ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **11** (2022), no.9, 759–775
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.11.9.2

ANALYSIS OF VISCOPLASTIC CONTACT PROBLEM

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ABSTRACT. In this paper, we analyze a quasistatic problem modeling frictional contact between a viscoplastic body and an obstacle, the so called foundation. The material constitutive relation is assumed to be non-linear. The boundary conditions of contact and friction are modeled respectively by the *Signorini* conditions and the generalized Coulomb's non-local law. We derive a variational formulation for the problem and prove the existence of its unique weak solution. The proof use, essentially, classical arguments of compactness, variational inequalities and Banach's fixed point theorem.

1. INTRODUCTION

The problems of contact are the results of a vast field in the world of industry. For this reason, in the last decades, there is considerable interest in studying this type of problem. The literature on this subject includes books like [1,2,5,6], also, there exists a large number of papers where the authors are essentially interested on existence and uniqueness results. However, there are works that have dealt with the numerical simulation of contact processes and others in optimal control [9,10].

The mathematical models differ according to the mechanical properties of the considered body and the boundary conditions. For elastic or viscoplastic materials,

2020 *Mathematics Subject Classification.* 74C10, 74m10, 74M15, 49J40, 35J87.
Key words and phrases. Frictionless contact, Viscoplastic material, variational inequality.
Submitted: 26.08.2022; *Accepted:* 10.09.2022; *Published:* 19.09.2022.

we distinguish unilateral or bilateral contact problems with or without friction, an analysis for a general model in the quasistatic case has been established in [11].

Our aim in this work is to study a problem that models the quasistatic contact between a viscoplastic body and a rigid base. The contact is unilateral with friction. A similar problem has been studied in [3] considering unilateral frictionless contact in the case of nonlinear elasticity and in [4] for elastic-viscoelastic materials with intern state variable considering Signiorini boundary conditions.

The version of Coulomb's law of friction which we consider (see [1]) in this paper, depends on the normal stress which presents a major difficulty for the study. Indeed, the variational problem obtained is not of the usual type.

The rest of the paper is structured as follows. In Section 2, we start with the description of the mechanical model, then we list functional preliminaries needed and some assumptions on the data. In Section 3, variational formulation is established by using the Green type formula, the constitutive law and boundary conditions. Finally, in section 4, we give the existence and uniqueness results of the weak problem. The proof is based in the first time on a temporal discretization technique with the goal to eliminate the derivative term, so, we obtain an intermediate problem in which the unknown is the displacement field. Subsequently, we complete the proof using a fixed point method and the theory of elliptic variational inequalities.

2. Problem statement and hypotheses

We consider a viscoplastic body occupies bounded a domain in $\mathbb{R}^d$, $(d = 1, 2, 3)$ denoted Ω. The boundary Γ is Lipschitz continuous boundary divided into three disjoint measurable sets: Γ_1, Γ_2 and Γ_3. Let $T > 0$, we suppose that $meas\Gamma_1 > 0$ and that the field of displacements vanished on $\Gamma_1 \times [0, T]$. We also assume that tractions g act on $\Gamma_2 \times [0, T]$ and that volume forces f act in $\Omega \times [0, T]$.

We study the case where the law of behaviour is viscoplastic of the form:

$$\dot{\sigma} = \xi(\varepsilon(\dot{u})) + G(\sigma, \varepsilon(u)), \tag{2.1}$$

where u is a displacement field, $\sigma = (\sigma_{ij})$ is the stress tensor, $\varepsilon = (\varepsilon_{ij})$ is the linearized deformation operator, ξ and G are constitutive functions.

The displacements and stresses verify the following Signorini conditions:

$$u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0, \tag{2.2}$$

where, u_ν and σ_ν are the normal components of u and σ respectively.

The law of friction considered is the following:

$$\begin{cases} |\sigma_\tau| \leq \mu p(|R\sigma_\nu|) \\ |\sigma_\tau| < \mu p(|R\sigma_\nu|) \Rightarrow \dot{u}_\tau = 0, \quad \text{on } \Gamma_3 \times [0,T] \\ |\sigma_\tau| = \mu p(|R\sigma_\nu|) \Rightarrow \exists \lambda \geq 0 \quad \text{such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases}. \tag{2.3}$$

In (2.3), the operator R is a normal regularizer, i.e. a linear continuous operator. The function p is positive and called: friction function, u_τ and σ_τ are the tangential displacements and stresses respectively; $\mu \geq 0$ is the friction coefficient.

We use here *Coulomb* friction law, which consists to taking the function p such as:

$$p(r) = r(1 - \alpha r)_+,$$

where α is a fairly small positive coefficient related to the hardness of the contact surface and $r_+ = \max\{0, r\}$. This law of friction means that when the regularized normal stress is very large, exceeding $\frac{1}{\alpha}$, the surface disintegrates and no longer offers resistance to movement.

Our problem is formulated as follows:

Problem P: Find a displacement field $u = (u_i) : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\sigma = (\sigma_{ij})\Omega \times [0,T] \to \mathcal{S}_d$ such that:

$$\dot{\sigma} = \xi(\varepsilon(\dot{u})) + G(\sigma, \varepsilon(u)) \quad \text{on} \quad \Omega \times [0,T],$$

$$Div\, \sigma + f = 0 \quad \text{in} \quad \Omega \times [0,T], \tag{2.4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times [0,T], \tag{2.5}$$

$$\sigma\nu = g \quad \text{on} \quad \Gamma_2 \times [0,T], \tag{2.6}$$

$$u_\nu \leq 0, \ \ \sigma_\nu \leq 0, \ \ \sigma_\nu u_\nu = 0,$$

$$\begin{cases} |\sigma_\tau| \le \mu p(|R\sigma_\nu|) \\ |\sigma_\tau| < \mu p(|R\sigma_\nu|) \Rightarrow \dot{u}_\tau = 0 \quad \text{on } \Gamma_3 \times [0,T] \\ |\sigma_\tau| = \mu p(|R\sigma_\nu|) \Rightarrow \exists \lambda \ge 0 \quad \text{such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases},$$

(2.7) $$u(0) = u_0 \quad \text{et} \quad \sigma(0) = \sigma_0 \quad \text{on} \quad \Omega.$$

Note that $\mathcal{S}^d$ is the the space of second order symmetric tensors on $\mathbb{R}^d$, (2.4) is the equation of equilibrium in which Div denotes the divergence operator and (2.7) is the initial condition.

We consider the standard Lebesgue and Sobolev spaces associated to Ω and Γ, moreover we will use the following spaces:

$$\begin{aligned} H =& \{u = (u_{ij}) : u_{ij} \in L^2(\Omega)\}, \quad H_1 = \{u = (u_i) : u_{ij} \in H^1(\Omega)\}, \\ \mathcal{H} =& \{\sigma = (\sigma_{ij}) : \sigma_{ij} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\sigma = (u_{ij}) : \sigma_{ij,j} \in H\}. \end{aligned}$$

These are real Hilbert spaces endowed with the canonical inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i.v_j dx, \quad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}.\tau_{ij} dx,$$
$$\langle u, v\rangle_{\mathcal{H}} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}},$$
$$\langle \sigma, \tau\rangle_{\mathcal{H}} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle Div(\sigma), Div(\tau)\rangle_{\mathcal{H}}$$

and the associated norms $|.|_H$, $|.|_{H_1}$, $|.|_{\mathcal{H}}$, and $|.|_{\mathcal{H}_1}$, respectively. Here, the deformation operator $\varepsilon : H_1 \to \mathcal{H}$ is linear, continuous and defined by

$$\varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i)}).$$

In addition, we consider $H_\Gamma = H^{1/2}(\Gamma)^d$ and $\gamma : H_1 \to H_\Gamma$ is the trace operator.

For the study of problem P, we consider the following hypotheses:

(2.8) $$\begin{cases} \text{The tensor } \xi : \Omega \times S_d \to S_d \text{ is positivly defined, a.e:} \\ \quad \text{a) } \xi_{ij\kappa l} \in L^\infty(\Omega) \text{ for all } i, j, \kappa, l = 1, d; \\ \quad \text{b) } \xi\sigma.\tau = \sigma.\xi\tau \text{ for all } \sigma, \tau \in S_d; \\ \quad \text{c) ther exists } m > 0 \text{ such that } \xi\sigma.\sigma \ge m|\sigma^2| \text{ for all } \sigma \in S_d. \end{cases}$$

$$(2.9)\qquad \begin{cases} \text{the operator } G:\Omega\times S_d\times S_d\to S_d \text{ fulfills:}\\ \quad \text{a) there is } L>0 \text{ such that:}\\ \quad |G(.,\sigma_1,\varepsilon(u_1))-G(.,\sigma_2,\varepsilon(u_2))|\le L(|\sigma_1-\sigma_2|+|u_1-u_2|)\\ \quad \text{for all } \sigma_1,\varepsilon_1,\sigma_2,\varepsilon_2\in S_d,\quad a.e \text{ in } \Omega.\\ \quad \text{b) } G(.,\sigma,\varepsilon) \text{ is a Lebesgue function measurable on } \Omega\\ \quad \text{for all } \sigma,\varepsilon\in S_d.\\ \quad \text{c) } G(.,0,0)\in\mathcal{H}. \end{cases}$$

$$(2.10)\qquad \begin{cases} \text{The friction function } p:\Gamma_3\times\mathbb{R}_+\to\mathbb{R}_+ \text{ obeys:}\\ \text{(a) there exist } M>0 \text{ such that: } |p(x,r_1)-p(x,r_2)|\le M|r_1-r_2|\\ \text{for all } r_1,r_2\in\mathbb{R}_+\quad x\in\Gamma_3.\\ \text{(b) } x\mapsto p(x,r) \text{is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R}_+.\\ \text{(c) } p(x,0)=0\quad a.e \text{ on } \Gamma_3. \end{cases}$$

We also assume that:

$$(2.11)\qquad f\in W^{1,\infty}(0,T,H),$$

and

$$(2.12)\qquad g\in W^{1,\infty}(0,T,L^2(\Gamma_2)^d),$$

while μ satisfies

$$(2.13)\qquad \mu\in L^\infty(\Gamma_3),\qquad \mu(x)\ge 0\qquad \text{on } \Gamma_3.$$

We consider the closed subspace V of H_1 defined by:

$$(2.14)\qquad V=\{u\in H_1 \text{ such that } \gamma\mu=0 \text{ on } \Gamma_1\}.$$

We endow V with the following scalar product:

$$(2.15)\qquad \langle v,w\rangle=\langle\varepsilon(v),\varepsilon(w)\rangle_{\mathcal{H}},$$

since $meas\,\Gamma_1>0$, Korn's inequality holds on V:

$$\exists C>0,\quad |\varepsilon(v)|_{\mathcal{H}}\ge C|v|_H,\quad \forall v\in V.$$

By means of Korn's inequality, we can verify that the norm on V denoted by $|.|_V$ and the standard $|.|_{H_1}$ are equivalent. Then V endowed with the scalar product defined by (2.15) is a real Hilbert space.

We denote by $l : [0,T] \to V$ the element of V defined by:

$$\langle l(t), v\rangle_V = \langle f(t), v\rangle_H + \langle g, \gamma v\rangle_{L^2(\Gamma_2)^d} \quad \forall v \in V. \tag{2.16}$$

From (2.11), (2.12) and (2.16) we deduce that:

$$l \in W^{1,\infty}(0,T,V). \tag{2.17}$$

Let $j : \mathcal{H} \times V \to R$ be the functional defined by:

$$j(\sigma, v) = \int_{\Gamma_3} \mu p(|R\sigma_\nu|)|v_\tau| da. \tag{2.18}$$

We have $R\sigma_\nu$ is an element of $L^2(\Gamma_2)^d$, then from (2.12) and (2.13) we deduce that the integral (2.18) is well defined. According to (2.12) and the continuity of R we obtain:

$$|j(\sigma, v)| \leq C\ |\mu|_{L^\infty(\Gamma_3)}|\ \sigma\ |_{\mathcal{H}_1}|\ v|_V \ \ \forall \sigma \in \mathcal{H}_1, v \in V. \tag{2.19}$$

We also assume that the functional j satisfies the condition

$$\exists \alpha \left[0, \frac{mc}{2}\right] \text{ where } c \text{ is the constant of Korn such that:}$$

$$\begin{aligned} &\int_0^t (j(\sigma_1(s), \dot{u}(s)) - j(\sigma_1(s), \dot{v}(s)) + j(\sigma_2(s), \dot{u}(s)) \\ &\quad - j(\sigma_2(s), \dot{v}(s)))ds \leq \alpha|(\sigma_1 - \sigma_2)(t)||(u-v)(t)|. \end{aligned} \tag{2.20}$$

Finally, we denote by U_{ad} the set of admissible displacements defined by:

$$U_{ad} = \{v \in V/v_\nu \leq 0 \text{ on } \Gamma_3\}. \tag{2.21}$$

3. Variational Formulation

In this section, we are going to establish variational formulation of the mechanical problem P. Let us first give an equivalent form of the boundary conditions (2.3):

Lemma 3.1. *The condition* (2.3) *is equivalent to:*

$$\begin{cases} u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0 \\ |\sigma_\tau| \leq \mu p(|R\sigma_\nu|) \text{ on } \Gamma_3 \times [0,T]\,, \\ \dot{u}_\tau.\sigma_\tau + \mu p(|R\sigma_\nu|)|\dot{u}_\tau| = 0 \end{cases} \tag{3.1}$$

Proof. We assume that $\dot{u}_\tau = 0$ then $\sigma_\tau = -\lambda\dot{u}_\tau$. Then we have: $\dot{u}_\tau.\sigma_\tau + \mu|\dot{u}_\tau|p(|R\sigma_\nu|)|\dot{u}_\tau| = 0$, now suppose that (3.1) holds.

- If: $|\sigma_\tau| = \nu p(|R\sigma_\nu|)$ then:

$$\dot{u}_\tau.\sigma_\tau = |\dot{u}_\tau||\sigma_\tau| \Longrightarrow \exists\lambda > 0,\ \text{such that: } \dot{u}_\tau = -\lambda\sigma_\tau.$$

- If:

$$|\sigma_\tau| < \nu p(|R\sigma_\nu|) \text{ on } \Gamma_3 \times [0,T]\,,$$

so:

$$0 = \dot{u}_\tau.\sigma_\tau + |\dot{u}_\tau\mu|p(|R\sigma_\nu|) \geq -|\dot{u}_\tau||\sigma_\tau| + |\dot{u}_\tau\mu|p(|R\sigma_\nu|),$$

but $|\mu p(|R\sigma_\nu||) - |\sigma_\tau| > 0$. It results that $\dot{u}_\tau = 0$ on $\Gamma_3 \times [0,T]$.

□

To establish the variational formulation we need the following result:

Lemma 3.2. *If the pair of functions* (u,σ) *is a fairly regular solution of the mechanical problem P then:*

$$\begin{cases} u \in U_{ad} \\ \langle\sigma(t),\varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(\sigma(t).v) - j(\sigma(t),\dot{u}(t)) \geq \langle l(t), \\ \nu - \dot{u}(t)\rangle_V + \int_{\Gamma_3}\sigma_\nu(v_\nu - \dot{u}_\nu(t)), \quad \forall v \in V, \quad t \in [0,T] \\ \int_{\Gamma_3}\sigma_\nu(z - u_\nu(t)) \geq 0, \quad \forall z \in U_{ad}, \\ u(0) = u_0. \end{cases} \tag{3.2}$$

Proof. Let $v \in V$, we have by Green's formula:

$$\langle\sigma(t),\varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} = -\langle Div\sigma,\varepsilon(v) - \varepsilon(\dot{u}(t))\rangle + \int_\Gamma \sigma_\nu.(v - \dot{u}(t)).$$

From (2.4), we get:

$$\langle\sigma(t).\varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} = \langle f(t).\varepsilon(v) - \varepsilon(\dot{u}(t))\rangle + \int_\Gamma \sigma_\nu.(v - \dot{u}(t)),$$

and with (2.17), (2.5) and (2.7) it follows

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} = \langle l(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle + \int_{\Gamma} \sigma_\nu.(v - \dot{u}(t)). \tag{3.3}$$

Using the decomposition $\sigma_\nu.v = \sigma_\nu \nu_\nu + \sigma_\tau v_\tau$ in (3.3), we get

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} = \langle l.v - \dot{u}(t) \rangle_{\mathcal{H}} + \int_{\Gamma_3} (\sigma_\nu(v_\nu - \dot{u}(t)_\nu) + \sigma_\tau(v_\tau - \dot{u}(t)_T))da.$$

By adding $\int_{\Gamma_3} \mu p(|R\sigma_\nu|)(|v_\tau| - |\dot{u}_\tau|)da$ in each member of the last equality we obtain:

$$\begin{cases} \langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} + \int_{\Gamma_3} \mu p(|R\sigma_\nu|)(|v_\tau| - |\dot{u}_\tau|)da - \langle l.v - \dot{u}(t) \rangle_{\mathcal{H}} \\ = \int_{\Gamma_3} (\sigma_\nu(v_\nu - \dot{u}(t))) + (v_\tau.\sigma_\tau + \mu |v_\tau| p(|R\sigma_\nu|)) - (\dot{u}_\tau.\sigma_\tau + \mu |\dot{u}_\tau| p(|R\sigma_\nu|))da, \end{cases}$$

since: $v_\tau.\sigma_\tau \geq |v_\tau||\sigma_\tau| \geq -\mu |v_\tau| p(|R\sigma_\nu|)$, which implies

$$\sigma_\tau.v_\tau + \mu |v_\tau| p(|R\sigma_\nu|) \geq 0.$$

Since then $u \in U_{ad}$, from (2.18), (2.7) and (2.2), we deduce:

$$\langle \sigma(t), \varepsilon_v \rangle_{\mathcal{H}} + j(\sigma, \varepsilon) - j(\sigma, \dot{u}(t)) \geq \langle l(t), v - \dot{u}(t) \rangle_V + \int_{\Gamma_3} \sigma_\nu(v_\nu - \dot{u}_\nu(t)); \forall v \in V, \tag{3.4}$$

$$\int_{\Gamma_3} \sigma_\nu(z - u_\nu(t)) \geq 0 \ \forall (z \in U_{ad}). \tag{3.5}$$

□

We can now give the variational formulation of the mechanical problem P:

Problem P$_v$: find the displacement field $u : \Omega \times [0, T] \to R^d$ and the stress field $\sigma : \Omega \times [0, T] \to \mathcal{H}$ such that:

$$\dot{\sigma} = \xi(\varepsilon(\dot{u})) + G(\sigma, \varepsilon(u)),$$

$$\begin{cases} u \in U_{ad}, \\ \langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} + j(\sigma(t), v) - j(\sigma(t), \dot{u}(t)), \\ \geq \langle l(t), v - \dot{u}(t) \rangle_V + \int_{\Gamma_3} \sigma_\nu(v_\nu - \dot{u}_\nu(t)) \quad \forall v \in V \quad t \in [0, T], \\ \int_{\Gamma_3} \sigma_\nu(v_\nu - u_\nu(t)) \geq 0 \quad \forall v \in U_{ad}, \\ u(0) = u_0. \end{cases} \tag{3.6}$$

Remark 3.1. *The variational problem obtained above is formally equivalent to the mechanic problem. Indeed, if we suppose that (u, σ) is a regular solution of the problem P, we take $v = (\dot{u} \pm \varphi) \in U_{ad}$, $\forall \varphi \in \mathcal{D}(\Omega)^d$ in* (3.6)*, we get:*

$$\langle \sigma, \varepsilon(\varphi) \rangle_{\mathcal{H}} = \langle l, \varphi \rangle_V ,$$

and from the inner product properties in H_1 (see the formula (2.5) and (2.6) in [3])

$$Div\, \sigma + f = 0 \;\; in\; H.$$

Now using (2.16)*,* (2.17)*,* (3.6) *and (2.5) in [3], we get:*

$$(3.7) \qquad \begin{cases} \langle \sigma\nu, \gamma(v - \dot{u}) \rangle_{H'_\Gamma \times H_V} + \int_{\Gamma_3} \mu p(|R\sigma_\nu|)(|v_\tau| - |\dot{u}_\tau|) da \geq \\ \langle g, \gamma(v - \dot{u}) \rangle + \int_{\Gamma_3} \sigma_\nu (v_\nu - \dot{u}_\nu(t)), \forall v \in V, \end{cases}$$

for $v = 2\dot{u}$ then $v = 0$, we find

$$\langle \sigma\nu, \gamma(\nu - \dot{u}) \rangle_{H'_\Gamma x H_V} + \int_{\Gamma_3} \mu p(|R\sigma_\nu|)(|\dot{u}_\tau|) da = \langle g, \gamma \dot{u} \rangle + \int_{\Gamma_3} \sigma_\nu (\dot{u}_\nu(t)).$$

Now taking $v = (\pm w) \in U_{ad}$, for all $w \in H_1$ such that $w = 0$ on $\Gamma_1 \cup \Gamma_3$ in (3.7) *we deduce that $\sigma\nu = g$ on Γ_2. For $v = w$, $\forall w \in H_1$ such that: $w = 0$ on $\Gamma_1 \cup \Gamma_2$ and $w_\tau = 0, w_\nu \leq 0$ on Γ_3, given* (3.7)*, it follows that $\sigma\nu \leq 0$ on Γ_3.*

From $\int_{\Gamma_3} \sigma_\nu (\nu_\nu - u_\nu(t)) \geq 0$, for $v \in U_{ad}$ such that: $v_\nu = 2u_\nu$ then $v = u_\nu$, it results that: $\sigma_\nu u_\nu = 0$ on Γ_3, because $\sigma_\nu = g$ on Γ_2, $\sigma_\nu \leq 0$ on Γ_3 and $u \in U_{ad}$. We now set $v_\nu = \dot{u}_\nu$ on Γ_3, given (3.7) *and the equality $\sigma_\nu u_\nu = 0$ on Γ_3, we deduce*

$$\int_{\Gamma_3} (v_\tau.\sigma_\tau + \mu |v_\tau| p |R\sigma_\nu|) da - \int_{\Gamma_3} (\dot{u}_\tau.\sigma_\tau + \mu |\dot{u}_\tau| p |R\sigma_\nu|) da \geq 0,$$

then we put $v_\tau = 2\dot{u}_\tau$ and $v_\tau = 0$ in the last inequality, we get

$$\int_{\Gamma_3} (\dot{u}_\tau \sigma_\tau + \mu |\dot{u}_\tau| p(|R\sigma_\nu|)) da = 0,$$

so

$$\int_{\Gamma_3} (v_\tau \sigma_\tau + \mu |\dot{u}_\tau| p(|R\sigma_\nu|)) da \geq 0,$$

if we choose $v_\tau = -\sigma_\tau$ we end up with:

$$-|\sigma_\tau|^2 + \mu \, |\sigma_\tau| \, p(|R\sigma_\nu|) da \geq 0 \;\; sur\; \Gamma_3,$$

in other words $|\sigma_\tau| \leq p(|R\sigma_\nu|)da$ *and* $-\dot{u}_\tau \sigma_\tau \leq |\dot{u}_\tau||\sigma_\tau$*, so*

$$-\dot{u}_\tau \sigma_\tau + \mu\,|\dot{u}_\tau|\,p(|R\sigma_\nu|) \geq 0 \text{ on } \Gamma_3,$$

we conclude

$$-\dot{u}_\tau \sigma_\tau + \mu\,|\dot{u}_\tau|\,p(|R\sigma_\nu|) = 0 \text{ on } \Gamma_3.$$

Thus we have found the boundary condition (2.21) *of the lemma 3.1 which is equivalent to that of the mechanical problem.*

Conversely, if $\nu, u \in U_{ad}$ *are such that*

$$\int_{\Gamma_3} \sigma_\nu(\nu_\nu - u_\nu(t)) \geq, \quad \forall \nu \in U_{ad},$$

then, using the definition of $\dot{u}$*,* $\forall t \in [0,T]\,\forall t \in [0,T]$ *and* $\forall \Delta t > 0$*, we obtain*

$$\int_{\Gamma_3} \sigma_\nu \dot{u}_\nu(t) = \lim_{\Delta t \to 0} \int_{\Gamma_3} \sigma_\nu \frac{u_\nu(t+\Delta t) - u_\nu(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\Gamma_3} \sigma_\nu u_\nu(t+\Delta t) - u_\nu(t),$$

we have $u_\nu(t+\Delta t) \in U_{ad}$*, so, from* (3.4)*, it results* $\int_{\Gamma_3} \sigma_\nu \dot{u}_\nu \geq 0$*. Similarly for* $\lim_{\Delta t \to 0} \int_{\Gamma_3} \dfrac{u_\nu(t-\Delta t) - u_\nu(t)}{-\Delta t}$*, we find* $\int_{\Gamma_3} \sigma_\nu \nu_\nu \geq 0$*,* (3.4) *becomes*

$$u \in U_{ad},\ \langle \sigma, \varepsilon_\nu - \varepsilon_{\dot{u}} \rangle_{\mathcal{H}} + j(\sigma, \nu) - j(\sigma, \dot{u}) \geq \langle f, \nu - \dot{u} \rangle_V \quad \nu \in U_{ad}.$$

4. Existence and uniqueness result

In this section, we give the existence and uniqueness results for the variational problem P_v.

Theorem 4.1. *Assumptions that* (2.9)-(2.14) *hold Then, there exists a constant* $\alpha_0 > 0$ *depending on* Ω, Γ, G, ξ *and* p *such that: if* $\alpha < \alpha_0$ *then* P_v*; has a unique solution having the regularity* $u \in W^{1,\infty}(0,T,V)$ *and* $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$.

The demonstration is carried out in several steps:

It is clear that P_v, is not a problem of the usual type, so we cannot directly apply the theorems of existence and uniqueness of elliptic variational inequalities, because the functional j depends on the term $p(|R\sigma_\nu,|)$. For this reason, we assume that the regularity of the stress on the boundary Γ_3 is given, we denote it by g. For all $\eta \in L^\infty(0,T,\mathcal{H})$, we assume that the inelastic part of the stress is given and denoted by z_η:

$$z_\eta(t) = \int_0^t \eta(s)ds + z_0 \quad \forall t \in [0,T]\,, \tag{4.1}$$

with

$$z_0 = \sigma_0 - \xi\varepsilon(u(0)). \tag{4.2}$$

Thus we obtain an intermediate problem $P_v^{g\eta}$. Once we prove the existence and uniqueness of $P_v^{g\eta}$, we use the fixed point method for the application defined of $\mathcal{H}_1$ in $\mathcal{H}_1$ by $g \to \sigma_{g\eta}$, then for application defined from: $L^\infty(0,T,\mathcal{H}) \to L^\infty(0,T,\mathcal{H})$ by:

$$\eta(t) \to G(\varepsilon(u(t)), \sigma(t)).$$

Remark 4.1. *For the problem $P_v^{g\eta}$, we have the same equations verified by $u_{g\eta}$, as well as the boundary conditions except on Γ_3, where we will have:*

$$\begin{cases} u_\nu^{g\eta} \leq 0\,,\ \sigma_\nu^{g\eta} u_\nu^{g\eta} = 0,\ \sigma_\nu^{g\eta} \leq 0 \quad |\sigma_\tau^{g\eta}| \leq \mu p(|Rg_\nu|) \\ |\sigma_\tau^{g\eta}| = \mu p(|Rg_\nu|) \Longrightarrow \exists \lambda > 0, \text{ such that } \sigma_\tau^{g\eta} = -\lambda u_\tau^{g\eta} \quad \text{ on } \Gamma_3\,. \\ |\sigma_\tau^{g\eta}| < \mu p(|Rg_\nu|) \Longrightarrow u_\tau^{g\eta} = 0 \quad . \end{cases}$$

Let the intermediate problem:

Problem $P_v^{g\eta}$: Finding the field of displacements $u_{g\eta} : \Omega \times [0,T] \to \mathbb{R}^n$ and the stress field $\sigma_{g\eta} : \Omega \times [0,T] \to \mathcal{H}$ such that:

$$\sigma^{g\eta}(t) = \xi\varepsilon(u^{g\eta}(t)) + z_\eta(t), \tag{4.3}$$

$$\begin{cases} u^{g\eta} \in U_{ad}, \\ \langle \xi\varepsilon(u^{g\eta}) + z_\eta, \varepsilon(v) - \varepsilon(\dot{u}^{g\eta}(t)) \rangle_{\mathcal{H}} + j(g,v) - j(g,\dot{u}^{g\eta}(t)) \geq \\ \langle l(t), v - \dot{u}^{g\eta}(t) \rangle_V + \int_{\Gamma_3} \sigma_\nu^{g\eta}(v_\nu - \dot{u}_\nu^{g\eta}(t)), \quad \forall v \in V, \\ \int_{\Gamma_3} \sigma_\nu^{g\eta}(v_\nu - \dot{u}_\nu^{g\eta}(t)) \geq 0, \quad \forall v \in U_{ad}, \quad \forall t \in [0,T] \\ u^{g\eta}(0) = u_0^{g\eta}. \end{cases} \tag{4.4}$$

Lemma 4.1. *For all $g \in \mathcal{H}_1$, the problem $P_v^{g\eta}$ has a unique solution $(u_{g\eta}, \sigma_{g\eta})$ such that: $u_{g\eta} \in W^{1,\infty}(0,T,V)$ and $\sigma_{g\eta} \in W^{1,\infty}(0,T,\mathcal{H}_1)$.*

The proof of this lemma is obtained by using a technique of temporal discretization because of the appearance of the term $\dot{u}(t)$. This allows us to consider only a system equation which will be written only in displacement " type ".

Proof. Let $N \in \mathbb{N}^*$, $h = T/N$, $t_n = nh$ and $f_n = f(t_n)$, $\forall n = 0, N$. Consider the following problem $P^{g\eta}_{v,n}$:

$$\begin{cases} u \in U_{ad}, \\ \left\langle \xi\varepsilon(u^{g\eta}_{n+1}(t)) + z_n, \varepsilon(v) - \varepsilon((u^{g\eta}_{n+1} - u^{g\eta}_n)/h) \right\rangle_{\mathcal{H}} + j(g, v) \\ j(g, ((u^{g\eta}_{n+1} - u^{g\eta}_n)/h)) \geq \left\langle l(t), v - ((u^{g\eta}_{n+1} - u^{g\eta}_n)/h) \right\rangle_V, \quad \forall v \in U_{ad}. \end{cases} \tag{4.5}$$

We denote by u^g_N the function defined by, $u^g_N : [0, T] \to V$ such that:

$$u^g_N = \frac{(t - t_n)}{h}(u^{g\eta}_{n+1} - u^{g\eta}_n) + u^{g\eta}_n. \tag{4.6}$$

It follows from the theory of variational inequalities that (u^g_N) is a bounded sequence in the space $W^{1,\infty}(0, T, V)$, whence, by classical compactness arguments there exists $(u^g_k)_{k\in N}$ subsequence of (u^g_N) convergent in $W^{1,\infty}(0, T, V)$ i.e.:

$$u^{g\eta}_k \to u^{g\eta} \quad \text{in } L^\infty(0, T, V) \quad \text{weak}, \tag{4.7}$$

$$u^{g\eta}_k \to \dot{u}^{g\eta} \quad \text{in } L^\infty(0, T, V) \ \text{weak}. \tag{4.8}$$

Using (4.5), (4.6) and (4.7) it follows that $u^{g\eta}$ satisfies (4.4). So, for every $t \in [0, T]$, there is a unique pair $(u^{g\eta}(t), \sigma^{g\eta}(t)) \in V \times \mathcal{H}$ solution of (4.3) and (4.4). Putting now $r = \dot{u}_g \pm \varphi$ in (4.4), it comes that

$$\langle \sigma^{g\eta}(t), \varepsilon(\varphi)\rangle_{\mathcal{H}} = \langle l, \varphi\rangle_{\mathcal{H}} \quad \forall \varphi \in \mathcal{D}(\Omega)^d,$$

and from (2.17) we deduce that

$$Div\, \sigma^{g\eta} + f = 0 \quad \text{in} \quad \Omega \times [0, T]\,.$$

So from (2.12) and the equality above it follows that

$$\sigma^{g\eta}(t) \in \mathcal{H}_1. \tag{4.9}$$

Let now $t_1, t_2 \in [0, T]$ (for the sake of simplicity), we note $\sigma_{g\eta}(t_i) = \sigma^i_{g\eta}$, $u_{g\eta}(t_i) = u^i_{g\eta}$ and $z(t_i) = z_i$, then from (4.3) and (2.9), we obtain

$$|\sigma^1_{g\eta} - \sigma^2_{g\eta}|_{\mathcal{H}} \leq C(|u^1_{g\eta} - u^2_{g\eta}|_V + |z_1 - z_2|_{\mathcal{H}}). \tag{4.10}$$

Finally from (4.10), (4.8),(4.7) and (4.1) we deduce that $\sigma^{g\eta} \in W^{1,\infty}(0,T,\mathcal{H}_1)$. □

Remark 4.2. *The expression of the sequence* (u_N^g) *is inspired by the scheme of divided differences to approach the derivative.*

From the lemma above and (4.9), we conclude that, we can consider the operator

$$\Lambda_\eta : L^\infty(0,T,\mathcal{H}_1) \to L^\infty(0,T,\mathcal{H}_1),$$

defined by

$$\Lambda_\eta g = \sigma_{g\eta} \quad \forall g \in L^\infty(0,T,\mathcal{H}_1). \tag{4.11}$$

Now, we will prove that Λ_η has a fixed point.

Lemma 4.2. *There exists a constant* $\alpha_1 > 0$ *depending on* Ω, Γ, ξ *and* p *such that: if* $\alpha < \alpha_1$ *then the operator* Λ_η *has a single fixed point* g^*.

Proof. Let $g_1, g_2 \in L^\infty(0,T,\mathcal{H}_1)$, $(u_i^\eta, {\sigma_i}^{eta})$ the solution of the problem $P_v^{g\eta}, i = 1, 2$..

In (4.4), taking $v = \dot{u}_1^\eta$ in $P_v^{g_2\eta}$ then $v = \dot{u}_2^\eta$ in $P_v^{g_2\eta}$, we get

$$\begin{aligned} &\langle \xi\varepsilon(u_1^\eta) - \xi\varepsilon(u_2^\eta), \varepsilon(\dot{u}_1^\eta(t)) - \varepsilon(\dot{u}_2^\eta(t))\rangle_{\mathcal{H}} \leq j(g_1, \dot{u}_2^\eta(t)) - \\ &j(g_1, \dot{u}_1^\eta(t)) + j(g_2, \dot{u}_1^\eta(t)) - j(g_2, \dot{u}_2^\eta(t)). \end{aligned} \tag{4.12}$$

Also, we have:

$$\frac{d}{dt}\langle \xi\varepsilon(u_1^\eta - u_2^\eta), \varepsilon(u_1^\eta - u_2^\eta)\rangle = \langle \xi\varepsilon(u_1^\eta - u_2^\eta), \varepsilon(\dot{u}_1^\eta - \dot{u}_2^\eta)\rangle + \langle \xi\varepsilon(\dot{u}_1^\eta - \dot{u}_2^\eta), \varepsilon(u_1^\eta - u_2^\eta)\rangle,$$

from (2.9) it results:

$$\langle \xi\varepsilon(u_1^\eta - u_2^\eta), \varepsilon(\dot{u}_1^\eta - \dot{u}_2^\eta)\rangle = \frac{1}{2}\frac{d}{dt}\langle \xi\varepsilon(u_1^\eta - u_2^\eta), \varepsilon(u_1^\eta - u_2^\eta)\rangle.$$

So, (4.12) becomes:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\langle \xi\varepsilon(u_1^\eta - u_2^\eta)(t), \varepsilon(u_1^\eta - u_2^\eta)(t)\rangle \\ &\leq j(g_1, \dot{u}_2^\eta(t)) - j(g_1, \dot{u}_1^\eta(t)) + j(g_2, \dot{u}_1^\eta(t)) - j(g_2, \dot{u}_2^\eta(t)), \end{aligned}$$

and by integrating the last inequality with respect from 0 to t, we obtain:

$$\begin{cases} \langle \xi\varepsilon(u_1^\eta - u_2^\eta)(t), \varepsilon(u_1^\eta - u_2^\eta)(t)\rangle - \langle \xi\varepsilon(u_1^\eta - u_2^\eta)(0), \varepsilon(u_1^\eta - u_2^\eta)(0)\rangle \leq \\ \int_0^t j(g_1, \dot{u}_2^\eta(s)) - j(g_1, \dot{u}_1^\eta(s)) + j(g_2, \dot{u}_1^\eta(s)) - j(g_2, \dot{u}_2^\eta(s))ds. \end{cases}$$

Now from (2.9), (2.20), Korn's inequality and the continuity of the tensor ε, we get:

$$\frac{mc}{2}\,|(u_1^\eta - u_2^\eta)(t)|^2_{W^{1,\infty}(0,T,V)} \leq \alpha|g_1 - g_2|_{W^{1,\infty}(0,T,\mathcal{H})}|(u_1^\eta - u_2^\eta)(t)|_{W^{1,\infty}(0,T,V)} + C\,|(u_1^\eta - u_2^\eta)(0)|^2_{W^{1,\infty}(0,T,V)}, \tag{4.13}$$

knowing that

$$ab \leq \frac{a^2}{2b}\frac{\delta b^2}{2} \quad \delta \neq 0, \tag{4.14}$$

then (4.13) becomes

$$\frac{mc}{2}\,|(u_1^\eta - u_2^\eta)(t)|^2_{W^{1,\infty}(0,T,V)} \leq \frac{\alpha}{2b}|g_1 - g_2|_{W^{1,\infty}(0,T,\mathcal{H})} + (\frac{\alpha\delta}{2} + C)|(u_1^\eta - u_2^\eta)(t)|_{W^{1,\infty}(0,T,V)},$$

so, for $\delta \in [0, (mc - 2C)/\alpha]$ and $C < mc/2$, from (4.10), (4.11) and from Banach's fixed point theorem, it follows that Λ_η has a unique point g^*. □

In the following, we assume that $\alpha < \alpha_1$. For $\eta \in L^\infty(0,T,\mathcal{H})$ we denote by g^* the fixed point of the operator Λ_η given by lemma 4.2 and let $u_\eta \in W^{1,\infty}(0,T,V)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$ the functions defined by

$$u_\eta = u_{\eta g^*}, \quad \sigma_\eta = \sigma_{\eta g^*}. \tag{4.15}$$

For all $t \in [0,T]$, we define the operator $\Lambda : L^\infty(0,T,\mathcal{H}) \to L^\infty(0,T,\mathcal{H})$ by

$$\Lambda\eta(t) = G(\sigma(t), \varepsilon(u(t))). \tag{4.16}$$

Let us show that Λ has a unique fixed point:

Lemma 4.3. *there exists a constant $\alpha_0 > 0$ depending on Ω, Γ and p such that: if $\alpha < \alpha_0$, then the operator Λ admits a unique fixed point $\eta^* \in L^\infty(0,T,\mathcal{H})$.*

Proof. Let $\eta_1, \eta_2 \in L^\infty(0,T,\mathcal{H})$, $u_i = u_{\eta_i}$, $\sigma_i = \sigma_{\eta_i}$, $g_i = g_{\eta_i}$, for $i = 1.2$. Using the lemma 4.2 we have $g_i = \sigma_i$, from (4.4) and (4.3), we have:

$$\sigma_i = \xi\varepsilon(u_i) + \eta_i. \tag{4.17}$$

$$\begin{cases} u_i \in U_{ad} \quad \langle \sigma_i, \varepsilon(v) - \varepsilon(\dot{u}_i(t))\rangle_{\mathcal{H}} + j(\sigma_i, v) + \\ j(\sigma_i, \dot{u}_i(t)) \geq \langle l(t), v - \dot{u}_i(t)\rangle_V, \quad \forall v \in U_{ad}. \end{cases} \tag{4.18}$$

For $t \in [0,T]$ and $i = 1,2$. Using (4.17), (4.18) and similar estimates to those used in the proof of the previous lemma (see (4.10),(4.13), and (4.12)) we end up with:

$$|\sigma_1 - \sigma_2|^2_{W^{1,\infty}(0,T,\mathcal{H})} \leq C_1(|u_1 - u_2|^2_{W^{1,\infty}(0,T,V)} + |z_1 - z_2|^2_{W^{1,\infty}(0,T,\mathcal{H})}), \tag{4.19}$$

and

$$|u_1 - u_2|^2_{W^{1,\infty}(0,T,\mathcal{H})} \leq C_2(|\sigma_1 - \sigma_2|^2_{W^{1,\infty}(0,T,V)} + |z_1 - z_2|^2_{W^{1,\infty}(0,T,\mathcal{H})}), \tag{4.20}$$

from (4.19) and (4.20). It results that

$$(1 - C_1C_2)|u_1 - u_2|^2_{W^{1,\infty}(0,T,\mathcal{H})} \leq C_1C_2|z_1 - z_2|^2_{W^{1,\infty}(0,T,\mathcal{H})}, \tag{4.21}$$

using (4.16), (4.1) and (2.10) we get

$$|\Lambda\eta_1 - \Lambda\eta_2|_{W^{1,\infty}(0,T,\mathcal{H})} \leq C\int_0^t |\eta_1(s) - \eta_2(s)|_{W^{1,\infty}(0,T,\mathcal{H})}ds, \tag{4.22}$$

an iteration of order n will give

$$|\Lambda^n\eta_1 - \Lambda^n\eta_2|_{W^{1,\infty}(0,T,\mathcal{H})} \leq \frac{C^nT^n}{n!}|\eta_1 - \eta_2|_{W^{1,\infty}(0,T,\mathcal{H})},$$

which implies that, for n large enough. Λ^n is a contraction. Thus, there is a unique $\eta^* \in L^\infty(0,T,\mathcal{H})$ fixed point for the operator Λ. □

Proof. **Proof of Theorem 4.1.**

Existence.

Let's η^* be the fixed point of the operator Λ and $(u_{\eta^*}, \sigma_{\eta^*})$ is a solution of the problem P_v.

Choosing $\eta = \eta^*$ and $g = g^*$ in (4.4), (4.3) and using (4.15), we get:

$$\sigma_{\eta^*} = \xi\varepsilon(u_{\eta^*}(t)) + z_{\eta^*}, \tag{4.23}$$

$$\begin{aligned} &\langle\sigma_{\eta^*}(t), \varepsilon(v) - \varepsilon(\dot{u}_{\eta^*}(t))\rangle_{\mathcal{H}} + j(g^*_{\eta^*}, v) \\ &- j(g^*_{\eta^*}, \dot{u}_{\eta^*}) \geq \langle l(t), v - \dot{u}_{\eta^*}(t)\rangle_V\,, \quad \forall v \in V, \quad a.e\ t \in [0,T]\,. \end{aligned} \tag{4.24}$$

Then, using (4.23), (4.16) and since $G(\sigma_{\eta^*}, \varepsilon(u_{\eta^*})) = \eta^* = \Lambda\eta^*$ we obtain

$$\dot{\sigma}_{\eta^*} = \xi(\varepsilon(\dot{u}_{\eta^*})) + G(\sigma_{\eta^*}, \varepsilon(u_{\eta^*})),$$

while the inequality (3.6) follows from (4.24), (4.11), (4.15) and from the fact that:

$$g^*_{\eta^*} = \Lambda_{\eta^*}g^*_{\eta^*} = \sigma_{\eta^*}.$$

Finally the initial conditions result from (4.2) and the regularity $u_{\eta*} \in W^{1,\infty}(0,T,H_1)$, $\sigma_{\eta^*} \in W^{1,\infty}(0,T,\mathcal{H}_1)$, is a consequence of lemma 4.1.

Uniqueness:

To prove the uniqueness of the solution, we assume that $(u_{\eta^*}, \sigma_{\eta^*})$, obtained in the existence part is a solution of the variational problem P_v and (u, σ) is another solution of P_v, such that $u \in W^{1,\infty}(0,T,H_1)$ and $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$.
We denote by $\eta \in L^\infty(0,T,\mathcal{H}\times Y)$ the function defined by:

$$\eta(t) = G(\sigma(t), \varepsilon(u(t))), \tag{4.25}$$

and let

$$\dot{z} = G(\sigma(t), \varepsilon(u(t))) \quad \text{and} \quad g = \sigma. \tag{4.26}$$

Then from (4.3) and (4.4) we deduce that (u, σ) is a solution of the problem $P^v_{\eta g}$, and since this problem has a unique solution, we conclude that:

$$u = u_{\eta g}, \quad \sigma = \sigma_{\eta g}. \tag{4.27}$$

Thus, $g = g^*_\eta$, from (4.27), it comes that:

$$u = u_{\eta g^*_\eta}, \quad \sigma = \sigma_{\eta g^*_\eta}, \tag{4.28}$$

so (4.15) and (4.28) imply that

$$u = u_\eta, \quad \sigma = \sigma_\eta. \tag{4.29}$$

Now using (4.16), (4.25) and (4.29), we obtain $\Lambda\eta = \eta$, but Λ admits a unique fixed point, hence:

$$\eta = \eta^*, \tag{4.30}$$

so the uniqueness of the solution follows from (4.29)-(4.30), which ends the proof.

□

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