathbb{R}, t \in [0,1]\}$, and $QNx = 0$, thus,

$$\int_0^1 (1-s)^{n-1} f(s, cs^{n-2}, (n-2)cs^{n-3}, \ldots, (n-2)!c)ds$$
$$- \frac{1}{n\eta^{n-1}} \int_0^\eta (\eta - s)^2 f(s, cs^{n-2}, (n-2)cs^{n-3}, \ldots, (n-2)!c)ds < 0.$$

From (H2), $\|\omega\|_\infty = |c|$, so $\|\omega\| = |c| \leq M$, thus Ω_2 is bounded.

STEP 3. We show that the set $\Omega_3 = \{\omega \in \ker L : -\lambda Jx + (1-\lambda)JQNx = 0, \lambda \in [0,1]\}$, where, $J : \ker L \to ImQ$ is the linear isomorphism given by

$$J(c) = ct^{n-2}, \forall c \in \mathbb{R}, t \in (0,1].$$

Then, Ω_3 is bounded.

I) If the first part of (H3) holds, that is, there exists $M^* > 0$,such that, for any $c \in \mathbb{R}$, if $|c| > M^*$, then,

$$\frac{n(n+1)(n+2)c}{n+2-\eta^{n-1}} \Big[\int_0^1 (1-s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \ldots, c)ds$$
$$- \frac{1}{n\eta^{n-1}} \int_0^\eta (\eta - s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \ldots, c)ds \Big] < 0. \tag{3.15}$$

Since, for $\omega = c_0 t^{n-2}$, then, for $t \in (0,1]$, we obtain

$$\lambda c_0 = c(1-\lambda)\frac{n(n+1)(n+2)}{n+2-\eta^{n-1}} \Big[\int_0^1 (1-s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \ldots, (n-2)!c)ds$$
$$- \frac{1}{n\eta^{n-1}} \int_0^\eta (\eta - s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \ldots, (n-2)!c)ds \Big] < 0.$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, then in view of (3.15) one has

$$\lambda c_0^2 = c(1-\lambda)\frac{n(n+1)(n+2)}{n+2-\eta^{n-1}}\Big[\int_0^1 (1-s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \dots, c)ds$$
$$-\frac{1}{n\eta^{n-1}}\int_0^\eta (\eta-s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \dots, c)ds\Big] < 0,$$

which contradicts $\lambda c_0^2 \geq 0$. Thus, $\Omega_3 \subset \{\omega \in KerL : \|\omega\| \leq M^*\}$ is bounded.

II) If the second part of (H3) holds, that is, there exists $M^* > 0$, such that, for any $c \in \mathbb{R}$, if $|c| > M^*$, then,

$$c\frac{n(n+1)(n+2)}{n+2-\eta^{n-1}}\Big[\int_0^1 (1-s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \dots, c)ds$$
$$-\frac{1}{n\eta^{n-1}}\int_0^\eta (\eta-s)^{n-1} f(s, cs^{n-2}, cs^{n-3}, \dots, c)ds\Big] > 0$$

Similarly, we can verify Ω_3 is bounded.

Step 4. Let Ω be a bounded open subset of X, such that $\cup_{i=1}^{i=3}\overline{\Omega_i} \subset \Omega$. By the Ascoli-Arzela theorem, we can show that $Kp(I-QN) : \overline{\Omega} \to Y$ is compact, thus, N is $L-$compact on $\overline{\Omega}$. Then, by the above argument, we have

(i) $L\omega \neq \lambda Nx$, for every $(x,\lambda) \in ((domL \backslash \ker L) \cap \partial\Omega] \times (0,1)$.
(ii) $N\omega \notin ImL$, for every $\omega \in \ker L \cap \partial\Omega$.
(iii) Let $H(\omega,\lambda) = \pm\lambda\omega + (1-\lambda)QN\omega = 0$.

According to the above argument, we know $H(x,\lambda) \neq 0$, for $\omega \in \ker L \cap \partial\Omega$, by the homotopy property of degree, we get

$$\begin{aligned}\deg(JQN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(H(.,0), \Omega \cap \ker L, 0)\\ &= \deg(H(.,1), \Omega \cap \ker L, 0)\\ &= \deg(\pm J, \Omega \cap \ker L, 0)\\ &\neq 0.\end{aligned}$$

According to definition of degree on a space which is isomorphic to $\mathbb{R}$, and

$$\Omega \cap KerL = \{ct : |c| < d\}.$$

We have

$$\begin{aligned}\deg(-I, \Omega \cap \ker L, 0) &= \deg(-J^{-1}IJ, J^{-1}(\Omega \cap \ker L), J^{-l}\{0\}) \\ &= \deg(-I, (-d, d), 0) \\ &= -1 \neq 0.\end{aligned}$$

If the second part of condition (iii) of Theorem 2.1 holds, let

$$H(x, \lambda) = -\lambda x + (1-\lambda)JQNx.$$

Similar to the above argument, we have

$$\begin{aligned}\deg(JQN_{|\ker L}, \Omega \cap \ker L, 0) &= \deg(H(.,0), \Omega \cap \ker L, 0) \\ &= \deg(H(.,1), \Omega \cap \ker L, 0) \\ &= \deg(I, \Omega \cap \ker L, 0) \\ &= 1.\end{aligned}$$

Then, we have

$$\deg(JQN_{|\ker L}, \Omega \cap \ker L, 0) \neq 0$$

Then by, Theorem 2.1, $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$, so that the BVP (1.9)(1.10) has at least one solution in $C^1[0,1]$. The proof is completed. □

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