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EPICONVERGENCE OF A SEQUENCE OF INTEGRAL FUNCTIONALS DEFINED ON $H^1(A)$

Mahmoud Brahimi[1] and Mounia Laouar

ABSTRACT. In this paper, we are interested by the study of asymptotic behaviour of the sequence of integral functionals using the direct method.

1. INTRODUCTION

Recently, intensive works by several authors (mathematicians, physicists...) [4, 6] are devoted to the asymptotic behavior of the sequences of integral functionals coming from mechanics, electrostatics, ... There are two methods to determine the limit integral functional. Either by using epiconvergence techniques, in which case it is necessary to already know the functional candidate to be epilimit of such a sequence. Either by a compactness argument, we show a compactness theorem for a class of studied functional, then from to the additional properties of the studied sequence (periodicity, convexity, ...). We identify the unique adherence value of the sequence.

[1] *corresponding author*
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In this work, we propose to study by epiconvergence technic the asymptotic behavior when $\varepsilon_n \underset{n\to+\infty}{\to} 0$ of the following integral functional sequence

$$F_n(u,\ A) = \begin{cases} \int_A f\left(\frac{u(x)}{\varepsilon_n},\ \nabla u(x)\right) dx, \text{ if } u \in H^1(A)\,; \\ +\infty, \text{ if } u \in L^2(A) - H^1(A)\,. \end{cases}$$

Here A is a bounded open of $\mathbb{R}^d$, and

$$\begin{cases} f : \mathbb{R}^d \times \mathbb{R}^d \to [0,\ +\infty[\,; \\ \quad (x,\ y) \longmapsto f(x,\ y)\,, \end{cases}$$

$x-$measurable, $x-$periodic, $y-$convex and satisfied:

$$\alpha\,|y|^2 \leq f(x,\ y) \leq \beta\left(1+|y|^2\right), \tag{1.1}$$

where $0 < \alpha \leq \beta < +\infty$. We apply the technics used by [2, 3] and [1] we shaw that F_n epiconverges to F^{hom} for the strong topology of $L^2(A)$, where:

$$F^{\text{hom}}(u,\ A) = \begin{cases} \int_A f^{\text{hom}}(\nabla u(x))\, dx, \text{ if } u \in H^1(A)\,; \\ +\infty, \text{ if } u \in L^2(A) - H^1(A)\,, \end{cases}$$

where, for all $a \in \mathbb{R}^d$,

$$f^{\text{hom}}(a) = \min\left\{\int_Y f(at + u(t),\ a + \nabla u(t))\, dt,\ u \in H^1_{per}(Y)\right\}, \tag{1.2}$$

with $Y = [0,\ 1]$ and $H^1_{per}(Y) = \{u \in H^1(Y)\,;\ u\, Y - \text{periodic}\}$.

This paper is organised as follows. In section 2, we present a definitions and preliminaries. In section 3, we are going to study the epiconvergence of a sequence of functional integrals. Finally, a conclusion is drawn in section 4.

2. Definitions and preliminaries [5]

Let be E a Banach space, E^* be dual space, and φ a function defined from E to $\mathbb{R} \cup \{+\infty\}$.

Definition 2.1. *We call under differential of φ at point $x_0 \in E$, the set*

$$\partial\varphi(x_0) = \{f \in E^* : \langle f,\ y - x\rangle + \varphi(x_0) \leq \varphi(y)\}\,.$$

The continuous linear form f is called the under gradient of φ in x_0.

2.1. **Properties.** If $\varphi(x,\ .)$ is convex and bounded, then $\varphi(x,\ .)$ is under differentiable and satisfies:

(1) For $p \in \mathbb{R}^d;\ \varphi(x,\ p) \geq \varphi(x,\ y) + \langle \partial\varphi(x,\ y),\ p - y\rangle$.
(2) $\forall (x,\ y) \in \mathbb{R}^d \times \mathbb{R}^d;\ |\partial\varphi(x,\ y)| \leq M(1 + |y|)$, where M is a constant.

Let be now τ a topology of E and $F_n,\ F_n : E \longrightarrow \mathbb{R} \cup \{+\infty\},\ n \in \mathbb{N}$ a family of functions.

Definition 2.2. *For all $x_0 \in E$, the following two quantities:*

$$epi \liminf F_n(x_0) = \sup_{V \in J_\tau(x_0)} \liminf_{n \to +\infty} \inf_{x \in V} F_n(x);$$

$$epi \limsup F_n(x_0) = \sup_{V \in J_\tau(x_0)} \limsup_{n \to +\infty} \inf_{x \in V} F_n(x),$$

where, $J_\tau(x_0)$ is the set of neighborhoods of x_0, are called respectively the $\tau-$lower epilimity and upper $\tau-$epilimit of the sequence $(F_n)_{n\in\mathbb{N}}$ at point x_0.

- If these two quantities are equal, the sequence F_n called $\tau-$epiconvergent at point x_0, and the common value:

$$epi \liminf F_n(x_0) = epi \limsup F_n(x_0) = F(x_0),$$

is called $\tau-$epilimit of F_n at point x_0.

- If F_n $\tau-$epiconverge at all points $x \in E$, we say that the sequence F_n $\tau-$ epiconverge to the function F defined from E to $\overline{\mathbb{R}}$ and we write:

$$F = \tau - epi \lim F_n.$$

An equivalent definition in the case of normed space will be given by the following theorem.

Theorem 2.1. *[4] the sequence $(F_n)_{n\in\mathbb{N}}$ $\tau-$epiconverge to F if and only if the two following assertions are satisfied:*

(1) *For all $x \in E$, there exist a sequence $(x_n)_{n\in\mathbb{N}}$ which $\tau-$converges to x such that*

$$F(x) \geq \limsup_{n\to+\infty} F_n(x_n).$$

(2) *For all $x \in E$ and for all sequence $(x_n)_{n\in\mathbb{N}}$ which $\tau-$converges to x, we have*

$$F(x) \leq \liminf_{n\to+\infty} F_n(x_n),$$

the variational property of epiconvergence is given by the following theorem.

Theorem 2.2. *[4] Let be $F = \tau - epi \lim F_n$, then:*

(1) *If $(x_n)_{n\in\mathbb{N}}$ is a minimizing sequence of problem $\inf\limits_{x\in E} F_n(x)$ which is $\tau-$relatively compact. Then any accumulation point $\overline{x}$ of the sequence $(x_n)_{n\in\mathbb{N}}$ is a minimum point of F and we have the infinite continuity*

$$F(\overline{x}) = \min_{x\in E} F(x) = \lim_{n\to+\infty} \inf_{x\in E} F_n(x).$$

(2) *For all function $G : E \to \mathbb{R}$ we have:*

$$\tau - epi \lim (F_n + G) = F + G,$$

stability by continuous perturbation.

3. Main result

The main result of this paper is the following theorem.

Theorem 3.1. *F_n epiconverges to $F^{\hom}$ for the strong topology of $L^2(A)$. Moreover $f^{\hom}$ is convex and satisfies (1.1).*

Proof. To prove the theorem it suffices to verify the two assertions in theorem 2.1.

I. We start with verification of assertion 1, by steps.

Step 1. Suppose that $u_a(t) = at + b$ is an affine function, and suppose that u_a is a solution of (1.2) i.e.

$$f^{\hom}(a) = \int_Y f(at + u_a(t),\ a + \nabla u_a(t))\, dt.$$

We pose $u_n(t) = at + b + \varepsilon_n u_a\left(\frac{t}{\varepsilon_n}\right)$ so $\|u_n - u\|^2_{L^2(A)} \to 0$, on the other hand we have:

$$\begin{aligned} F_n(u_n,\ A) &= \int_A f\left(\frac{at}{\varepsilon_n} + m_{\varepsilon_n} + u_a\left(\frac{t}{\varepsilon_n}\right),\ a + \nabla u_a\left(\frac{t}{\varepsilon_n}\right)\right) dt \\ &= \int_A f\left(\frac{at}{\varepsilon_n} + u_a\left(\frac{t}{\varepsilon_n}\right),\ a + \nabla u_a\left(\frac{t}{\varepsilon_n}\right)\right) dt, \end{aligned}$$

converges to

$$\begin{aligned}\int_A dt \int_Y f\left(ax+u_a(x),\ a+\nabla u_a(x)\right) dx &= \int_A f^{\mathrm{hom}}(a)\, dt \\ &= \int_A f^{\mathrm{hom}}\left(\nabla u_a(t)\right) dt = F^{\mathrm{hom}}\left(u_a,\ A\right).\end{aligned}$$

where $m_{\varepsilon_n} \in \mathbb{Z}^d$ and $\varepsilon_n m_{\varepsilon_n} \to b$.

Step 2. In this step we assume that u is a piecewise affine function. Let A_i be a partition of A and $u^i = a_i t + b_i$ an element of $L^2(A_i)$, let's introduce Σ_δ as follows

$$\Sigma_\delta = \left\{x \in A / dis\left(x,\ \Sigma\right) < \delta\ \right\},$$

$\delta > 0$ and $\varphi_\delta \in C_0^\infty(A)$ such as:

$$\begin{cases} \varphi_\delta = 1,\ \text{on } \Sigma_\delta; \\ \varphi_\delta = 0,\ \text{on } A - \Sigma_\delta; \\ 0 \leq \varphi_\delta \leq 1,\ \text{otherwise,} \end{cases}$$

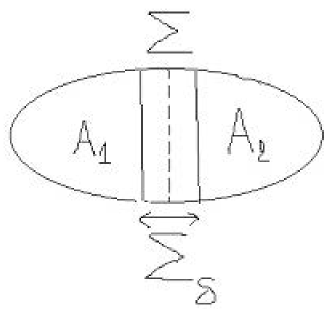

FIGURE 1. The set Sigma

We put:

$$u_n^\delta = \begin{cases} (1-\varphi_\delta)\, u_n^1 + \varphi_\delta u;\ \text{on } A_1, \\ (1-\varphi_\delta)\, u_n^2 + \varphi_\delta u;\ \text{on } A_2, \end{cases}$$

where:

$$u_n^i(t) = u(t) + \varepsilon_n u_{a_i}\left(\frac{t}{\varepsilon_n}\right).$$

On Σ_δ we have:

$$(1-\varphi_\delta)\, u_n^1 + \varphi_\delta u = (1-\varphi_\delta)\, u_n^2 + \varphi_\delta u = u,$$

and for all $0 < t < 1$:

$$f(tu_n^\delta) = \sum_{i=1}^{2} \int_{A_i} f\left(\frac{tu_n^\delta}{\varepsilon_n},\ t\left(1-\varphi_\delta\right) \nabla u_n^i + t\varphi_\delta \nabla u + (1-t)\,\frac{t}{1-t}\left(u-u_n^i\right)\nabla\varphi_\delta\right) dt,$$

as $t\left(1-\varphi_\delta\right) + t\varphi_\delta + (1-t) = 1$ and $f\left(x,\ .\right)$ is convex then:

$$\begin{aligned} f(tu_n^\delta) &\leq \sum_{i=1}^{2} \int_{A_i} f\left(\frac{tu_n^\delta}{\varepsilon_n},\ \nabla u_n^i\right) dt + \int_{\Sigma_{2\delta}} f\left(\frac{tu_n^\delta}{\varepsilon_n},\ \nabla u\right) dt \\ &+ \sum_{i=1}^{2} (1-t) \int_{A_i} f\left(\frac{tu_n^\delta}{\varepsilon_n},\ \frac{t}{1-t}\left(u-u_n^i\right)\nabla\varphi_\delta\right) dt, \end{aligned}$$

using (1.1), we obtain:

$$\begin{aligned} f(tu_n^\delta) &\leq \sum_{i=1}^{2} \int_{A_i} f\left(\frac{tu_n^\delta}{\varepsilon_n},\ \nabla u_n^i\right) dt + \int_{\Sigma_{2\delta}} \beta\left(1+|\nabla u|^2\right) dt \\ &+ \sum_{i=1}^{2} (1-t) \int_{A_i} \beta\ \left(1 + \left(\frac{t}{1-t}\right)^2 \left(u-u_n^i\right)^2 |\nabla\varphi_\delta|^2\right) dt, \end{aligned}$$

by step 1, we have $\lim\limits_{n\to+\infty} u_n^i = u^i$ in $L^2\left(A\right)$, then:

$$\limsup_{n\to+\infty} F_n\left(tu_n^\delta\right) \leq \sum_i \int_{A_i} f^{\text{hom}}\left(\nabla u\right) dt + \int_{\Sigma_{2\delta}} \beta\left(1+|\nabla u|^2\right) dt + (1-t)\,\beta\mu\left(A\right),$$

and

$$\limsup_{\substack{\delta\to 0\\ t\to 1}} \limsup_{n\to+\infty} F_n\left(tu_n^\delta\right) \leq F^{\text{hom}}\left(u\right).$$

According to the diagonalization lemma (see [3]), there exist an application $\varepsilon_n \mapsto \left(\delta\left(\varepsilon_n\right),\ t\left(\varepsilon_n\right)\right)$ such as

$$\begin{cases} \lim\limits_{n\to+\infty} \delta\left(\varepsilon_n\right) = 0, \\ \lim\limits_{n\to+\infty} t\left(\varepsilon_n\right) = 1, \end{cases}$$

then

$$\limsup_{n\to+\infty} F_n\left(t(\varepsilon_n)\, u_n^{\delta(\varepsilon_n)}\right) \le \limsup_{\substack{\delta\to 0\\ t\to 1}} \limsup_{n\to+\infty} F_n\left(t u_n^{\delta}\right),$$

if we put $u_n = t(\varepsilon_n)\, u_n^{\delta(\varepsilon_n)}$ we find $u_n \underset{n\to+\infty}{\to} u$ in $L^2(A)$ and

$$\limsup_{n\to+\infty} F_n(u_n) \le F^{\text{hom}}(u).$$

Step 3. For $u \in W^{1,2}(A)$, by density of piecewise affine function in $W^{1,2}(A)$, there exists a sequence (u_k) of affine functions such that $\|u_k - u\|_{W^{1,2}(A)} \to 0$, as F^{hom} is continuous we have

$$F^{\text{hom}}(u_k) \underset{k\to+\infty}{\to} F^{\text{hom}}(u).$$

By step 2, for $u_k \in W^{1,2}(A)$ there exist a sequence $(u_{n,\,k})_n$ such as

$$\begin{cases} u_{n,\,k} \underset{n\to+\infty}{\to} u_k \text{ strong in } L^2(A); \\ F^{\text{hom}}(u_k) \ge \limsup\limits_{n\to+\infty} F_n(u_{n,\,k}), \end{cases}$$

thus

$$F^{\text{hom}}(u) \ge \limsup_{k\to+\infty} \limsup_{n\to+\infty} F_n(u_{n,\,k}),$$

according to the diagonalization lemma (see [3]) there exist an application $n \mapsto k(\varepsilon_n)$ such as $\lim\limits_{n\to+\infty} k(\varepsilon_n) = +\infty$.
We put $u_n = u_{n,\,k(\varepsilon_n)}$, then we find

$$\begin{cases} u_n \underset{n\to+\infty}{\to} u \text{ strong in } L^2(A); \\ F^{\text{hom}}(u) \ge \limsup\limits_{n\to+\infty} F_n(u_n). \end{cases}$$

II. Remains to verify the assertion 2, by steps.

Step 1. We assume that u is affine, so we use a proof similar to **step 1** of **I**.

Step 2. u is a piecewise affine.

The idea of the proof is to lower $F_n(u_n)$ by $F_n(v_n)$, where $F_n(v_n)$ converges to $F^{\text{hom}}(v)$ after **I**, we then put:

$$v_n^i(t) = v(t) + \varepsilon_n u_{a_i}\left(\frac{t}{\varepsilon_n}\right),\ t \in A_i.$$

According to the convexity inequality applied to points u_n and v_n, we have:

$$f\left(\frac{u_n}{\varepsilon_n},\ \nabla u_n\right)\varphi_i \geq f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right)\varphi_i + \left\langle \partial f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right),\ \nabla u_n - \nabla v_n^i\right\rangle \varphi_i,$$

where $\varphi_i \in \mathcal{D}(A_i)$ and $0 < \varphi_i < 1,\ i = 1,\ 2$. So, on each A_i we have:

$$\begin{aligned}\int_{A_i} f\left(\frac{u_n}{\varepsilon_n},\ \nabla u_n\right)\varphi_i dt \geq \int_{A_i} f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right)\varphi_i dt \\ + \int_{A_i}\left\langle \partial f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right),\ \nabla u_n - \nabla v_n^i\right\rangle \varphi_i dt,\end{aligned}$$

hence

$$\begin{aligned}\text{(3.1)}\quad \int_A f\left(\frac{u_n}{\varepsilon_n},\ \nabla u_n\right) dt \ &\geq\ \sum_{i=1}^{2}\int_{A_i} f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right)\varphi_i dt \\ &+\ \sum_{i=1}^{2}\int_{A_i}\left\langle \partial f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right),\ \nabla u_n - \nabla v_n^i\right\rangle \varphi_i dt,\end{aligned}$$

as after step 1:

$$\lim_{n\to+\infty}\sum_{i=1}^{2}\int_{A_i} f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right)\varphi_i dt = \sum_{i=1}^{2}\int_{A_i} f^{\text{hom}}(\nabla v)\varphi_i dt,$$

and after Green's formula we have:

$$\begin{aligned}&\sum_{i=1}^{2}\int_{A_i}\left\langle \partial f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right),\ \nabla\left(u_n - v_n^i\right)\right\rangle \varphi_i dt \\ =\ & -\sum_{i=1}^{2}\int_{A_i}\left\langle \partial f\left(\frac{v_n^i}{\varepsilon_n},\ \nabla v_n^i\right),\ \left(u_n - v_n^i\right)\nabla\varphi_i\right\rangle dt.\end{aligned}$$

So, (3.1) is written as follow:

$$\liminf_{n\to+\infty} F_n(u_n) \geq \sum_{i=1}^{2}\int_{A_i} f^{\text{hom}}(\nabla v)\varphi_i dt - \sum_{i=1}^{2}\int_{A_i}\left\langle \partial f^{\text{hom}}(\nabla v),\ u - v\right\rangle \nabla\varphi_i dt.$$

On the other hand:

$$\sum_{i=1}^{2}\int_{A_i}\left\langle \partial f^{\hom}\left(\nabla v\right),\ u-v\right\rangle \nabla\varphi_i dt=-\sum_{i=1}^{2}\int_{A_i}\left\langle \partial f^{\hom}\left(\nabla v\right),\ \nabla\left(u-v\right)\right\rangle \varphi_i dt,$$

then:

$$\liminf_{n\to+\infty}F_n\left(u_n\right)\geq\sum_{i=1}^{2}\int_{A_i}f^{\hom}\left(\nabla v\right)\varphi_i dt+\sum_{i=1}^{2}\int_{A_i}\left\langle \partial f^{\hom}\left(\nabla v\right),\ \nabla\left(u-v\right)\right\rangle \varphi_i dt,$$

when $\varphi_i\to 1$ we have:

$$\begin{cases}\sum_{i=1}^{2}\int_{A_i}f^{\hom}\left(\nabla v\right)\varphi_i dt\to\int_A f^{\hom}\left(\nabla v\right)dt,\\ \text{and}\\ \sum_{i=1}^{2}\int_{A_i}\left\langle \partial f^{\hom}\left(\nabla v\right),\ \left(u-v\right)\right\rangle \nabla\varphi_i dt\to\int_A\left\langle \partial f^{\hom}\left(\nabla v\right),\ \nabla\left(u-v\right)\right\rangle dt.\end{cases}$$

So:

$$\liminf_{n\to+\infty}F_n\left(u_n\right)\geq\int_A f^{\hom}\left(\nabla v\right)dt+\int_A\left\langle \partial f^{\hom}\left(\nabla v\right),\ \nabla\left(u-v\right)\right\rangle dt.$$

According to the density of the piecewise affine function in $H^1\left(A\right)$, the continuity of $f^{\hom}$ and the bound of $\partial f^{\hom}\left(a\right)$ we find that:

$$\liminf_{n\to+\infty}F_n\left(u_n\right)\geq F^{\hom}\left(u\right).$$

□

As a consequence of the two theorems 2.2 and 3.1 we have:

Theorem 3.2. *For all $g\in L^2\left(A\right)$, we have:*

$$\min_{u\in H^1_{per}(A)}\left(F_n\left(u,\ A\right)-\int_A gudx\right)\to\min_{u\in H^1_{per}(A)}\left(F^{\hom}\left(u,\ A\right)-\int_A gudx\right).$$

Moreover, there exist a subsection (u_{n_k}) and u respectively solutions of the two minimization problems, such as $u_{n_k}\to u$ in $L^2\left(A\right)$.

Proof. Let

$$\begin{cases}H_n\left(u\right)=F_n\left(u,\ A\right)-\int_A gudx;\\ H^{\hom}\left(u\right)=F^{\hom}\left(u,\ A\right)-\int_A gudx,\end{cases}$$

and $v_n \in \arg\min H_n$. Then:

$$\left\{ \begin{array}{c} H_n\left(v_n\right) \leq \beta\mu\left(A\right); \\ \text{and} \\ \left\|\nabla v_n\right\|^2_{L^2(A)} - \dfrac{1}{\alpha}\int_A \left|g\right|\left|v_n\right| dx \leq \dfrac{\beta}{\alpha}\mu\left(A\right). \end{array} \right. \tag{3.2}$$

After (1.1), (3.2) and Young inequality $ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ applied on $a = |v_n|$ and $b = |g|$ we have, for all $n \in \mathbb{N}$:

$$\begin{aligned} \left\|\nabla v_n\right\|^2_{L^2(A)} &\leq \frac{\beta}{\alpha}\mu\left(A\right) + \frac{1}{\alpha}\int_A \left|g\right|\left|v_n\right| dx \\ &\leq \frac{\beta}{\alpha}\mu\left(A\right) + \frac{1}{\alpha}\int_A \left(\left(\frac{\alpha}{2C}\right)^{-1}\left|g\right|^2 + \frac{\alpha}{2C}\left|v_n\right|^2\right) dx \\ &\leq \frac{\beta}{\alpha}\mu\left(A\right) + \frac{1}{\alpha}\left(\frac{\alpha}{2C}\right)^{-1}\int_A \left|g\right|^2 dx + \frac{1}{2C}\int_A \left|v_n\right|^2 dx \\ &\leq \frac{\beta}{\alpha}\mu\left(A\right) + \frac{1}{\alpha}\left(\frac{\alpha}{2C}\right)^{-1}\int_A \left|g\right|^2 dx + \frac{1}{2}\left\|\nabla v_n\right\|^2_{L^2(A)}. \end{aligned}$$

Then $\left\|\nabla v_n\right\|^2_{H^1_{per}(A)} \leq M$. Where, C is the Poincaré constant and M is a constant depends on n.

After Rellich-Kondracov, there exist a subsequence (v_{n_k}) of (v_n) and $v \in H^1\left(A\right)$ such that:

$$\left\{ \begin{array}{c} v_{n_k} \rightharpoonup v \text{ in } H^1\left(A\right), \\ \text{and} \\ v_{n_k} \to v \text{ strongly in } L^2\left(A\right). \end{array} \right.$$

Since H_n epiconverge to $H^{\hom}$ for the strong topology of $L^2\left(A\right)$, then $v \in \arg\min H^{\hom}$. □

Note that we have used in this proof the fact that:

$$\min_{u \in H^1_{per}(A)} \left(F_n\left(u,\ A\right) - \int_A gudx\right) \to \min_{u \in H^1_0(A)} \left(F_n\left(u,\ A\right) - \int_A gudx\right).$$

4. Conclusion

In this work, a detailed study of a sequence of integral functionals defined on functional spaces not necessarily reflexive was done. Using the epiconvergence technic, in which case the epilimit must be designed in advance and the two assertions must be verified in the definition of epiconvergence.

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LABORATORY OF PARTIAL DIFFERENTIAL EQUATIONS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BATNA2
BATNA,
ALGERIA.
Email address: brahimim347@gmail.com

Email address: m.laouar@univ-batna2.dz