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UNIVERSAL ATTRACTOR FOR A NONLOCAL REACTION-DIFFUSION PROBLEM WITH DYNAMICAL BOUNDARY CONDITIONS

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ABSTRACT. A nonlocal reaction-diffusion equation is presented in this article, based on a model proposed by J. Rubinstein and P. Sternberg [6] with a nonlinear strictly monotone operator. A dynamical boundary condition is considered, rather then the usual ones such as Neumann or Dirichlet boundary conditions.

The well-posedness and the existence of a universal attractor of this problem, which describes the long time behavior of the solution, are established.

1. INTRODUCTION

The Allen Cahn equation is one of the appropriate mathematical models, that describes the behavior of the phases in the absence of temperature variations and mechanical stresses (see S. M. Allen and J. W. Cahn [1]). It is represented by a reaction-diffusion equation with a reaction term equal to a derivative of a double well potential. The problem of phase separation in binary mixtures was presented and analysed by J. Rubinstein and P. Sternberg [6], where a nonlocal reaction-diffusion equation with nul-flux boundary conditions was proposed. For $u(t) : \Omega \to \mathbb{R}$ an order parameter in the mixture or simply the concentration of one of the species, the nonlocal model proposed by J. Rubinstein and P. Sternberg [6] is given by the equation

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$$u_t - \Delta u = f(u(x,t)) - \frac{1}{|\Omega|}\int_\Omega f(u(x,t))dx, \quad x \in \Omega, t > 0, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ an open bounded domain with a smooth boundary $\Gamma = \partial\Omega$, supplimented by the Neumann boundary condition

$$\partial_n u = \frac{\partial u}{\partial n} = 0, \quad x \in \Gamma, t \geq 0, \tag{1.2}$$

and initial data

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

A very important property of this problem is the mass conservation, that is

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx = m, \quad \forall t > 0.$$

S. Boussaïd, D. Hilhorst and T. N. Nguyen [2] proved the existence of a unique global solution and its convergence to a steady state by means of a Lojasiewicz inequality.

The present article is devoted to the study of the well-posedness and the asymptotic behavior of the solution of

$$u_t - \operatorname{div}(A(\nabla u)) = f(u(x,t)) - \frac{1}{|\Omega|}\int_\Omega f(u(x,t))dx, \quad x \in \Omega, t > 0, \tag{1.4}$$

with the assumption that $A = \nabla_v \Psi(v)$: $\mathbb{R}^n \to \mathbb{R}^n$ for some strictly convex function $\Psi \in C^{1,1}$ (i.e. $\Psi(v) \in C^1(\mathbb{R}^n)$ and $\nabla\Psi(v)$ is Lipschitz continuous) satisfying

$$\begin{cases} A(0) = \nabla\Psi(0) = 0, \Psi(0) = 0 \\ \|D^2\Psi\|_{L^\infty(\mathbb{R}^n;\, \mathbb{R}^{n\times n})} \leq c_1 \end{cases}, \tag{1.5}$$

for some constant $c_1 > 0$. Remark that (1.5) implies that

$$|A(a) - A(b)| \leq C|a-b|, \tag{1.6}$$

for all $a, b \in \mathbb{R}^n$, where C is a positive constant and that the strict convexity of Ψ implies that A is strictly monotone, namely there exists a positive constant C_0 such that

$$(A(a) - A(b))(a-b) \geq C_0|a-b|^2, \tag{1.7}$$

for all a,b $\in \mathbb{R}^n$.

Remark that if A is the identity matrix, the nonlinear diffusion operator $-\operatorname{div}(A(\nabla v))$ reduces to the linear operator $-\triangle v.$

The function $f = -F'$ is such that F is a smooth double well potential, more precisely

$$F(s) = \frac{1}{2}\left(s^2 - 1\right)^2,$$

and f satisfies the hypotheses

(H_1) $f(s)s \leq -C_1 s^4 + C_2,$
(H_2) $f'(s) \leq C_3,$
(H_3) $-F(s) \leq -C_4(s^4 - 1)$, with $F(s) = -\int_0^s f(\tau)d\tau$,
(H_4) $|f(s)| \leq C_5(|s|^3 + 1),$

where $C_i,$ for $i = 1, \cdots, 5$, are positive constants.

Usually equation (1.4) is supplemented by the homogeneous Neumann boundary condition

$$\partial_n A(\nabla u) = \frac{\partial A(\nabla u)}{\partial n} = 0 \quad x \in \Gamma, t \geq 0, \tag{1.8}$$

which means that there cannot be any exchange of the mixture constituents through the boundary Γ. From equation (1.4) and the usual boundary condition (1.8) the following bulk free energy $E(u(t))$ decreases

$$E_\Omega(u(t)) = \int_\Omega \left\{\frac{1}{2}A(|\nabla u|^2) + F(u)\right\} dx. \tag{1.9}$$

Recently physicists (see [3]) proposed the surface free energy functional

$$E_\Gamma(u(t)) = \frac{1}{2}\int_\Gamma \left\{A(|\nabla_\Gamma u|^2) + u^2\right\} d\Sigma, \tag{1.10}$$

happening when the effective interaction between the wall (the boundary Γ) and both mixture components is short-ranged. Thus this surface free energy functional is added to the bulk energy to form the total energy functional as

$$E(u(t)) = E_\Omega(u(t)) + E_\Gamma(u(t)).$$

The total energy functional is supposed to be decreasing for all time $t \geq 0$; using integration by parts we formally have

$$\frac{d}{dt}E(u) = \frac{d}{dt}E_\Omega(u) + \frac{d}{dt}E_\Gamma(u) = \frac{d}{dt}\int_\Omega \left\{\frac{1}{2}A(|\nabla u|^2) + F(u)\right\} dx$$

$$+\int_\Gamma\left\{\left(-\operatorname{div}_\Gamma(A(\nabla_\Gamma u))+\frac{\partial A(\nabla_\Gamma u)}{\partial n}\right)+u\right\}u_t d\Sigma;$$

then for $u_{|\Gamma}=v$ the following dynamical boundary condition is considered and ensures that $\frac{d}{dt}E_\Gamma(u)\le 0$

$$v_t=\operatorname{div}_\Gamma(A(\nabla_\Gamma v))-\frac{\partial}{\partial n}A(\nabla_\Gamma v)-v, \quad \text{on } \Gamma. \tag{1.11}$$

Remark that

$$\frac{d}{dt}E(u(t))+\int_\Omega u_t^2 dx+\int_\Gamma v_t^2 d\Sigma=0. \tag{1.12}$$

This paper is organized as follows. Section 2 is devoted to the proof of the existence and uniqueness of a global solution. In section 3 the asymptotic behavior of this solution is discribed by the existence of a universal attractor.

2. Existence of a solution

In this section a proof of the existence of a unique global solution is given for the following problem

$$(P)\begin{cases} u_t-\operatorname{div}(A(\nabla u))=f(u(x,t))-\frac{1}{|\Omega|}\int_\Omega f(u(x,t))dx, & x\in\Omega, t>0\\ u_{|\Gamma}=v, v_t=\operatorname{div}_\Gamma(A(\nabla_\Gamma v))-\frac{\partial}{\partial n}A(\nabla_\Gamma v)-v, & x\in\Gamma, t\ge 0\\ u(x,0)=u_0(x), & x\in\Omega\\ v(x,0)=v_0(x), & x\in\Gamma\end{cases}.$$

The nonlinear operator A satisfies properties (1.5)-(1.7) and the function f fulfills hypotheses (H_1)-(H_4). The well-posedness of Problem (P) is proved by compactness and monotonicity arguments.

Set

$$K=\left\{(z,z_\Gamma)\in H^1(\Omega)\times H^1(\Gamma), \int_\Omega z dx=\int_\Omega z_0 dx \text{ and } \int_\Gamma z_\Gamma d\Sigma=\int_\Gamma z_{0\Gamma} d\Sigma\right\},$$

a convex closed set in $H^1(\Omega)\times H^1(\Gamma)$.

If $(u,u_\Gamma)\in H^1(\Omega)\times H^1(\Gamma)$ is a critical point of $E(u,u_\Gamma)$ over K, then $(u,u_\Gamma)\in C^\infty$ and it is a solution to Problem (P). Moreover the functional $E(u,u_\Gamma)$ has at least a minimizer $(\omega, \omega_\Gamma) \in K$ such that

$$E(\omega, \omega_\Gamma) = \inf_{(u,u_\Gamma)\in K} E(u, u_\Gamma).$$

Thus Problem (P) admits at least a classical solution.

Let's define the spaces

$$\mathbb{H} = \left\{\psi \in L^2(\Omega), \int_\Omega \psi(x,t)dx = 0\right\}, \mathbb{H}_\Gamma = \left\{\psi \in L^2(\Gamma), \int_\Gamma \psi(x,t)d\Sigma = 0\right\},$$

$$\mathbb{L}^4(\Omega) = \{\psi \in L^4(\Omega), \int_\Omega \psi(x,t)dx = 0\},$$

$$\mathbb{V} = H^1(\Omega) \cap \mathbb{H}, \mathbb{V}_\Gamma = H^1(\Gamma) \cap \mathbb{H}_\Gamma,$$

$$\mathcal{V} = \{z \in \mathbb{V}, \quad z_{|\Gamma} \in \mathbb{V}_\Gamma\}.$$

Note that the inclusion $\mathbb{V} \subset \mathbb{H}$ is compact and dense and that $\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}^*$ with dense and compact injection and also $\mathbb{V}_\Gamma \subset \mathbb{H}_\Gamma \subset \mathbb{V}_\Gamma^*$.

Let $(.,.)$ and $\|.\|_\mathbb{H}$ be the inner product and the induced norm in $L^2(\Omega)$. Also denote by $\|.\|_\mathbb{V}$, $\|.\|_{\mathbb{H}_\Gamma}$ and $\|.\|_{\mathbb{V}_\Gamma}$ the norms in the corresponding spaces. The symbol $< .,. >$ stands for the duality pairing between $\mathbb{V}^*$ and $\mathbb{V}$. We endow the space $\mathcal{V} \subset \mathbb{V}$ with the following inner product and the induced graph norm

$$\left\{ \begin{array}{l} (\omega, z)_\mathcal{V} = (\omega, z)_\mathbb{V} + (\omega_{|\Gamma}, z_{|\Gamma})_{\mathbb{V}_\Gamma} = \int_\Omega \nabla\omega\nabla z dx + \int_\Gamma (\nabla_\Gamma\omega\nabla_\Gamma z + \omega z)\, d\Sigma \\ \|\omega\|^2_\mathcal{V} = \|\omega\|^2_\mathbb{V} + \|\omega_{|\Gamma}\|^2_{\mathbb{V}_\Gamma} \end{array} \right. , \tag{2.1}$$

$\omega, z \in \mathcal{V}$. Multiply equation (1.4) with a test function $z \in H^1(\Omega)$, $(z_\Gamma \in H^1(\Gamma))$ and integrate by parts on Ω (taking the boundary condition (1.11) into account)

$$\begin{aligned} \int_\Omega u_t.z dx + \int_\Gamma v_t.z d\Sigma + \int_\Omega A(\nabla u).\nabla z dx + \int_\Gamma A(\nabla_\Gamma v).\nabla_\Gamma z_\Gamma d\Sigma + \int_\Gamma v.z_\Gamma d\Sigma \\ = \int_\Omega z f(u) dx - \frac{1}{|\Omega|}\int_\Omega z dx \int_\Omega f(u) dx. \end{aligned} \tag{2.2}$$

A first result for the existence and uniqueness of the solution is

Theorem 2.1. *For u_0 given in $\mathbb{H}$ and v_0 given in $\mathbb{H}_\Gamma$, there exists a unique solution (u, v) to Problem (P), satisfying*

$$u \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^4(0,T;\mathbb{L}^4(\Omega)),$$

$$v \in L^\infty(0,T;\mathbb{H}_\Gamma) \cap L^2(0,T;\mathbb{V}_\Gamma), \textit{ for all } T > 0,$$

$$u \in C(\mathbb{R}_+; \mathbb{H}) \textit{ and } v \in C(\mathbb{R}_+; \mathbb{H}_\Gamma).$$

Before going into the proof of this theorem, note that it allows to define the semigroup

$$S(t) : u_0 \in \mathbb{H} \longrightarrow u(t) \in \mathbb{H}$$
$$v_0 \in \mathbb{H}_\Gamma \longrightarrow u(t) \in \mathbb{H}_\Gamma.$$

Proof. The proof relies on classical arguments (compacity and monotonicity), see J. L. Lions [4]. But before that let's give some apriori estimates on u and v.

(i-) Consider $u = z \in \mathbb{V}$ and $v = u_\Gamma = z_\Gamma \in \mathbb{V}_\Gamma$ in (2.2) to obtain

$$\begin{aligned} &\int_\Omega u_t.udx + \int_\Gamma v_t.vd\Sigma + \int_\Omega A(\nabla u)\nabla udx + \int_\Gamma A(\nabla_\Gamma v)\nabla_\Gamma vd\Sigma \\ &\qquad + \int_\Gamma v^2 d\Sigma = \int_\Omega uf(u)dx, \end{aligned} \tag{2.3}$$

with property (1.7) and hypothesis (H_1) equation (2.3) will be for $\overline{C} = \min\{C_0, 1\}$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_\Omega u^2dx\right) + \frac{1}{2}\frac{d}{dt}\left(\int_\Gamma v^2d\Sigma\right) + \overline{C}\int_\Omega |\nabla u|^2dx + \overline{C}\int_\Gamma |\nabla_\Gamma v|^2 d\Sigma \\ &\qquad + \overline{C}\int_\Gamma v^2d\Sigma + C_1\int_\Omega u^4dx \le C_1|\Omega|, \end{aligned} \tag{2.4}$$

an integration from 0 to T yields

$$\begin{aligned} &\frac{1}{2}\int_\Omega u^2(x,T)dx + \frac{1}{2}\int_\Gamma v^2(x,T)d\Sigma + \overline{C}\int_0^T\int_\Omega |\nabla u|^2dxds \\ &+ \overline{C}\int_0^T\int_\Gamma |\nabla_\Gamma v|^2d\Sigma ds + \overline{C}\int_0^T\int_\Gamma v^2d\Sigma ds \\ &+ C_1\int_0^T\int_\Omega u^4dxds \le C_2|\Omega|T + \frac{1}{2}\int_\Omega u_0^2(x)dx + \frac{1}{2}\int_\Gamma v_0^2(x)d\Sigma. \end{aligned} \tag{2.5}$$

From (2.5) set $K = C_2|\Omega|T + \frac{1}{2}\int_\Omega u_0^2(x)dx + \frac{1}{2}\int_\Gamma v_0^2(x)d\Sigma$, so

$$\sup_{t\in[0,T]}\left(\int_\Omega u^2(x,t)dx\right) \le 2K, \tag{2.6}$$

$$\sup_{t\in[0,T]}\left(\int_\Gamma v^2(x,t)d\Sigma\right) \le 2K, \tag{2.7}$$

$$\int_0^T \int_\Omega |\nabla u|^2 dxds + \int_0^T \int_\Gamma \left(|\nabla_\Gamma v|^2 + v^2\right) d\Sigma ds \leq K/\overline{C}, \tag{2.8}$$

$$\int_0^T \int_\Omega u^4 dxds \leq K/C_1. \tag{2.9}$$

(ii-) The proof relies on the Galerkin method and only the main steps are given. Set $B = -\operatorname{div}(A(\nabla)) : \mathbb{V} \longrightarrow \mathbb{V}^*$ the nonlinear operator and let $\omega_1, \cdots, \omega_m$ be the basis of $\mathbb{V}$. Introduce $u_m(t) \in [\omega_1, \cdots, \omega_m]$ an approximated solution of Problem (P), $u_m(t) = \Sigma_{i=1}^m u_m^i \omega_i$, as

$$\begin{cases} (u_m'(t), \omega_j) + (B(u_m(t)), \omega_j) = (f(t), \omega_j) \\ \qquad -\frac{1}{|\Omega|}(\int_\Omega f(t), \omega_j), j = 1, \cdots, m \\ u_m(0) = u_{0m}(x) \in [\omega_1, \cdots, \omega_m] \\ v_m(0) = u_{0m}(x) \in [\omega_1, \cdots, \omega_m] \end{cases} . \tag{2.10}$$

Problem (2.10) admits a unique solution on some interval $(0, T_m), T_m > 0$ (when the a priori estimates are uniformly independent $T_m = \infty$).

By the results in (i) a subsequence still denoted u_m exists such that

$$\begin{aligned} &u_m \rightharpoonup u \text{ in } L^2(0, T; \mathbb{V}) \text{ weakly,} \\ &u_m \rightharpoonup u \text{ in } L^\infty(0, T; \mathbb{H}) \text{ weak *,} \\ &f(u_m) \rightharpoonup \Phi \text{ in } L^{4/3}((0, T) \times \Omega) \text{ weakly,} \\ &B(u_m) \rightharpoonup \Psi \text{ in } L^2(0, T; \mathbb{V})^* \text{ weakly.} \end{aligned}$$

as $m \to \infty$.

Passing to the limit in (2.10) yields the problem

$$\frac{du}{dt} + \Psi u = \Phi. \tag{2.11}$$

According to Lemma 8.3 p 218 in [5], $\Phi = f$.

To prove that $\Psi = B$, a monotonicity method is used. Based on property (1.7) define

$$X_m = \int_0^T (B(u_m(t)) - B(v(t)), u_m(t) - v(t)) > 0.$$

Thus

$$X_m = \int_0^T (B(u_m), u_m)dt - \int_0^T (B(u_m), v)dt - \int_0^T (B(v), u_m - v)dt,$$

but according to (2.11)

$$\begin{aligned} X_m = &\int_0^T (f, u_m)dt + \frac{1}{2}\int_\Omega u_{0m}^2 dx - \frac{1}{2}\int_\Omega u_m(T)^2 dx \\ &- \int_0^T (B(u_m), v)dt - \int_0^T (B(v), u_m - v)dt, \end{aligned}$$

then

$$\begin{aligned} \limsup X_m \leq &\int_0^T (f, u)dt + \frac{1}{2}\int_\Omega u_0^2 dx - \frac{1}{2}\int_\Omega u^2(T)dx \\ &- \int_0^T (\Psi, v)dt - \int_0^T (B(v), u - v)dt. \end{aligned}$$

Again with $\int_0^T (\Psi, u)dt = \int_0^T (f, u)dt + \frac{1}{2}\int_\Omega u_0^2 dx - \frac{1}{2}\int_\Omega u^2(T)dx$, yields

$$\int_0^T (\Psi - B(v), u - v)dt > 0.$$

Setting $v = u - \lambda w$ for $w \in L^2(0, T; \mathbb{V})$ and by the hemicontinuity of B the result will be $B = \Psi$, when $\lambda \longrightarrow 0$. (iii-) To study the uniqueness of the solution, suppose the existence of two solutions u^1 and u^2 to Problem (P), where $u_\Gamma^1 = v^1$, $u_\Gamma^2 = v^2$ and $u_0^1(x) = u^1(x, 0) = u^2(x, 0) = u_0^2(x)$, $v_0^1(x) = v^1(x, 0) = v^2(x, 0) = v_0^2(x)$. Then replace u in (2.3) by $\omega = u^1 - u^2$, $f(u)$ by $f(u^1) - f(u^2)$ and v by $\rho = v^1 - v^2$ to obtain

$$\begin{aligned} &\int_\Omega \omega_t.\omega dx + \int_\Gamma \rho_t.\rho d\Sigma + \int_\Omega \left(A(\nabla u^1) - A(\nabla u^2)\right)\nabla\omega dx \\ &+ \int_\Gamma \left(A(\nabla_\Gamma v^1) - A(\nabla_\Gamma v^2)\right)\nabla_\Gamma \rho d\Sigma + \int_\Gamma \rho^2 d\Sigma = \int_\Omega \omega\left(f(u^1) - f(u^2)\right)dx, \end{aligned} \tag{2.12}$$

but $f(u^1) - f(u^2) = \int_{u^2}^{u^1} f'(\sigma)d\sigma$, thus with hypothesis (H_2) and property (1.7) equation (2.12) will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \frac{1}{2}\frac{d}{dt}\left(\int_\Gamma \rho^2 d\Sigma\right) \tag{2.13}$$

$$+C_0\int_\Omega |\nabla\omega|^2dx + C_0\int_\Gamma |\nabla_\Gamma\rho|^2 d\Sigma + \int_\Gamma \rho^2 d\Sigma \le C_3\int_\Omega \omega^2 dx,$$

integrate it over $(0,t)$ where $t\in(0,T]$. So

$$\frac{1}{2}\int_\Omega \omega^2(x,t)dx + \frac{1}{2}\int_\Gamma \rho^2(x,t)d\Sigma + C_0\int_0^t\int_\Omega |\nabla\omega|^2(x,s)dxds$$
$$+ C_0\int_0^t\int_\Gamma |\nabla_\Gamma\rho|^2(x,s)d\Sigma ds + \int_0^t\int_\Gamma \rho^2(x,s)d\Sigma ds \le \frac{1}{2}\int_\Omega \left(u_0^1-u_0^2\right)^2(x)dx$$
$$+\frac{1}{2}\int_\Gamma \left(v_0^1-v_0^2\right)^2(x)d\Sigma + C_3\int_0^t\int_\Omega \omega^2 dx.$$

An application of the Gronwall's lemma yields the uniqueness of the solution (u,v) knowing that $u_0^1(x)=u^1(x,0)=u^2(x,0)=u_0^2(x)$ and $v_0^1(x)=v^1(x,0)=v^2(x,0)=v_0^2(x)$. □

The next result gives a better regularity of (u,v).

Theorem 2.2. *Suppose u_0 given in $\mathbb{V}$ and v_0 given in $\mathbb{V}_\Gamma$ then $(u,v),(u_t,v_t)$ satisfy $u\in L^\infty(0,T;\mathbb{V}), v\in L^\infty(0,T;\mathbb{V}_\Gamma)$ and $u_t\in L^2(0,T;\mathbb{V}^*), v_t\in L^2(0,T;\mathbb{V}_\Gamma^*)$.*

Proof. Consider $u_t=z\in\mathbb{V}$ and $v_t=u_{\Gamma t}=z_\Gamma\in\mathbb{V}_\Gamma$ in (2.2). Thus

$$\int_\Omega u_t^2dx + \int_\Gamma v_t^2 d\Sigma + \frac{1}{2}\frac{d}{dt}\left(\int_\Omega A\left(|\nabla u|^2\right)dx\right)$$
$$+\frac{1}{2}\frac{d}{dt}\left(\int_\Gamma A\left(|\nabla_\Gamma v|^2\right)d\Sigma\right) + \frac{1}{2}\frac{d}{dt}\left(\int_\Gamma v^2 d\Sigma\right) = -\frac{d}{dt}\left(\int_\Omega F(u)dx\right). \tag{2.14}$$

Integrating (2.14) in time gives

$$\int_0^t\int_\Omega u_s^2dxds + \int_0^t\int_\Gamma v_s^2 d\Sigma ds + \overline{C}\left(\int_\Omega |\nabla u|^2dx\right)$$
$$+\overline{C}\left(\int_\Gamma |\nabla_\Gamma v|^2 + v^2 d\Sigma\right) \le C_4|\Omega| + \frac{C_0}{2}\left(\int_\Omega |\nabla u_0|^2dx\right) \tag{2.15}$$
$$+\frac{C_0}{2}\left(\int_\Gamma |\nabla_\Gamma v_0|^2 d\Sigma\right) + \frac{1}{2}\left(\int_\Gamma v_0^2 d\Sigma\right) + \int_\Omega F(u_0)dx,$$

with property (1.7), hypothesis (H_3) and setting $\overline{C}=\frac{1}{2}\min\{C_0,1\}$. From (2.15) set

$$\kappa = C_4|\Omega| + \frac{C_0}{2}\left(\int_\Omega |\nabla u_0|^2dx\right) + \frac{C_0}{2}\left(\int_\Gamma |\nabla_\Gamma v_0|^2 d\Sigma\right) + \frac{1}{2}\left(\int_\Gamma v_0^2 d\Sigma\right) + \int_\Omega F(u_0)dx,$$

to ensure that

$$\sup_{t\in[0,T]}\left(\int_\Omega |\nabla u|^2 dx\right) \leq \kappa/\overline{C}, \tag{2.16}$$

$$\sup_{t\in[0,T]}\left(\int_\Gamma \left\{|\nabla_\Gamma v|^2 + v^2 d\Sigma\right\}\right) \leq \kappa/\overline{C}, \tag{2.17}$$

$$\int_0^T \int_\Omega u_s^2 dx ds \leq \kappa, \tag{2.18}$$

$$\int_0^T \int_\Gamma v_s^2 dx ds \leq \kappa. \tag{2.19}$$

□

3. The universal attractor

Attractors are a very important tool for the study of the large time behavior of the solutions of reaction-diffusion problems. The universal attractor for dynamical systems is a compact subset of the phase space that attracts all trajectories, it exists for dissipative systems; those that have a compact absorbing sets. First an existence results of an absorbing sets are given bellow.

Lemma 3.1. *There exist an absorbing sets in* $\mathbb{H}\times\mathbb{H}_\Gamma$ *and* $\mathcal{V}$.

Proof.
i-**Absorbing set in** $\mathbb{H}\times\mathbb{H}_\Gamma$
Recall the energy estimate (2.4)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega u^2 dx\right) + \frac{1}{2}\frac{d}{dt}\left(\int_\Gamma v^2 d\Sigma\right) + \overline{C}\int_\Omega |\nabla u|^2 dx + \overline{C}\int_\Gamma |\nabla_\Gamma v|^2 d\Sigma\\ &+ \bar{C}\int_\Gamma v^2 d\Sigma + C_1\int_\Omega u^4 dx \leq C_2|\Omega|,\end{aligned} \tag{3.1}$$

where knowing that $s^4 \geq 2s^2 - 1$, gives

$$\frac{d}{dt}\left(\int_\Omega u^2 dx\right) + \frac{d}{dt}\left(\int_\Gamma v^2 d\Sigma\right) + 4C_1\int_\Omega u^2 dx \tag{3.2}$$

$$+ 2\overline{C}\int_\Gamma v^2 d\Sigma \leq 2\left(C_1 + C_2\right)|\Omega|. \tag{3.3}$$

Set $\overline{C}_1 = 2\min\{2C_1;\overline{C}\}$ and $\overline{C}_2 \geq 2\left(C_1 + C_3\right)|\Omega|$ to obtain

$$\frac{d}{dt}\left(\int_\Omega u^2 dx\right) + \frac{d}{dt}\left(\int_\Gamma v^2 d\Sigma\right) \leq -\overline{C}_1\left\{\int_\Omega u^2 dx + \int_\Gamma v^2 d\Sigma\right\} + \overline{C}_2, \tag{3.4}$$

thus an application of the Gronwall's Lemma yields

$$\|(u,v)(t)\|^2_{\mathbb{H}\times\mathbb{H}_\Gamma} \leq \frac{\overline{C}_2}{\overline{C}_1}\left(1-e^{-\overline{C}_1 t}\right) + \|(u_0,v_0)\|^2_{\mathbb{H}\times\mathbb{H}_\Gamma} e^{-\overline{C}_1 t}, \tag{3.5}$$

so

$$\sup_t \|(u,v)(t)\|^2_{\mathbb{H}\times\mathbb{H}_\Gamma} \leq \|(u_0,v_0)\|^2_{\mathbb{H}\times\mathbb{H}_\Gamma} - \frac{\overline{C}_2}{\overline{C}_1}, \tag{3.6}$$

$$\lim_{t\longrightarrow\infty} \sup_t \|(u,v)(t)\|^2_{\mathbb{H}\times\mathbb{H}_\Gamma} \leq \frac{\overline{C}_2}{\overline{C}_1} = \rho_1^2. \tag{3.7}$$

As a deduction, from (3.5) any ball of $\mathbb{H}\times\mathbb{H}_\Gamma$ centered at the origine and of radius $\rho_2 > \rho_1 = \sqrt{\overline{C}_2/\overline{C}_1}$ is an absorbing set in $\mathbb{H}\times\mathbb{H}_\Gamma$. Indeed if $\mathcal{B}_0$ is a bounded set of $\mathbb{H}\times\mathbb{H}_\Gamma$ included in a ball $B(0,R)$ of $\mathbb{H}\times\mathbb{H}_\Gamma$, then $S(t)\mathcal{B}_0 \subset B(0,\rho_2)$, for $t \geq t_0(\mathcal{B}_0)$ and $t_0 = \dfrac{1}{\overline{C}_1}\ln\left(\dfrac{R^2}{\rho_2^2-\rho_1^2}\right)$.

In conclusion the set $\mathcal{B} = B(0,\rho_2) = \left\{(u,v)\in\mathbb{H}\times\mathbb{H}_\Gamma : \|(u,v)(t)\|^2_{\mathbb{H}\times\mathbb{H}_\Gamma} \leq \rho_2^2\right\}$ is an absorbing set in $\mathbb{H}\times\mathbb{H}_\Gamma$.

Relation (3.1) yields

$$\begin{aligned}&\int_t^{t+r}\left\{\frac{d}{ds}\left(\int_\Omega u^2 dx\right) + \frac{d}{dt}\left(\int_\Gamma v^2 d\Sigma\right)\right\} ds \\ &+ 2\overline{C}\int_t^{t+r}\left\{\int_\Omega |\nabla u|^2 dx + \int_\Gamma |\nabla_\Gamma v|^2 d\Sigma + \int_\Gamma v^2 d\Sigma\right\} ds \leq 2rC_3|\Omega|,\end{aligned}$$

thus

$$\begin{aligned}&\int_\Omega u^2(x,t+r)dx + \int_\Gamma v^2(x,t+r)d\Sigma \\ &+ 2\overline{C}\int_t^{t+r}\left\{\int_\Omega |\nabla u|^2 dx + \int_\Gamma |\nabla_\Gamma v|^2 d\Sigma + \int_\Gamma v^2 d\Sigma\right\} ds \\ &\leq \int_\Omega u^2(x,t)dx + \int_\Gamma v^2(x,t)d\Sigma + 2rC_3|\Omega|.\end{aligned}$$

So

$$\int_t^{t+r} \|(u,v)\|^2_{\mathcal{V}} \leq \frac{2rC_3}{\overline{C}}|\Omega|, \tag{3.8}$$

for $(u_0, v_0) \in \mathcal{B}' \subset B(0,R)$ and $t \geq t_0$.

ii-**Absorbing set in** $\mathcal{V}$
Equation (2.14) gives

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega A\left(|\nabla u|^2\right)dx + \frac{1}{2}\int_\Gamma A\left(|\nabla_\Gamma v|^2\right)d\Sigma + \frac{1}{2}\int_\Gamma v^2 d\Sigma + \int_\Omega F(u)dx\right\} \leq 0,$$

an integration in time and with property (1.7) yields

$$\begin{aligned}&\frac{1}{2}C_0\int_\Omega \left(|\nabla u|^2(x,t) - |\nabla u_0|^2\right)dx + \frac{1}{2}C_0\int_\Gamma \left(|\nabla_\Gamma v|^2(x,t) - |\nabla_\Gamma v_0|^2\right)d\Sigma \\ &+ \frac{1}{2}\int_\Gamma v^2(x,t)d\Sigma + \int_\Omega F(u)dx \leq \frac{1}{2}\int_\Gamma v_0^2 d\Sigma + \int_\Omega F(u_0)dx,\end{aligned} \tag{3.9}$$

hypothesis (H_3) and setting $\overline{C} = \min(C_0, 1)$ yields

$$\begin{aligned}&\int_\Omega |\nabla u|^2(x,t)dx + \int_\Gamma |\nabla_\Gamma v|^2(x,t)d\Sigma + \int_\Gamma v^2(x,t)d\Sigma \leq \\ &\int_\Omega |\nabla u_0|^2 dx + \int_\Gamma |\nabla_\Gamma v_0|^2 d\Sigma + \frac{1}{2}\int_\Gamma v_0^2 d\Sigma + \int_\Omega F(u_0)dx + \frac{2C_4}{\overline{C}}|\Omega|.\end{aligned} \tag{3.10}$$

Then for $\theta = \int_\Omega |\nabla u_0|^2 dx + \int_\Gamma |\nabla_\Gamma v_0|^2 d\Sigma + \frac{1}{2}\int_\Gamma v_0^2 d\Sigma + \int_\Omega F(u_0)dx + \frac{2C_4}{\overline{C}}|\Omega|$,

$$\sup_t \left(\|(u,v)\|_{\mathcal{V}}^2\right) \leq \theta,$$

which is an estimation of (u,v) in $L^\infty(0,T;\mathcal{V})$ and with the uniform Gronwall's Lemma. Remark that for $t \geq t_0$, (3.8) yields

$$\|(u,v)\|_{\mathcal{V}}^2 \leq \frac{2C_2}{\overline{C}}|\Omega|. \tag{3.11}$$

Thus $B\left(0, \left(\frac{2C_2}{\overline{C}}|\Omega|\right)^{1/2}\right)$ is an absorbing set in $\mathcal{V}$ and relatively compact in $\mathbb{H} \times \mathbb{H}_\Gamma$.

Let $t_1 = t_1(R_0)$, where for $(u_0, v_0) \in \mathbb{H} \times \mathbb{H}_\Gamma$ and $(u_0, v_0) \in B(0, R_0) \subset B(0, \rho_2)$. Again for $t \geq t_1 + r : (u,v) \in B\left(0, \left(\frac{2C_2}{\overline{C}}|\Omega|\right)^{1/2}\right)$ is a bounded set in $\mathcal{V}$. □

Now we are in position to state the existence of a universal attractor.

Theorem 3.1. *Problem* (P) *admits a compact universal attractor* $\mathcal{A}$.

Proof. Part (ii) of Lemma (3.1) proved the existence of an absorbing set in $\mathcal{V}$, relatively compact in $\mathbb{H} \times \mathbb{H}_\Gamma$, which is connected thus the universal attractor is $\mathcal{A} = \omega\left(B\left(0, \left(\frac{2C_2}{\overline{C}}|\Omega|\right)^{1/2}\right)\right)$. □

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