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WEAK CONVERGENCE APPROACH TO THE MODERATE DEVIATIONS PRINCIPLE FOR POSITIVE DIFFUSION

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ABSTRACT. We consider the family of stochastic processes $X^\varepsilon = \{X_t^\varepsilon \text{ for } 0 \leq t \leq 1\}$, $\varepsilon > 0$ where X^ε is a solution of the Ito's differential equation.

$$X_t^\varepsilon = x + \sqrt{\varepsilon}\int_0^t \sigma(X_s^\varepsilon)dW_t + \int_0^t b(X_s^\varepsilon)dt; \quad x > 0. \tag{A}$$

In this paper, we deal with the weak convergence approach to prove a moderate deviation principle for X^ε solution of (A).

1. INTRODUCTION

In recent years, several authors have studied the large deviations theory [1,5,7, 8]. Like the large deviations, the moderate deviation problems arise in the theory of statistical inference quite naturally. The estimates of moderate deviations can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals.

We consider in this paper the following stochastic differential equation

$$\begin{cases} dX_t^\varepsilon = \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dW_t + b(X_t^\varepsilon)dt \\ X_0^\varepsilon = x \end{cases}, \tag{1.1}$$

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where $x > 0$, W is a standard one-dimentionnal Brownian motion and σ, b satisfy the following assumption.

(H): $\sigma : \mathbb{R} \to \mathbb{R}$ is Holder continuous with exponent $\gamma \in [\frac{1}{2}; 1]$, and is locally Lipschitz continuous on $]0, +\infty[$, vanishes at 0 and has a sublinear growth at ∞. The function $b : \mathbb{R} \to \mathbb{R}$ is C^1-continiuos and has a sublinear growth at ∞ and $b(0) > 0$. We asume that there exist an constant $C_1 > 0$ such that, for all $x, x' \in \mathbb{R}$, we have:

$$\begin{aligned} xb(x) &\leq C_1(1 + \|x\|), \\ \|b(x) - b(x') &\leq C_1(\|x - x'\|), \\ \|\sigma(x)\| &\leq C_1, \\ \|\sigma(x) - \sigma(x')\| &\leq C_1(\|x - x'\|^\gamma). \end{aligned}$$

Under assumption (H), the equation (1.1) has a unique strong solution X_t^ε and furthermore $X_t^\varepsilon > 0$ (see [2, 11]).

The applications to finance have attracted the attention to the study of these models that are based on diffusion processes whose state space is the positive half line. Letting $b(x) = \alpha(\beta - x)$ and $\sigma = \rho x^\gamma$ with some constants $\alpha > 0$, $\beta > 0$ and $\gamma \in [\frac{1}{2}; 1]$; (1.1) is a constant elasticity of variance (CEV) model and the special case $\gamma = \frac{1}{2}$ is the Cox-Ingersoll-Ross (CIR) model.

The type of moderate deviation of stochastic (partial) differential equation has been studied by several authors such that: in [4] the authors have estabilished the moderate deviations principle for small perturbation Wishart processes that is a non-Lipschitzian coefficients. For the Lipschitzian coefficients, the moderate deviations for martingale with bounded jumps have established by [6]; the moderate deviations for Volterra equation can be found in [10] and the moderate deviations for stochastic reaction-diffusion equation with multiplicative noise was presented in [13].

Let us consider a process Y^ε solution of the following equation

$$Y_t^\varepsilon := \frac{1}{\sqrt{\varepsilon} h(\varepsilon)}(X_t^\varepsilon - X_t^0), \quad t \in [0, 1], \tag{1.2}$$

where X^0 is the solution of the ordinary differential equation

$$\begin{cases} dX_t^0 = b(X_t^0)dt \\ X^0(0) = x > 0 \end{cases}$$

and $h(\varepsilon)$ is some deviation scale which strongly influences the asymptotic behavior of Y^ε.

(a) The case that $h(\varepsilon) = 1/\sqrt{\varepsilon}$ provides some large deviations estimates.
(b) If $h(\varepsilon) = 1$, we are in the domain of the central limit theorem (CLT in short).
(c) To fill in the gap between the CLT scale $[h(\varepsilon) = 1]$ and the large deviations scale $[h(\varepsilon) = 1/\sqrt{\varepsilon}]$, we will study the moderate deviation principle (MDP in short), that is, when the deviation scale satisfies

$$h(\varepsilon) \to \infty \; and \; \sqrt{\varepsilon}h(\varepsilon) \to 0, \; when \; \varepsilon \to 0, \tag{1.3}$$

For the positive diffusion, Baldi and Caramellino [2] are established the general Freidlin-Wentzell large deviations and li. Y and Zhang. S [9] are studied the moderate deviation and central limit theorem. Their approch are used the exponentially equivalent and some theorem in Dembo and Zeitouni [5] such as in R. Wang and T. Zhang [13] wich is shown that the processes exponentially equivalents satisfy the same large deviations principle (LDP in short). We purpose in this paper to establish the LDP in $C([0;1];R)$ for the process $Y^\varepsilon(t)$ defined by (1.2), for that we will use the weak convergence approch, our method consist to show only the MDP but without the CLT.

This paper is organised as follow, the next section consistes to introduice some general results for large deviations and contains the main result. We will establish the proof in section 3 with additionnal hypothesis.

2. Large deviation prrinciple

In this section, we recall a definition and some basic properties about LDP, see [3, 5, 12]. Let ξ be a Polish space with the Borel $\sigma-$field $\mathcal{B}(\xi)$.

Definition 2.1. *A family $\{X^\varepsilon\}_{\varepsilon>0}$ of $E-$valued random elements is said to satisfy the large deviations principle on E, with the rate function I and with the speed function $\lambda(\varepsilon)$ which is a sequence of positive numbers tending to $+\infty$ when $\varepsilon \to 0$, if the following conditions hold:*

(a) *for each $M < \infty$ thr level set $\{x \in \xi; I(x) \leq M\}$ is a compact subset of ξ,*
(b) *for each closed subset F of ξ, $\limsup_{\varepsilon\to 0} \frac{1}{\lambda(\varepsilon)} \log(\mathbb{P}(X^\varepsilon \in F)) \leq -\inf_{x\in F} I(x)$;*

(c) *for each closed subset* F *of* ξ*;* $\liminf_{\varepsilon\to 0} \frac{1}{\lambda(\varepsilon)} \log(\mathbb{P}(X^\varepsilon \in G)) \geq -\inf_{x\in G} I(x)$.

The Cameron–Martin space associated with the Brownian motion $\{W_t;\ t \in [0,T]\}$ is given by

$$\mathcal{H} = \left\{ h : [0,T] \to \mathbb{R}; h\ is\ absolutly\ coninuous\ and\ \int_0^T |\dot{h}_s|^2 ds < +\infty \right\}.$$

The space $\mathcal{H}$ is an Hilbert space with winner product $< h_1, h_2 >:= \int_0^T < \dot{h_1}(s), \dot{h_2}(s) > ds$. The Hilbert space $\mathcal{H}$ is endowed with the weak topology, i.e., for any h_n where $h \in \mathcal{H},\ n \geq 1$, we say that h_n converges to h in the weak topology, if for any $g \in \mathcal{H}$,

$$< h_n - h, g >= \int_0^T < \dot{h}_{ns} - \dot{h}_s, \dot{g}_s > ds \to 0,\ as\ n \to \infty.$$

Theorem 2.1 (see [12]). *The probability measures induced by* $\sqrt{\varepsilon}W$ *on* $C([0,1];\mathbb{R})$*, satisfy a LDP with the good rate function* I *defined by*

$$I(h) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{h}_s|^2 ds\ \ if\ \ h \in \mathcal{H} \\ \infty\ \ otherwise \end{cases}. \tag{2.1}$$

Let us $\mathcal{A}$ denote the class of real-valued $\{F_t\}-$predictable processes φ belonging to $\mathcal{H}$ a.s. Let

$$\mathcal{S}_N := \left\{ h \in \mathcal{H}; \int_0^T \|\dot{h}_s\|^2 ds \leq N \right\},$$

$\mathcal{S}_N$ is endowed with the weak topology induced from $\mathcal{H}$. Define the following

$$\mathcal{A}_N := \{\Phi \in \mathcal{A}; \Phi(\omega) \in \mathcal{S}_N, \mathbb{P} - a.s\}.$$

Theorem 2.2. *(see [3]) For any* $\varepsilon > 0$*, let* Γ^ε *be a measurable mapping from* $C([0,1];\mathbb{R})$ *into* ξ*. Suppose that* $\{\Gamma^\epsilon\}_{\varepsilon>0}$ *satisfies the following assumptions: there exists a measurable map* $\Gamma^0 : C([0,1];\mathbb{R}) \to \xi$ *such that*

(a) *for every* $N < +\infty$ *and any family* $\{h_\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_N$ *satisfying that* h_ε *converge in distribution as* $\mathcal{S}_N-$*valued random elements to* h *as* $\varepsilon \to 0$*,* $\Gamma^\varepsilon\left(W_. + \int_0^T \dot{h}^\varepsilon(s)ds\right)$ *converge in distribution to* $\Gamma^0\left(\int_0^T \dot{h}(s)ds\right)$ *as* $\varepsilon \to 0$*;*

(b) *for every* $N < \infty$*, the set* $\left\{\Gamma^0\left(\int_0^T \dot{h}(s)ds\right); h \in \mathcal{S}_N\right\}$ *is a compact subset of* ξ*. Then, the family* $\{\Gamma^\varepsilon(W(.))\}_{\varepsilon>0}$ *satisfies a LDP in* ξ *with the rate function*

I given by

$$I^*(g) = \inf_{\left\{g=\Gamma^0\left(\int_o^\cdot \dot{h}(s)ds\right)\right\}} I(h); \quad g \in \xi. \tag{2.2}$$

where I is given by (2.1).

We now present our main result.

Theorem 2.3. *Under assumption (H); the process Y^ε defined in* (1.2)*, such that the deviation scale $h(\varepsilon)$ respects* (1.3)*, satisfies an LDP in $C([0,1];\mathbb{R})$ with the rate function I^* given by* (2.2).

3. Proof of the moderate deviation principle

3.1. **Skeleton equation.** For any $h \in \mathcal{H}$, consider the deterministic integral equation

$$Y_t^h = \int_0^t \sigma(X_s^0)\dot{h}_s ds + \int_0^t b'_x(X_s^0) Y_s^h ds. \tag{3.1}$$

We suppose the following hypothesis

(L): The coefficient b' derivate of b is locally lipschitzian, that is there exists a positive constant $C_{b'}$ such that for all $x_1, x_2 \in \mathbb{R}$

$$\|b'(x_1) - b'(x_2)\| \leq C_{b'}\|x_1 - x_2\|; \tag{3.2}$$

combined with the uniform Lipschitz continuity of b, we have

$$\|b'(x)\| \leq C_{b'}, \ \forall x \in \mathbb{R}. \tag{3.3}$$

We begin by introducing the map Γ^0 that will be used to define the rate function and to verify the conditions in Theorem 2.2.

Lemma 3.1. *Under Hypotheses (H) and (L), for any $h \in \mathcal{H}$, the equation* (3.1) *admits a unique solution Y^h in $C([0,1];\mathbb{R})$, denote by $Y^h = \Gamma^0\left(\int_0^\cdot \dot{h}_s ds\right)$. Moreover, for any $N > 0$, there exists an constant $c(C_{b'}, C_1, N, T)$ wich depend in $C_{b'}, C_1, N$ and T such that*

$$\sup_{h \in \mathcal{S}_N} \left\{ \sup_{0 \leq t \leq T} \left\|Y^h(t)\right\| \right\} \leq c(C_{b'}, C_1, N, T). \tag{3.4}$$

Proof. The existence and uniqueness of the solution can be proved by hypothesis (H) and (L). The inequality (3.4) follows from the linear growth conditions of the coefficient functions and Gronwall's inequality. □

Proposition 3.1. *Under Hypotheses (H) and (L), for every positive number $N < \infty$, the family*

$$K_N := \left\{ \Gamma^0 \left(\int_0^{\cdot} \dot{h}_s ds \right) ; h \in \mathcal{S}_N \right\}$$

is compact in $C([0,1];\mathbb{R})$.

Proof. We first prove that the map Γ^0 defined in Lemma 3.1 is continuous from $\mathcal{S}_N$ to $C([0,1];\mathbb{R})$. Then for any $N < \infty$, the fact that K_N is compact follows from the compactness of $\mathcal{S}_N$ and the continuity of the map Γ^0 from $\mathcal{S}_N$ to $C([0,1];\mathbb{R})$. It remains to prove that Γ^0 is a continuous map from $\mathcal{S}_N$ to $C([0,1];\mathbb{R})$. Let $h_n \to h$ weakly in $\mathcal{S}_N$ as $n \to \infty$. Then,

$$Y_t^{h_n} - Y_t^h = \int_0^t \sigma(X_s^0)(\dot{h}_n(s) - \dot{h}(s))ds + \int_0^t b'_x(X_s^0)(Y_s^{h_n} - Y_s^h)ds = I_1^n + I_2^n.$$

By the linear growth condition of σ and the boundness of X^0, we know that for any fixed $t \in [0,T]$, the function $\sigma(X_\cdot^0) : s \in [0,t] \to \mathbb{R}$ belongs to $L^2([0,t];\mathbb{R})$. Since $h^n \to h$ weakly in $L^2([0,T];\mathbb{R})$, we know that

$$I_1^n(t) = \int_0^t \sigma(X_s^0)(\dot{h}_n(s) - \dot{h}(s))ds \to 0. \tag{3.5}$$

For any $0 \le t_1 < t_2 \le T$; by (H) and Cauchy-schwartz inequality, we have

$$\begin{aligned} \|I_1^n(t_2) - I_1^n(t_1)\| &= \left\| \int_{t_1}^{t_2} \sigma(X_s^0)(\dot{h}_n(s) - \dot{h}(s))ds \right\| \\ &\le C_1 \int_{t_1}^{t_2} \|\dot{h}_n(s) - \dot{h}(s)\| ds \\ &\le C_1 (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} \|\dot{h}_n(s) - \dot{h}(s)\|^2 ds \right)^{1/2} \\ &\le \sqrt{2} N C_1 |t_2 - t_1|^{1/2} \end{aligned} \tag{3.6}$$

Hence the functions I_2^n are equi-continuous in $C([0;1];\mathbb{R})$.

By the linear growth condition of σ and Cauchy-Schwartz inequality, we have

$$\begin{aligned} \sup_{t \in [0,T]} \|I_1^n(t)\| &\le \int_0^T \|\sigma(X_s^0)(\dot{h}_n(s) - \dot{h}(s)\| ds \\ &\le \left(\int_0^T \|\sigma(X_s^0)\|^2 \right)^{1/2} \left(\int_0^T \|\dot{h}_n(s) - \dot{h}(s)\| \right)^{1/2} \\ &\le c(C_1, N, T) < \infty \text{ and independant of } n. \end{aligned} \tag{3.7}$$

According to Arzéla-Ascoli Theorem, (3.6) and (3.7) imply that

$$\lim_{n\to\infty}\sup_{0\le t\le T}\|I_1^n(t)\|=0. \tag{3.8}$$

By (3.3), we have

$$\|I_2^n(t)\|\le C_{b'}\int_0^t\|Y_s^{h_n}-Y_s^h\|ds.$$

So by gronwall lemma apply to $\lambda^n(t)=\sup_{0\le s\le t}\|Y_s^{h_n}-Y_s^h\|$ and (3.8), its follow that

$$\sup_{t\in[0,T]}\|Y^{h_n}-Y^h\|\le\exp(C_{b'}T)\times\sup_{t\in[0,T]}\|I_1^n(t)\|\to 0,\ as\ n\to\infty.$$

And then, assertion follows. □

3.2. **The moderate deviations principle.** Let consider the process Y^ε define by (1.2) which is solution of the following equation:

$$\begin{aligned}Y_t^\varepsilon=&\frac{1}{h(\varepsilon)}\int_0^t\sigma(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^\varepsilon)dW_s\\&+\frac{1}{h(\varepsilon)\sqrt{\varepsilon}}\int_0^t[b(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^\varepsilon)-b(X_s^0)]ds.\end{aligned}\tag{3.9}$$

This equation admits an unique solution $Y^\varepsilon=\Gamma^\varepsilon(W.)$, where Γ^ε stands for the solution functionnal from $C([0;1];\mathbb{R})$ to $C([0;1];\mathbb{R})$.

The proof of the main theorem need the following lemma wich is a direct consequence of Girsanov's theorem and Ito's formula.

Lemma 3.2. *For every fixed $N\in\mathbb{N}$, let $\Phi^\varepsilon\in\mathcal{A}_N$ and Γ^ε be given by 3.9. Then $Y^{\Phi^\varepsilon,\varepsilon}:=\Gamma^\varepsilon\left(W.+h(\varepsilon)\int_0^{\cdot}\dot{\Phi}_s^\varepsilon ds\right)$ is the solution of the following equation*

$$\begin{aligned}Y_t^{\Phi^\varepsilon,\varepsilon}\;&=\frac{1}{h(\varepsilon)\sqrt{\varepsilon}}\int_0^t\left[b(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon})-b(X_s^0)\right]ds\\&\quad+\int_0^t\sigma(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon})\dot{\Phi}_s^\varepsilon ds\\&\quad+\frac{1}{h(\epsilon)}\int_0^t\sigma(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon})dW_s\end{aligned}\tag{3.10}$$

Furthermore, there exists an constant $c(C_1,N,T)$ independent of ε such that

$$\mathbb{E}\int_0^T|Y_t^{\Phi^\varepsilon,\varepsilon}|^2dt<c(C_1,N,T). \tag{3.11}$$

Proposition 3.2. *Assume that hypothesis (H) and (L) hold. For every fixed $N\in\mathbb{N}$, let $\Phi^\varepsilon,\Phi\in\mathcal{A}_N$ be such that Φ^ε convergence in distribution to Φ as ε goes to 0. Then*

$\Gamma^\varepsilon\left(W(.)+h(\varepsilon)\int_0^\cdot\dot{\Phi}^\varepsilon(s)ds\right)$ *converges in distribution to* $\Gamma^0\left(\int_0^\cdot\dot{\Phi}(s)ds\right)$ *in the space* $C([0,1],\mathbb{R}^d)$ *as* ε *goes to* 0.

Proof. By the Skorokhod representation theorem, there exist a probability $(\bar{\Omega},\bar{F},(\bar{\mathcal{F}}_t),\bar{\mathbb{P}})$, and on this basis, a Brownian motion $\bar{W}$ and also a family of $\bar{\mathcal{F}}_t-$predictable processes $\{\bar{\Phi}^\varepsilon;\varepsilon>0\}$, $\bar{\Phi}$ taking values on $\mathcal{A}_N$, $\bar{\mathbb{P}}-$a.s., such that the joint law of $(\Phi^\varepsilon,\Phi,W)$ under $\mathbb{P}$ coincides with that of $(\bar{\Phi}^\varepsilon,\bar{\Phi},\bar{W})$ under $\bar{\mathbb{P}}$ and

$$\lim_{\varepsilon\to0}\langle\bar{\Phi}_\varepsilon-\bar{\Phi},g\rangle=0,\ \forall g\in\mathcal{H},\bar{\mathbb{P}}-a.s..$$

Let $\bar{Y}_t^{\bar{\Phi}^\varepsilon,\varepsilon}$ be the solution to a similar equation as (3.10) replacing Φ^ε by $\bar{\Phi}^\varepsilon$ and W by $\bar{W}$, and let $\bar{Y}$ be the solution to a similar equation as (3.1) replacing h by $\bar{\Phi}$. Thus, to prove this proposition, it is sufficient to prove that

$$\lim_{\varepsilon\to0}\|\bar{Y}^{\bar{\Phi}^\varepsilon,\varepsilon}-\bar{Y}\|=0\text{ in probability.}\tag{3.12}$$

From now on, we drop the bars in the notation for the sake of simplicity.

Notice that

$$\begin{aligned}
&Y_t^{\Phi^\varepsilon,\varepsilon}-Y_t^\Phi\\
&=\int_0^t\left\{\frac{1}{h(\varepsilon)\sqrt{\varepsilon}}\left[b\big(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)-b(X_s^0)\right]-b'_x(X_s^0)Y_s^\Phi\right\}ds\\
&\quad+\int_0^t\left[\sigma\big(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)\dot{\Phi}_s^\varepsilon-\sigma(X_s^0)\dot{\Phi}_s\right]ds\\
&\quad+\frac{1}{h(\varepsilon)}\int_0^t\sigma\left(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\right)dW_s\\
&=:J_1+J_2+J_3.
\end{aligned}\tag{3.13}$$

By Taylor formula, there exists a random variable $\eta^\varepsilon(t)$ taking values in $(0;1)$ such that

$$\begin{aligned}
&\left\|\frac{1}{h(\varepsilon)\sqrt{\varepsilon}}\left[b\big(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)-b(X_s^0)\right]-b'(X_s^0)Y_s^\Phi\right\|\\
&=\left\|b'\big(X_s^0+\eta\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)Y_s^{\Phi^\varepsilon,\varepsilon}-b'(X_s^0)Y_s^\Phi\right\|\\
&\le\left\|b'\big(X_s^0+\eta\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)\right\|\left\|Y_s^{\Phi^\varepsilon,\varepsilon}-Y_s^0\right\|\\
&\quad-\left\|b'(X_s^0+\eta\sqrt{\varepsilon}h(\varepsilon)Y^{\Phi^\varepsilon,\varepsilon})-b'(X_s^0)Y_s^\Phi\right\|\|Y_s^\Phi\|.
\end{aligned}$$

This inequality together with (3.2) and (3.3), we have

$$\sup_{s\in[0;t]}\|J_1(s)\|\le C_{b'}\int_0^t\|Y_s^{\Phi^\varepsilon,\varepsilon}-Y_s^0\|ds+C_{b'}\sqrt{\varepsilon}h(\varepsilon)\int_0^t\|Y_s^{\Phi^\varepsilon,\varepsilon}\|.\|Y_0^s\|ds.\tag{3.14}$$

By condition (H), we obtain

$$\begin{aligned}
\|J_2(t)\| &\leq \left\|\int_0^t \left[\sigma\big(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)(\dot{\Phi}_s^\varepsilon-\dot{\Phi}_s)\right]ds\right\| \\
&\quad+\int_0^t \left[\sigma\big(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)-\sigma(X_s^0)\right]\dot{\Phi}_s ds \\
&\leq \left\|\int_0^t \sigma(X_s^0)(\dot{\Phi}^\varepsilon(s)-\dot{\Phi}(s))\right\| \\
&\quad+\int_0^t \left\|\sigma(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y^{\Phi^\varepsilon,\varepsilon})-\sigma(X_s^0)\right\|\left\|\dot{\Phi}_s^\varepsilon-\dot{\phi}_s\right\|ds \\
&\quad+\int_0^t \left\|\left[\sigma\big(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\big)-\sigma(X_s^0)\right]\dot{\Phi}_s\right\|ds \\
&\leq \left\|\int_0^t \sigma(X_s^0)(\dot{\Phi}^\varepsilon(s)-\dot{\Phi}(s))\right\|+2\sqrt{\varepsilon}h(\varepsilon)C_1\int_0^t \|Y^{\Phi^\varepsilon,\varepsilon}\|(\|\dot{\Phi}^\varepsilon(s)\|+\|\dot{\Phi}(s)\|ds.
\end{aligned}$$

Using the same argument as in the proof of (3.8); we obtain that

$$\sup_{t\in[0;T]}\left\|\int_0^t \sigma(X_s^0)(\dot{\Phi}^\varepsilon(s)-\dot{\Phi}(s))ds\right\|\to 0 \tag{3.15}$$

in probability as $\varepsilon\to 0$. By the Lemma 3.1 and Lemma 3.2 and using Cauchy schwartz inequality, we know that there exists an constant $c(C_1,C_{b'},N,T)$ such that

$$\begin{aligned}
&\mathbb{E}\left\{\int_0^T \left\|Y_s^{\Phi^\varepsilon,\varepsilon}\right\|(\|\dot{\Phi}^\varepsilon(s)\|+\|\dot{\Phi}(s)\|ds+\int_0^T \left\|Y_s^{\Phi^\varepsilon,\varepsilon}\right\|.\left\|Y_s^{\Phi}\right\|ds\right\} \\
&\leq c(C_1,C_{b'},N,T).
\end{aligned} \tag{3.16}$$

Hence, we obtain by Chebychev's inequality and (3.16)

$$\sup_{t\in[0;T]}\left\|J_2(t)\right\|\to 0 \text{ in probability as } \varepsilon\to 0. \tag{3.17}$$

For the third term, by the Burkholder-Gandy-Devis inequality and assumption (H), there exists some constant K such that, for $t\in[0;T]$ we obtain

$$\begin{aligned}
\mathbb{E}\|J_3(t)\| &= \mathbb{E}\left(\left\|\frac{1}{h(\varepsilon)}\int_0^t \sigma\left(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\right)dW_s\right\|\right) \\
&\leq \frac{K}{h(\varepsilon)}\mathbb{E}\left(\int_0^t \left\|\sigma\left(X_s^0+\sqrt{\varepsilon}h(\varepsilon)Y_s^{\Phi^\varepsilon,\varepsilon}\right)\right\|^2 ds\right)^{1/2} \\
&\leq \frac{1}{h(\varepsilon)}c(K,C_1,T)<\infty.
\end{aligned}$$

Combining this last inequality with the Chebychev's inequality, we have that

$$\sup_{t\in[0;T]}\left\|J_3(t)\right\|\to 0 \text{ in probability as } \varepsilon\to 0. \tag{3.18}$$

Combining (3.13) and (3.14) we have

$$\sup_{s\in[0;t]}\left\|Y_s^{\Phi^\varepsilon,\varepsilon}-Y_s^{\Phi}\right\|$$
$$\leq C_{b'}\int_0^t \sup_{u\in[0;s]}\left\|Y_u^{\Phi^\varepsilon,\varepsilon}-Y_u^{\Phi}\right\|ds + C_{b'}\sqrt{\varepsilon}h(\varepsilon)\int_0^t\left\|Y_s^{\Phi^\varepsilon,\varepsilon}\right\|.\left\|Y_s^{\Phi}\right\|ds$$
$$+\sup_{s\in[0;T]}\left\|J_2(t)\right\|+\sup_{s\in[0;T]}\left\|J_3(t)\right\|.$$

This inequality together with (3.14); (3.17),(3.18), Chebychev's inequality and Gronwall's inequality imply that $\sup_{t\in[0;T]}\left\|Y_t^{\Phi^\varepsilon,\varepsilon}-Y_t^{\Phi}\right\|\to 0$ in probability as $\varepsilon\to 0$.

The proof is complete. □

We finish this paper for the proof of MDP for positive diffusions.

Proof of Theorem 2.3. In the following part, we prove our main theorem. According to the Theorem 2.2, we need to prove that two conditions of this theorem are fulfilled. For that, condition (b) has been established in Proposition 3.1 but the verification of condition (a) is given by Proposition 3.2. □

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