Advances in Mathematics: Scientific Journal **11** (2022), no.8, 723–739
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.8.7

ON THE M/G/1 FEEDBACK RETRIAL QUEUEING WITH ORBITAL SEARCH OF CUSTOMERS

Zina Boussaha, Nadia Oukid, Halim Zeghdoudi[1], Sarah Soualhi, and Natalia Djellab

ABSTRACT. We consider a single server feedback retrial queueing system with orbital search. An arriving customer finding the server idle enters into service immediately; otherwise, the customer enters into an orbit. The service times are supposed to be arbitrarily distributed. An orbiting customer competes for service, the inter-retrial times are exponentially distributed. Upon completion of a service, the server stays idle for a while, waiting for either a new job or a job from the buffer. After the idle period, the server starts searching for blocked customers, according to the exponential distribution. During the searching time, the server cannot serve a customer. After the searching time the server gets a customer from the orbit if any, otherwise it stays idle again. After completing, the customer has an option to join the orbit as a feedback , customer or to leave the system forever. Various performance measures are derived, and numerical results are given.

1. INTRODUCTION

In many real service situations the customers whose service cannot start upon arrival must leave the service area. They do not give up their request but join a virtual waiting room, called orbit and retry to get service again after a random

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 58J65, 62P30, 62C05.
Key words and phrases. Retrial queue, orbit, feedback, search time.
Submitted: 31.07.2022; *Accepted:* 15.08.2022; *Published:* 28.08.2022.

time interval. This queueing phenomenon is known as models with retrials. Retrial queues arise from various real life situations as well as telecommunication, computer networks and data communication systems. A review of the main results on this topic can be found in [1–4, 9, 20]. Among the most recent and interesting applications, we can mention a new queuing theory approach for cost reduction in product-service design [16]. For applications of queueing theory in health care, we refer the reader to [14], and readers with motivation in communication systems should consult [21].

One additional feature that has been widely discussed in retrial queueing systems is the Bernoulli feedback of customers. Many queueing situations have the feature that the customers may be served repeatedly for a certain reason. When the service of a customer is unsatisfied, it may be retried again and again until a successful service completion. These queueing models arise in the stochastic modeling of many real-life situations.The feedback is common property of communication networks in which data (packets, frames, etc.) are re-transmitted if errors occurred during their initial transmission. It also often appear in production systems where issues that are not fully machined are re-processed. Choi and Kulkarni [7] have studied an M/G/1 retrial queue with feedback of customers. Some of the authors like [8, 18, 19] have discussed the concept of feedback.

In some scenarios, idle servers are able to inform orbiting customers of their status. This allows servers to fetch customers directly from the orbit with some probability if there are no other customers waiting in the queue at service completion instant. This behavior is called orbital search and introduced in the case of classical queue by Neuts and Ramalhoto [15]. In the case of $M/G/1$ queues with retrials, search for orbital customers was introduced by Artalejo et al. in [5] as follows. Upon completion of a service, with probability $p, 0 \leq p \leq 1$, the server takes a customer, if any, from the orbit for service. The search time for the customer is assumed to be insignificant. With probability $1-p$ the server will become idle until an orbital customer or a new arrival captures the server. Chakravarthy et al. analyzed multi server queues with search of customers from the orbit in [6]. More literature related to orbital search can be found in [10–13, 17].

Although results have been reported seperately on retrial queueing systems, retrial queues with orbital search of customers after idle time and Bernoulli feedback no work has been found in the literature which studies retrial queues taking into account all the above mentioned features. In the present paper a more realistic feedback retrial queueing system with orbital search of customers is studied.

This work considers a single server feedback retrial queueing model with orbit search of customers. Search for customers in orbit is introduced to reduce their waiting time. The condition for system stability is established. Steady state analysis of the model has been done and some important measures of performance has been evaluated.

The model is motivated by the following:

- The M/G/1 Feedback Retrial Queueing With Orbital Search Of Customers use several parameters, but it is easy to apply.
- The explicit expressions for the average queue length of orbit and system of M/G/1 Feedback Retrial Queueing With Orbital Search Of Customers can be determined in an explicit form.
- This new model has advantages including many parameters which we can modeled engineering and actuarial science problems.
- The objectives of insurance companies to modernize and improve reception performance are underpinned by a quality approach based on strictly quantitative indicators. For this, the call of M/G/1 Feedback Retrial Queueing With Orbital Search Of Customers are more than necessary.

The rest of the paper is organized as follows. Section 2 describes the queueing model in details while Section 3 is devoted to the analysis of the model. In section 4, several numerical results are presented and some comments are made. Finally, the paper ends with a conclusion and directions for future work.

2. THE MODEL

In this paper, we consider a single server feedback retrial queueing system with orbital search of customers from the orbit. The detailed description of model is given as follows:

The arrival process: The primary customers arrive at the system according to a Poisson process of rate λ.

The service process: Service times are independent with distribution function $B(x), (B(0)=0)$. Let $\widetilde{B}(s) = \int_0^\infty e^{-sx} dB(x)$ be the Laplace-Stieltjes transform of the $B(x), \beta_k = (-1)^k \widetilde{B}^{(k)}(0)$ be the k^{th} moment of the service time about the origin, $\gamma = \frac{1}{\beta_1} > 0$, be the service rate, $b(x) = \frac{B'(x)}{1-B(x)}$ be the instantaneaus service intensity given that the elapsed service time is equal to x.

The retrial rule: We assume that there is no waiting space and therefore if an arriving customer finds that the server is free, the customer occupies it immediately. Otherwise, the server is busy (serving a customer or searching); the arrivals join the pool of blocked customers called an orbit in accordance with FCFS discipline. That is, only one customer at the head of the orbit queue is allowed access to the server. We assume that inter-retrial times follow an exponentially distributed time with mean $\frac{1}{\theta}$.

The idle time: After the completion of a service the server stay idle for an exponentially distributed time with mean $\frac{1}{\alpha}$.During this idle time, an arriving customer (eiter a new customer or a repeaded one) is immmediately served.

The search rule: After the idle time, the server starts searching for a customer in the orbit.The searching time follows the exponential distribution with mean $\frac{1}{\mu}$.

The feedback rule: After the customer is served completly, he may decide either to, leave the system with probability $\overline{c} = 1 - c$ or to join the retrial orbit again for another service with complementary probability c.

The flow of primary arrivals, the service times, the intervals between repeated attempts, the idle times of service, and searching times of customers are assumed to be mutually independent.

3. Stability condition

In this section, we find the steady state queue size distribution at departure epochs. Let ξ_n be the time when the server enters the idle state for the $n^{\ th}$ time. The sequence of random variables $\{q_n = N_o(\xi_n), n \geq 1\}$ forms a Markov chain which is the embedded Markov chain for our model. Its state space is $S = \mathbb{Z}^+$ and its fundamental equation is defined as

$$q_{n+1} = q_n - \delta_{q_n} + v_{n+1} + u. \tag{1}$$

Here, the random variable v_{n+1} represents the number of primary customers arriving at the system during the $(n+1)^{th}$service time interval. It does not depend on events which have occurred before the beginning of the $(n+1)^{th}$ service, and its distribution is given by

$$P\left(v_n=i\right)=k_i=\int_0^{\infty}\frac{(\lambda x)^i}{i!}e^{-\lambda x}dB\left(x\right),$$

with generating function $K\left(z\right)=\sum_{n=0}^{\infty}k_n z^n=\widetilde{B}\left(\lambda-\lambda z\right)$ .The random variable $\delta_{q_n}$is defined as:

$$\delta_{q_n}=\begin{cases}1 & \text{if the } (n+1)^{th} \text{ served customer is the orbiting one,}\\ 0 & \text{if the } (n+1)^{th} \text{ served customer is the primary one arriving.}\end{cases}$$

Its conditional distribution is given by

$$\begin{aligned}P\left(\delta_{q_n}=1/q_n=i\right) &= \frac{i\theta}{\lambda+i\theta}, i\neq 0.\\ P\left(\delta_{q_n}=0/q_n=i\right) &= \frac{\lambda}{\lambda+i\theta}.\end{aligned}$$

the random variable u is defined as

$$u=\begin{cases}1 \text{ the served customer decid to joind the orbit,}\\ 0 \text{ the served customeer decid to lived the system.}\end{cases}$$

thus $P\left[u=0\right]=\overline{c}$ and $P\left[u=1\right]=c.$

We have the following one-step transition probabilities

$$r_{ij}=k_{j-i}\frac{\lambda}{\lambda+i\theta}\overline{c}+k_{j-i+1}\frac{i\theta}{\lambda+i\theta}\overline{c}+k_{j-i-1}\frac{\lambda}{\lambda+i\theta}c+k_{j-i}\frac{i\theta}{\lambda+i\theta}c.$$

Note that $r_{ij}\neq 0$ for $i=0,1,2,...,j+1.$

Theorem 3.1. *The embedded Markov chain $\{q_n,n\geq 1\}$ is ergodic if and only if the inequality $\rho=\lambda\beta_1+c<1$ holds.*

Proof. From ($\backslash ref\{1\}$), we can see that $\{q_n,n\geq 1\}$is an irreducible and aperiodic Markov chain. To find a sufficient condition, we use Foster's criterion. According to the latter, we show the existence of a non negative function $f(k),k\in S$,(in our case $f\left(q_n\right)=q_n$) and $\varepsilon>0$ such that the mean drift $\chi_k=E[f\left(q_{n+1}\right)-f\left(q_n\right)/q_n=k]$ is finite for all $k\in S$ and $\chi_k\leq-\varepsilon$ for all $k\in S$ except perhaps a finite number, that is $\chi_k=E\left[q_{n+1}-q_n/q_n=k\right]=\lambda\beta_1-\frac{k\theta}{\lambda+k\theta}+c$.Let $\chi=\lim_{k\to\infty}\chi_k$.Then $\chi=\lambda\beta_1-1+c<0$. Therefore the sufficient condition is $\lambda\beta_1+c<1$. To prove that there is also a necessary condition for our embedded Markov chain, we apply Kaplan's condition: an irreducible and aperiodic Markov chain is not ergodic if

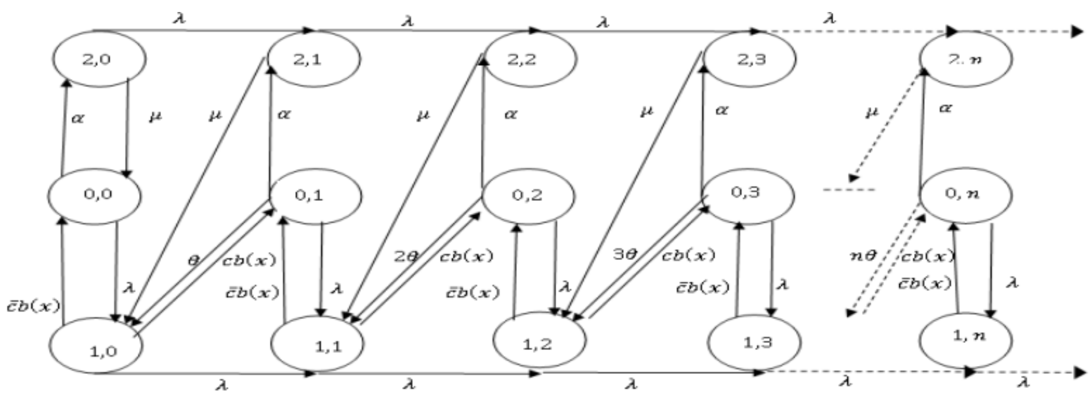
$\chi_k < \infty$ for all $k \geq 0$ and $k_0 \in \mathbb{N}$ exists such that $\chi_k \geq 0$ for $k \geq k_0$ in our case, this condition is verified because $r_{ij} = 0$ for $j < i - 1$ and $i > 0$. Therefore, $\lambda\beta_1 + c \geq 1$ gives the non ergodicity of the embedded Markov chain $\{q_n, n \geq 1\}$. Finally, $\{q_n, n \geq 1\}$ is ergodic if and only if $\rho = \lambda\beta_1 + c < 1$. $\square$

4. Steady state distribution

Let $C(t)$ denote the state of the server at time $t \geq 0$.

$$C(t) = \begin{cases} 0, & \text{the server is idle,} \\ 1, & \text{the server is serving a customer,} \\ 2, & \text{the server is searching for a customer.} \end{cases}$$

Let $N(t)$ denote the number of customers in the orbit at time $t \geq 0$. Note that the state space of the the process $X(t) = \{C(t), N(t); t \geq 0\}$ is $S = \{0, 1, 2\} \times \mathbb{N}$. The transitions among states are shown in Fig. 1.



If the system is in the steady state $\rho < 1$, then the joint distribution of the server state and queue (orbit) length.

$$\begin{aligned} p_{0,n} &= P\{C(t) = 0, N(t) = n\}, \\ p_{1,n}(x) &= P\{C(t) = 1, \xi(t) < x, N(t) = n\}, \\ p_{2,n} &= P\{C(t) = 2, N(t) = n\}. \end{aligned}$$

The PGFs technique is used here to obtain the steady state solution of the retrial queueing model. To solve the above equations, we define the generating functions for $|z| \leq 1$, as follows:

$$\begin{aligned} P_0(z) &= \sum_{n=0}^{\infty} z^n p_{0,n}, P_1(z) = \sum_{n=0}^{\infty} z^n p_{1,n}, P_1(z,x) \\ &= \sum_{n=0}^{\infty} z^n p_{1,n}(x), P_2(z) = \sum_{n=0}^{\infty} z^n p_{2,n}. \end{aligned}$$

The set of statistical equilibrium equations are obtained as:

$$(\lambda+\alpha) p_{0,0} = \bar{c}\int_0^{\infty} p_{1,0}(x) b(x) dx + \mu p_{2,0},$$

and

$$\begin{aligned} (\lambda+\alpha+n\theta) p_{0,n} &= (1-(1-\delta_{n,0}) c)\int_0^{\infty} p_{1,n}(x) b(x) dx \\ +c\int_0^{\infty} p_{1,n-1}(x) b(x) dx, n &\geq 1, \end{aligned}$$

Where

$$p'_{1,n}(x) = -(\lambda+b(x)) p_{1,n}(x) + \lambda p_{1,n-1}(x), n \geq 0, \tag{2}$$

$$p_{1,n}(0) = \lambda p_{0,n} + (n+1)\theta p_{0,n+1} + \mu p_{2,n+1}, n \geq 0, \tag{3}$$

$$(\lambda+\mu) p_{2,n} = \lambda p_{2,n-1} + \alpha p_{0,n}, n \geq 0. \tag{4}$$

and $p_{i,-1} = 0\,(i=1,2)$. Transforming the above balance equations to generating functions we obtain,

$$(\lambda+\alpha) P_0(z) + \theta z P'_0(z) = (1-c(1-z))\int_0^{\infty} P_1(z,x) b(x) dx + \mu p_{2,0}, \tag{5}$$

$$\frac{\partial P_1(z,x)}{\partial x} = -(\lambda-\lambda z+b(x)) P_1(z,x), \tag{6}$$

$$P_1(z,0) = \lambda P_0(z) + \theta\frac{dP_0(z)}{dz} + \frac{\mu}{z}(P_2(z)-p_{2,0}), \tag{7}$$

So

$$(\lambda-\lambda z+\mu) P_2(z) = \alpha P_0(z).$$

Solving (7) yields,

$$P_1(z,x) = P_1(z,0)(1-B(x)) e^{-\lambda(1-z)x}. \tag{8}$$

Combining (10), (8), (9) and (6) and after some algebra we get:

$$\begin{aligned} & \theta z\left[(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z\right]P_0'(z) \\ = \; & z\left[\lambda\left(1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)\right)+\alpha\left(1-\frac{\mu(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{z(\lambda-\lambda z+\mu)}\right)\right]P_0(z) \\ & +\mu p_{2,0}\left((\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z\right). \end{aligned} \tag{9}$$

Coefficient of $P_0'(z)$ has two zeros $z_1=0$ and $z_2=1$. We chosse an arbitrary point $a\in(0,1)$ and solving (11) for $z\in(0,a]$ then for $z\in[a,1)$.

We consider the function $f(z)=(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z$. We have:

$$\begin{aligned} f(1) &= \widetilde{B}(0)-1=1-1=0, \\ f'(z) &= -\lambda(\bar{c}+cz)\widetilde{B}'(\lambda-\lambda z)+c\widetilde{B}(\lambda-\lambda z)-1, \\ f'(1) &= -\lambda\widetilde{B}'(0)+c\widetilde{B}(0)-1=\rho-1<0, \\ f''(z) &= \lambda^2(\bar{c}+cz)\widetilde{B}''(\lambda-\lambda z)-2\lambda c\widetilde{B}'(\lambda-\lambda z), \\ f''(1) &= \lambda^2\beta_2+2\lambda c\beta_1\geq 0. \end{aligned}$$

Therefore the function $f(z)$ is decreasing on the interval $[0,1]$, $z_2=1$ is the only zero there and for $z\in[a,1)$ we have: $z<(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)\leq 1$.

Beside

$$\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}=\frac{\rho}{1-\rho}<\infty. \tag{10}$$

That is, the function $\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}$ can be defined at the point $z_2=1$ as $\frac{\rho}{1-\rho}$.This means that for $z\in(0,a]$ and for $z\in[a,1)$ we can rewrite equation (11) as:

$$P_0'(z)=\begin{bmatrix} \frac{\lambda}{\theta}\left(\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}\right) \\ +\frac{\alpha}{\theta}\left(\frac{1-\frac{\mu(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{z(\lambda-\lambda z+\mu)}}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}\right)\end{bmatrix}P_0(z)+\mu p_{2,0}\frac{1}{z}. \tag{11}$$

First we find the general solution of the homogeneous equation, which is transformed to

$$\frac{P_0'(z)}{P_0(z)} = \frac{\lambda}{\theta}\left(\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}\right) \tag{12}$$

$$+\quad \frac{\alpha}{\theta}\left(\frac{\frac{\lambda^2}{\lambda+\mu}}{\lambda+\mu-\lambda z}-\frac{\frac{\mu}{\lambda+\mu}}{z}+\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}\frac{\lambda}{\lambda+\mu-\lambda z}\right). \tag{13}$$

we pose

$$s(t_1,t_2) = \exp\left\{\frac{\lambda}{\theta}\int_{t_1}^{t_2}\frac{1-(\bar{c}+cu)\widetilde{B}(\lambda-\lambda u)}{(\bar{c}+cu)\widetilde{B}(\lambda-\lambda u)-u}du\right\}. \tag{14}$$

Solving (14) for $z\in(0,a]$ we get:

$$P_0(z) \;=\; c(z)s(a,z)^{\left(1+\frac{\alpha}{\lambda+\mu-\lambda z}\right)}\left[\left(\frac{a}{z}\right)^{\mu}\left(\frac{\lambda+\mu-\lambda a}{\lambda+\mu-\lambda z}\right)^{\lambda}\right]^{\frac{\alpha}{\theta(\lambda+\mu)}} \tag{15}$$

$$\times\exp\left\{-\alpha\lambda\int_a^z\frac{\ln s(a,u)}{(\lambda+\mu-\lambda u)^2}du\right\}. \tag{16}$$

By substituting this solution into the nonhomogeneous differential equation (13), we can determine the function $c(z)$ as:

$$\begin{aligned} c(z) \;=\; & \frac{\mu p_{2,0}}{\left[a^{\mu}(\lambda+\mu-\lambda a)^{\lambda}\right]^{\frac{\alpha}{\theta(\lambda+\mu)}}}\int_a^z s(a,t)^{-\left(1+\frac{\alpha}{\lambda+\mu-\lambda t}\right)}t^{\left(\frac{\alpha\mu}{\theta(\lambda+\mu)}-1\right)} \\ & \cdot(\lambda+\mu-\lambda t)^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}}\exp\left\{\alpha\lambda\int_a^t\frac{\ln s(a,u)}{(\lambda+\mu-\lambda u)^2}du\right\}dt+P_0(a). \end{aligned}$$

As $z\to 0^+, P_0(0)<\infty$ and $\left(\frac{a}{z}\right)^{\frac{\mu\alpha}{\theta(\lambda+\mu)}}$ diverge. Thus,

$$\begin{aligned} P_0(a) \;=\; & \frac{\mu p_{2,0}}{\left[a^{\mu}(\lambda+\mu-\lambda a)^{\lambda}\right]^{\frac{\alpha}{\theta(\lambda+\mu)}}}\int_0^a s(a,t)^{-\left(1+\frac{\alpha}{\lambda+\mu-\lambda z}\right)}t^{\left(\frac{\alpha\mu}{\theta(\lambda+\mu)}-1\right)} \\ & \cdot(\lambda+\mu-\lambda t)^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}}\exp\left\{\alpha\lambda\int_a^t\frac{\ln s(a,u)}{(\lambda+\mu-\lambda u)^2}du\right\}dt. \end{aligned} \tag{17}$$

On the other hand, solving (14) for $z \in [a, 1)$, and taking limit as $z \to 1^{-}$, with (13) we get:

$$P_0(a) = \left[a^{\mu}(\lambda+\mu-\lambda a)^{\lambda}\right]^{-\frac{\alpha}{\theta(\lambda+\mu)}} \tag{18}$$
$$\left[\begin{array}{c} P_0(1)\, s(a,1)^{-\left(1+\frac{\alpha}{\mu}\right)} \mu^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}} \exp\left\{\alpha\lambda \int_a^1 \frac{\ln s(a,u)}{(\lambda+\mu-\lambda u)^2} du\right\} \\ -\mu p_{2,0} \int_a^1 s(a,t)^{-\left(1+\frac{\alpha}{\lambda+\mu-\lambda t}\right)} t^{\left(\frac{\alpha\mu}{\theta(\lambda+\mu)}-1\right)} (\lambda+\mu-\lambda t)^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}} \\ \times \exp\left\{\alpha\lambda \int_a^t \frac{\ln s(a,u)}{(\lambda+\mu-\lambda u)^2} du\right\} dt \end{array}\right].$$

Equating (17) and (18) we get:

$$P_0(1) = p_{2,0}\mu^{1-\frac{\lambda\alpha}{\theta(\mu+\lambda)}} \int_0^1 s(1,t)^{-\left(1+\frac{\alpha}{\lambda+\mu-\lambda t}\right)} t^{\left(\frac{\alpha\mu}{\theta(\lambda+\mu)}-1\right)} (\lambda+\mu-\lambda t)^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}}$$
$$\exp\left\{\alpha\lambda \int_1^t \frac{\ln s(1,u)}{(\lambda+\mu-\lambda u)^2} du\right\} dt, \tag{19}$$

then we can rewrite the solution of (13) as:

$$P_0(z) = \mu p_{2,0} \left(\frac{1}{z^{\mu}(\lambda+\mu-\lambda z)^{\lambda}}\right)^{\frac{\alpha}{\theta(\lambda+\mu)}} \int_0^z s(z,t)^{-\left(1+\frac{\alpha}{\lambda+\mu-\lambda t}\right)} t^{\left(\frac{\alpha\mu}{\theta(\lambda+\mu)}-1\right)}$$
$$\cdot (\lambda+\mu-\lambda t)^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}} \exp\left\{\alpha\lambda \int_z^t \frac{\ln s(1,u)}{(\lambda+\mu-\lambda u)^2} du\right\} dt. \tag{20}$$

Combining(9), (8) and (10), we get

$$P_1(z,x) = \left[\lambda\left(1+\frac{\alpha}{\lambda+\mu-\lambda z}\right)\left(1+\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}\right) P_0(z) + \frac{\mu p_{2,0}}{z}(\theta-1)\right]$$
$$\cdot (1-B(x))\, e^{-\lambda(1-z)x}.$$

Since $P_1(z) = \int_0^{\infty} P_1(z,x)\, dx$, we obtain

$$P_1(z) = \frac{1-\widetilde{B}(\lambda-\lambda z)}{1-z} \left[\left(1+\frac{\alpha}{\lambda+\mu-\lambda z}\right)\left(1+\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}\right) P_0(z) \right.$$
$$\left. + \frac{\mu p_{2,0}}{\lambda z}(\theta-1)\right]. \tag{21}$$

Now applying the normalizing condition $P_0(1) + P_1(1) + P_2(1) = 1$, and we taken (21) with $\lim\limits_{z\to 1} \frac{1-\widetilde{B}(\lambda-\lambda z)}{1-z} = \lambda\beta_1$ we get:

$$P_0(1) = \frac{1-\beta_1(\theta-1)\mu p_{2,0}}{\left(1+\frac{\alpha}{\mu}\right)\left(1+\frac{\lambda\beta_1}{1-\rho}\right)}. \tag{22}$$

Using (22) and (19), we obtain the expression for $p_{2,0}$ as:

$$p_{2,0}=\left[\left(1+\frac{\alpha}{\mu}\right)\left(1+\frac{\lambda\beta_1}{1-\rho}\right)\mu^{1-\frac{\lambda\alpha}{\theta(\mu+\lambda)}}\int_0^1 s\left(1,t\right)^{-\left(1+\frac{\alpha}{\lambda+\mu-\lambda t}\right)}t^{\left(\frac{\alpha\mu}{\theta(\lambda+\mu)}-1\right)}\right.$$
$$\left.\cdot\left(\lambda+\mu-\lambda t\right)^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}}\exp\left\{\alpha\lambda\int_1^t\frac{\ln s\left(1,u\right)}{\left(\lambda+\mu-\lambda u\right)^2}du\right\}dt+\beta_1\left(\theta-1\right)\mu\right]^{-1}. \tag{23}$$

Now we summarize the above results in following theorem.

Theorem 4.1. *Under the stationary condition $\rho=\lambda\beta_1+c<1$, the generating functions of the stationary joint distribution of the orbit size and the server state are given by:*

$$P_0\left(z\right)=\mu p_{2,0}\left(\frac{1}{z^{\mu}\left(\lambda+\mu-\lambda z\right)^{\lambda}}\right)^{\frac{\alpha}{\theta(\lambda+\mu)}}\times\Theta(t)$$

Here

$$\begin{aligned}\Theta(t) &= \int_0^z s\left(z,t\right)^{-\left(1+\frac{\alpha}{\lambda+\mu-\lambda t}\right)}t^{\left(\frac{\alpha\mu}{\theta(\lambda+\mu)}-1\right)}\left(\lambda+\mu-\lambda t\right)^{\frac{\lambda\alpha}{\theta(\mu+\lambda)}}\\ &\quad\times\exp\left\{\alpha\lambda\int_z^t\frac{\ln s\left(1,u\right)}{\left(\lambda+\mu-\lambda u\right)^2}du\right\}dt.\end{aligned}$$

where $p_{2,0}$ is given in equation (23).

Theorem 4.2.

$$P_1\left(z\right)=\tfrac{1-\widetilde{B}(\lambda-\lambda z)}{1-z}\left[\begin{array}{c}\left(1+\frac{\alpha}{\lambda+\mu-\lambda z}\right)\left(1+\frac{1-(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)}{(\bar{c}+cz)\widetilde{B}(\lambda-\lambda z)-z}\right)P_0\left(z\right)\\+\frac{\mu p_{2,0}}{\lambda z}\left(\theta-1\right)\end{array}\right]$$

$$P_2\left(z\right)=\frac{\alpha}{\left(\lambda-\lambda z+\mu\right)}P_0\left(z\right).$$

Corollary 4.1. *Under the stability condition $\rho<1$,*
(i) The generating function of the orbit size, $P(z)$, is given by

$$P(z)=P_0(z)+P_1(z)+P_2(z).$$

(ii) The generating function of the system size, $\phi\left(z\right)$, is given by

$$\phi\left(z\right)=P_0(z)+zP_1(z)+P_2(z)=P(z)+\left(z-1\right)P_1(z).$$

From above results, we can get some performance measures of the system in steady state.

Corollary 4.2.
(1) The probability that the server is idle but the system is not empty, denote by P_0, is given by

$$P_0 = P_0(1) = \frac{1-\beta_1\left(\theta-1\right)\mu p_{2,0}}{\left(1+\frac{\alpha}{\mu}\right)\left(1+\frac{\lambda\beta_1}{1-\rho}\right)}.$$

(2) The probability that the server is busy, denote by P_1, is given by

$$P_1 = P_1(1) = \lambda\beta_1\left[\left(1+\frac{\alpha}{\mu}\right)\left(1+\frac{\rho}{1-\rho}\right)P_0 + \frac{\mu p_{2,0}}{\lambda}\left(\theta-1\right)\right].$$

(3) The probability that the server is searching for a customer, denote by P_2, is given by

$$P_2 = P_2(1) = \frac{\alpha}{\mu}P_0.$$

(4) The mean orbit size, L_q, is given by

$$L_q = P^{'}(1).$$

(5) The mean system size, L_s, is given by

$$L_s = \phi^{'}\left(1\right) = L_q + P_1(1).$$

(6) The average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) are found by using the Little's formula

$$W_s = \frac{L_s}{\lambda} \text{ and } W_q = \frac{L_q}{\lambda}.$$

5. Numerical Results

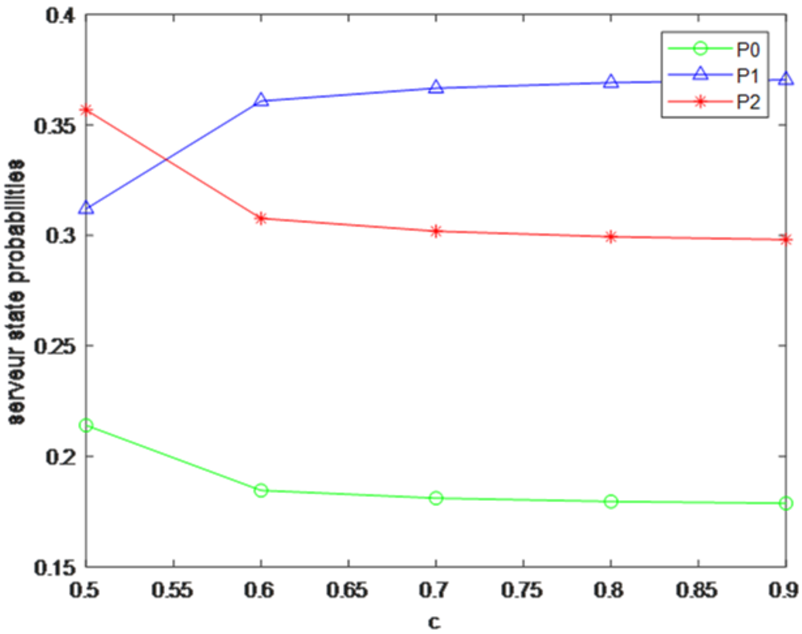
Based on the results obtained in previous sections, in this section we shall present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system. The arbitrary values to the parameters are so chosen such that they satisfy the stability condition. We assume that the service time follow the exponential distribution.

A. Effect of c on the server state probabilities:

We assume that: $\lambda = 0.06, \theta = 1.25, \alpha = 2.5, \mu = 1.5, \beta_1 = 1.$

TABLE 1.

c	P_0	P_1	P_2
0.5	0.2141	0.3120	0.3568
0.6	0.1846	0.3608	0.3077
0.7	0.1811	0.3666	0.3019
0.8	0.1796	0.3691	0.2994
0.9	0.1788	0.3704	0.2981



For Table 1 and Fig. 2, we can see that the probabilities that the server is idle P_0, in search of customers P_2 decrease monotonously. The probability P_1 that the server is busy increase, which agree with our expectations.

B. Effect of μ on performance measures:

We assume that: $\lambda = 6, \theta = 1, \alpha = 2, \mu = 1.5, \beta_1 = \frac{1}{30}, .c = 0.5$.

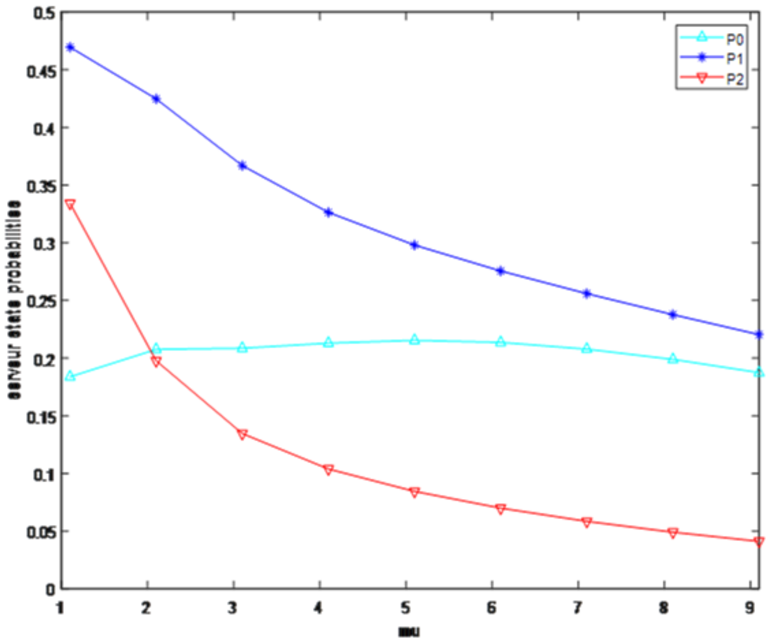
From Table 2 and Fig. 3, we can observe that as the orbit search rate μ increases, the probability that the server is idle increases monotonously, when the probabilities that de server is busy and searching for a customer are decreases.

C. Effect of α on the server state probabilities

We assume that: $\lambda = 5, \theta = 2, \mu = 2, c = 0.2, \beta_1 = \frac{1}{7}$.

TABLE 2.

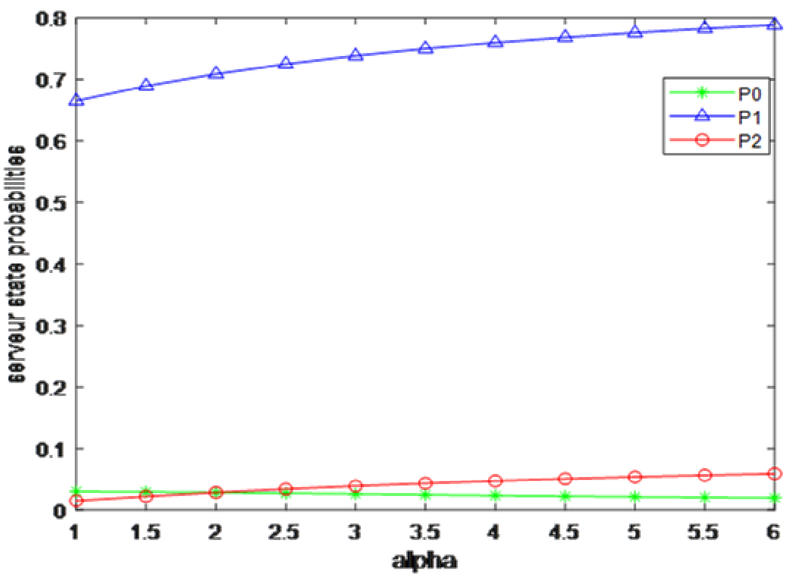
μ	P_0	P_1	P_2
0.1	0.0285	0.4179	0.5704
1.1	0.1838	0.4695	0.3342
2.1	0.2076	0.4247	0.1977
3.1	0.2086	0.3669	0.1346
4.1	0.2130	0.3263	0.1039
5.1	0.2154	0.2980	0.0845
6.1	0.2136	0.2755	0.0700
7.1	0.2078	0.2559	0.0585
8.1	0.1989	0.2377	0.0491
9.1	0.1874	0.2204	0.0412



According to Table 3 and Fig. 4, we can observe that, the probabilities that the server is busy P_1, in search of customers P_2 increase monotonously with the increase of α. Idle time after the completion of a service decrease (exponentially distributed time with mean $(\frac{1}{\alpha})$).

TABLE 3.

α	P_0	P_1	P_2
1	0.0300	0.6651	0.0150
1.5	0.0295	0.6892	0.0221
2	0.0285	0.7087	0.0285
2.5	0.0272	0.7248	0.0341
3	0.0260	0.7383	0.0390
3.5	0.0248	0.7498	0.0434
4	0.0236	0.7597	0.0472
4.5	0.0225	0.7683	0.0507
5	0.0215	0.7759	0.0537
5.5	0.0206	0.7826	0.0565
6	0.0197	0.7885	0.0590



6. CONCLUSION

This paper studies a $M/G/1$ feedback retrial queueing system with search for customers from the orbit. The necessary and sufficient condition for the system to be stable is obtained.The probability generating functions of the number of customers in the system when it is idle, busy, and searching for customers from the orbit is found. The explicit expressions for the average queue length of orbit and system have been obtained. The analytical results are validated with the help of numerical illustrations. Moreover, our model can be considered as a generalized version of many existing queueing models equipped with many features and associated with many practical situations. This model finds practical real life application in health Insurance.

As further future study, we can include the features like priority, active breakdown, delaying repair, etc. We plan to extend this model for cost optimization.

References

[1] N.K. Arrar, N. Djellab, J.B. Baillon: *On the asymptotic behaviour of MX/G/1 retrial queues with bath arrivals and impatience phenomenon*, Mathematical and computer modeling, (2012), 654-665.

[2] N.K. Arrar, N. Djellab, J.B. Baillon: *On stochastic decomposition property of single server retrial queueing systems*, Turkish Journal of Mathematics, **41** (2017), 918-932.

[3] J.R. Artalejo: *Accessible bibliography on retrial queues: progress in 2000-2009*, Math. Comput. Model. **51** (2010), 1071-1081.

[4] J.R. Artalejo, A. Gomez-Corral: *Retrial Queueing Systems: A Computational Approach* Springer, Berlin, 2008.

[5] J.R. Artalejo, V.C. Joshua, A. Krishnamoorthy: *An $M/G/1$ retrial queue with orbital search by server*, Advances in Stochastic Modelling, Notable Publications, New Jersey, (2002), 41-54.

[6] S.R. Chakravarthy, A. Krishnamoorthy, V.C. Joshua: *Analysis of a multi-server retrial queue with search of customers from the orbit*, Perform. Eval. **63** (2006), 776-798.

[7] B.D. Choi, V.G. Kulkarni: *Feedback retrial queueing systems*, Queueing and Related Models, U.N. Bhat and I.V. Basawa, eds., Oxford University Press, New York, (1992), 93-105.

[8] N. Djellab: *On the $M/G/1$ retrial queue with feedback*, Proceedings of the International Conference on Mathematical Methods of Optimization of Telecommunication Networks, (2005), 32-35. (2005).

[9] G.T. Falin, J.G.C. Templeton: *Retrial Queues*, Chapman & Hall, London, (1997).

[10] V. Joshua, A. Krishnamoorthy: *Retrial queues with orbital search*, 2003.

[11] A. Krishnamoorthy, T.G. Deepak, V.C. Joshua: *An $M/G/1$ retrial queue with non persistent customers and orbital search*, Stochast. Anal. Appl. **23** (2005), 975-997.

[12] A. Krishnamoorthy, V.C. Joshua, A.P. Mathew: *A retrial queueing system with abandonment and search for priority customers*, In Springer, Vishnevskiy, V.M., Samouylov, K.E., Kozyrev, D.V. (eds.) DCCN **700** (2017), 98-107.

[13] A. Krishnamoorthy, V.C. Joshua, D. Babu: *A $MAP/PH/1$ retrial queue with search and abandonment*, accepted for ECQT conference, 2018.

[14] C. LAKSHMI A.I. SIVAKUMAR: *Application of queueing theory in health care: a literature review*, Oper. Res. Health Care **2**(1) (2013), 25-39.

[15] M.F. NEUTS, M.F. RAMALHOTO: *A service model in which the server is required to search for customers*, J. Appl. Probab. **21** (1084), 57-166.

[16] D.S. NGUYEN: **Application of queuing theory in service design**, in: 2017 IEEE International Conference on Industrial Engineering and Engineering Management (IEEM), 2017, pp.837-840.

[17] T. PHUNG-DUC: *Retrial Queue for Cloud Systems with Separated Processing and Storage Units.*, Advances in Intelligent Systems and Computing, **383** (2015), 143-151.

[18] P. RAJADURAI, K. INDHIRA, V.M. CHANDRASEKARAN, M.C. SARAVANARAJAN: *Analysis of an $M^x/G/1$ Feedback Retrial Queue with Two Phase Service, Bernoulli Vacation, Delayed Repair and Orbital Search*, Advances in Physics Theories and Applications, **40** (2015), 1-9.

[19] P. RAJADURAI, V.M. CHANDRASEKARAN, M.C. SARAVANARAJAN: *Analysis of an $M^x/G/1$ unreliable retrial G-queue with orbital search and feedback under Bernoulli Vacation schedule*, OPSEARCH 10/2015; **53** (2015), 197-233.

[20] K. SASIKALA, S. INDIRA: *Bulk service queueing models- a survey*, Int. J. Pure Appl. Math. **106**(6) (2016), 43-56.

[21] P. ZEEPHONGSEKUL, A. BEDFORD,. J. BROBERG, P. DIMOPOULOS, Z. TARI: *Queuing Theory Applications to Communication Systems: Control of Traffic Flows and Load Balancing*, Springer, London, (2006), 991-1022.

DEPARTMENT OF MATHEMATICS, BADJI-MOKHTAR UNIVERSITY, ANNABA, ALGERIA.

DEPARTMENT OF MATHEMATICS, SAAD DAHLAB UNIVERSITY, BLIDA 1, ALGERIA.

DEPARTMENT OF MATHEMATICS, BADJI-MOKHTAR UNIVERSITY, ANNABA, ALGERIA.

DEPARTMENT OF MATHEMATICS, MOHAMED KHIDER UNIVERSITY, BISKRA, ALGERIA.

DEPARTMENT OF MATHEMATICS, BADJI-MOKHTAR UNIVERSITY, ANNABA, ALGERIA.