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EQUIVALENT DEFINITIONS OF MULTIVECTOR FIELDS ON WEIL BUNDLE

Nkou Vann Borhen[1], Mahoungou Moukala Norbert, and Maloko Mavambou Ange

ABSTRACT. In this paper, we generalize the notion of vector fields on Weil bundle. Let $q \geq 2$ be an integer, we give equivalent definitions of a q-vector field on Weil bundle in terms of q-derivations. Further, we construct a Lie graded algebra structure of multivector fields on Weil bundle.

1. INTRODUCTION

The notion of multivector field of degree $q \geq 2$ (an integer) or q-vector field on manifold M is a generalization of the notion of a vector field on a manifold. A multivector field is a section of the qth exterior power of the tangent bundle, $\Lambda^q TM$, and at each point x of M it assigns a q-vector in $\Lambda^q T_x M$ [3]. The set of all smooth q-vector fields on M defines a structure of module, denoted by $\mathfrak{X}^q(M)$ or $\Gamma(\Lambda^q TM)$. Moreover, since the tangent bundle is dual to the cotangent bundle, a multivector fields of degree q are dual to q-forms, and both are particular cases of the general concept of a tensor field (see [4], [5]), which is a section of some tensor bundle, often consisting of exterior powers of tangent and cotangent bundles. We consider a multivector field of degree q as a multi-derivation of degree q, simply called q-derivation.

[1]*corresponding author*
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The notion of infinitely near points manifolds has been introduced by André Weil in 1953 [9], [7]. In the working out of this theory, André Weil considers as local algebra, a real algebra A which is associative, commutative, with unit of finite dimension and admitting a unique maximal ideal $\mathfrak{m}$ of codimension 1 over $\mathbb{R}$, i.e., $A = \mathbb{R} \oplus \mathfrak{m}$. Let M be a smooth manifold and $C^\infty(M)$ be the algebra of smooth functions on M. An infinitely near point to x of M of kind A is a homomorphism of algebras (simply called Weil morphism), $\xi : C^\infty(M) \longrightarrow A$ such that $\xi(f) - f(x) \in \mathfrak{m}$, for any $f \in C^\infty(M)$. One denotes M_x^A the set of all infinitely near points to x of kind A and $M^A = \bigcup_{x \in M} M_x^A$. The set M^A is a smooth manifold of dimension $\dim M \times \dim A$, and the triplet (M^A, π, M) is a bundle of infinitely near points or called simply Weil bundle [6].

The main goal of this work is to generalize the notion of vector fields on Weil bundle: basically we study the structure of the set of all multivector fields on Weil bundle and the prolongations of multivector fields on a manifold to the Weil bundle associated with this manifold.

This work is organized as follows. In Section 2, we recall the notion of Weil bundle, some properties and definitions. In section 3, we give equivalent definitions of tangent q-vector at point ξ of the Weil bundle in terms of q-derivations and consequence relating to its equivalences. In section 4, we also give equivalent definitions of a q-vector field on Weil bundle in terms of q-derivations, consequence relating to its equivalences and others properties on Weil bundle. In section 5, we define, the wedge product and the Schouten-Nijenhuis bracket of multivector fields on Weil bundle, we show that exterior algebra of multivector fields on Weil bundle equipped with the wedge product and the Schouten-Nijenhuis bracket is an associative graded commutative graded algebra and a Lie graded algebra over A, respectively. In section 6, we show that if θ is a multivector field and $\{., \ldots, .\}_M$ is a Leibniz bracket on M, then there exists a unique multivector field θ^A and a unique Leibniz bracket $\{., \ldots, .\}_{M^A}$ on the Weil bundle M^A, respectively. We end the last section, with the study of the prolongation to the Weil bundle of multivector fields which coming from multi-derivations of the Weil algebra.

2. Preliminary

In what follows, we denote A, a local algebra (in the sense of André Weil) or simply Weil algebra. Let M be a smooth manifold, $C^\infty(M)$ an algebra of smooth functions on M and M^A the manifold of infinitely near points of kind A (see [9]). The triplet (M^A, π, M) is a bundle called bundle of infinitely near points or simply Weil bundle.

If $f : M \longrightarrow \mathbb{R}$ is a smooth function then the mappping

$$f^A : M^A \longrightarrow A, \xi \longmapsto \xi(f)$$

is also a smooth function. The set, $C^\infty(M^A, A)$, of smooth functions on M^A with values on A, is a commutative algebra over A with unit and the mapping

$$C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A$$

is an injective homomorphism of algebras. Then, we have:

$$(f+g)^A = f^A + g^A;$$
$$(\lambda \cdot f)^A = \lambda \cdot f^A;$$
$$(f \cdot g)^A = f^A \cdot g^A.$$

The map

$$C^\infty(M^A) \times A \longrightarrow C^\infty(M^A, A), (F, a) \longmapsto F \cdot a : \xi \longmapsto F(\xi) \cdot a$$

is bilinear and induces a unique linear map

$$\sigma : C^\infty(M^A) \otimes A \longrightarrow C^\infty(M^A, A).$$

Proposition 2.1. *Let $(a_\alpha)_{\alpha=1,2,\ldots,\dim A}$ be a basis of A and $(a^*_\alpha)_{\alpha=1,2,\ldots,\dim A}$ its dual basis. For $\varphi \in C^\infty(M^A, A)$, the following writing*

$$\varphi = \sum_{\alpha=1}^{\dim A} (a^*_\alpha \circ \varphi)\, a_\alpha \tag{2.1}$$

does not depend of a choisen basis. Moreover, for all $\varphi, \psi \in C^\infty(M^A, A)$ and $a \in A$, we have:

i)

$$a_\alpha^* \circ (a\varphi) = \sum_{\beta,\gamma=1}^{\dim A} a_\beta^*(a)\left(a_\gamma^* \circ \varphi\right) a_\alpha^*(a_\beta a_\gamma). \tag{2.2}$$

ii)

$$a_\alpha^* \circ (\varphi \cdot \psi) = \sum_{\beta,\gamma=1}^{\dim A} \left(a_\beta^* \circ \varphi\right)\left(a_\gamma^* \circ \psi\right) a_\alpha^*(a_\beta a_\gamma). \tag{2.3}$$

Proof. Let $(b_\beta)_{\beta=1,\ldots,\dim A}$ be another basis of A. For $\varphi \in C^\infty(M^A, A)$, we have

$$\begin{aligned}
\varphi &= \sum_{\alpha=1}^{\dim A} \left(b_\beta^* \circ \varphi\right) b_\beta \\
&= \sum_{\beta=1}^{\dim A} \left(b_\beta^* \circ \varphi\right)\left(\sum_{\alpha=1}^{\dim A} a_\alpha^*(b_\beta) a_\alpha\right) \\
&= \sum_{\alpha=1}^{\dim A} a_\alpha^* \left(\sum_{\beta=1}^{\dim A} \left(b_\beta^* \circ \varphi\right) b_\beta\right) a_\alpha \\
&= \sum_{\alpha=1}^{\dim A} \left(a_\alpha^* \circ \varphi\right) a_\alpha.
\end{aligned}$$

i) On one hand,

$$a \cdot \varphi = \sum_{\alpha=1}^{\dim A} a_\alpha^* \circ (a \cdot \varphi)\, a_\alpha.$$

On the other hand,

$$\begin{aligned}
a \cdot \varphi &= \sum_{\beta=1}^{\dim A} a_\beta^*(a)\, a_\beta \cdot \sum_{\gamma=1}^{\dim A} (a_\gamma^* \circ \varphi) a_\gamma \\
&= \sum_{\beta,\gamma=1}^{\dim A} a_\beta^*(a)\left(a_\gamma^* \circ \varphi\right) a_\beta a_\gamma \\
&= \sum_{\beta,\gamma=1}^{\dim A} a_\beta^*(a)\left(a_\gamma^* \circ \varphi\right)\left(\sum_{\alpha=1}^{\dim A} a_\alpha^*(a_\beta a_\gamma)\, a_\alpha\right) \\
&= \sum_{\alpha=1}^{\dim A}\left(\sum_{\beta,\gamma=1}^{\dim A} a_\beta^*(a)\left(a_\gamma^* \circ \varphi\right) a_\alpha^*(a_\beta a_\gamma)\right) a_\alpha.
\end{aligned}$$

Since $(a_\alpha)_{\alpha=1,\ldots,\dim A}$ is a basis of A, then we get (2.2).

ii) On one hand,

$$\varphi \cdot \psi = \sum_{\alpha=1}^{\dim A} a_\alpha^* \circ (\varphi \cdot \psi)\, a_\alpha.$$

On the other hand,

$$\begin{aligned}\varphi \cdot \psi &= \left(\sum_{\beta=1}^{\dim A} \left(a_\beta^* \circ \varphi\right) a_\beta\right) \cdot \left(\sum_{\gamma=1}^{\dim A} \left(a_\gamma^* \circ \psi\right) a_\gamma\right)\\ &= \sum_{\beta,\gamma=1}^{\dim A} \left(a_\beta^* \circ \varphi\right)\left(a_\gamma^* \circ \psi\right) a_\beta a_\gamma\\ &= \sum_{\beta,\gamma=1}^{\dim A} \left(a_\beta^* \circ \varphi\right)\left(a_\gamma^* \circ \psi\right)\left(\sum_{\alpha=1}^{\dim A} a_\alpha^* \left(a_\beta a_\gamma\right)\right) a_\alpha,\end{aligned}$$

thus

$$\varphi \cdot \psi = \sum_{\alpha=1}^{\dim A}\left(\sum_{\beta,\gamma=1}^{\dim A} \left(a_\beta^* \circ \varphi\right)\left(a_\gamma^* \circ \psi\right) a_\alpha^* \left(a_\beta a_\gamma\right)\right) a_\alpha$$

Since $(a_\alpha)_{\alpha=1,\ldots,\dim A}$ is a basis of A, we get (2.3).

□

The mapping

$$\sigma^{-1} : C^\infty(M^A, A) \longrightarrow A \otimes C^\infty(M^A), \varphi \longmapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes \left(a_\alpha^* \circ \varphi\right) \tag{2.4}$$

is an isomorphism of A-algebras [1]. That isomorphism does not depend of a choisen basis and the map

$$\gamma : C^\infty(M) \longrightarrow A \otimes C^\infty(M^A), f \longmapsto \sigma^{-1}(f^A), \tag{2.5}$$

is a homomorphism of algebras.

If (U, φ) is a local chart of M with coordinate system $(x_1, \ldots, x_n)$, the map

$$\varphi^A : U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), \ldots, \xi(x_n)) \tag{2.6}$$

is a bijection from U^A onto an open set of A^n. In addition, if $(U_i, \varphi_i)_{i\in I}$ is an atlas of M^A, then $(U_i^A, \varphi_i^A)_{i\in I}$ is also an atlas of M^A (see [1]).

One denotes, $\mathfrak{X}(M^A)$, the set of all smooth sections of TM^A. The set, $\mathfrak{X}(M^A)$, is a module of vector fields on M^A over $C^\infty(M^A)$ and $C^\infty(M^A, A)$ [1].

An A-covector field at $\xi \in M^A$ is a linear form on the A-module $T_\xi M^A$. The set, $T_\xi^* M^A$, of A-covectors at $\xi \in M^A$ is an A-free module of dimension n and

$$T^* M^A = \bigcup_\xi T_\xi^* M^A$$

is an A-manifold of dimension $2n$. The set, $\Omega^1(M^A, A)$, of differential sections of $T^* M^A$ is a $C^\infty(M^A, A)$-module and one says that $\Omega^1(M^A, A)$ is the $C^\infty(M^A, A)$-module of differential A-forms of degree 1.

For $p \in \{0\} \cup \mathbb{N}$ and for $\xi \in M^A$, one denotes $\mathcal{L}^p_{sks}(T_\xi M^A, A)$ the A-module of skew-symmetric multilinear forms of degree p on the A-module $T_\xi M^A$. One has, $\mathcal{L}^0_{sks}(T_\xi M^A, A) = A$. For tow integers p and q, one defines the wedge product

$$\wedge : \mathcal{L}^p(T_\xi M^A, A) \times \mathcal{L}^q(T_\xi M^A, A) \to \mathcal{L}^{p+q}(T_\xi M^A, A), (\alpha,\, \beta) \mapsto \alpha \wedge \beta.$$

The set,

$$A^p(T_\xi^* M^A,\, A) = \bigcup \mathcal{L}^p_{sks}(T_\xi M^A, A)$$

is A-manifold of dimension $n + \mathcal{C}^p_n$. The set, $\Omega^p(M^A, A)$, of differential sections of $A^p(T^* M^A,\, A)$ is a $C^\infty(M^A, A)$-module. One says that $\Omega^p(M^A, A)$ is the $C^\infty(M^A, A)$-module of A-differential forms of degree p on M^A and

$$\Omega^\bullet(M^A, A) = \bigoplus_{p=0}^{n} \Omega^p(M^A, A)$$

is the algebra of A-differential forms on M^A. The algebra $\Omega^\bullet(M^A, A)$ of A-differential forms on M^A is canonicaly isomorphic to $A \otimes \Omega^\bullet(M^A)$. One has, $\Omega^0(M^A, A) = C^\infty(M^A, A)$.

If η is a differential form of degree p on M, according to [1], then there exists a unique differential A-form of degree p,

$$\eta^A : \mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \cdots \times \mathfrak{X}(M^A) \longrightarrow C^\infty(M^A, A)$$

such that, for all $\theta_1, \ldots, \theta_p \in \mathfrak{X}(M)$ and $f_1, \ldots, f_p \in C^\infty(M)$, one has

$$\eta^A(f_1^A \theta_1^A, \ldots, f_p^A \theta_p^A) = f_1^A \cdots f_p^A [\eta(\theta_1, \ldots, \theta_p)]^A.$$

The mapping $\Omega^\bullet(M) \to \Omega^\bullet(M^A, A), \omega \mapsto \omega^A$, is a morphism of graded $\mathbb{R}$-algebras, and if

$$d : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

is an exterior differential operator, accoding to [1], one denotes

$$d^A : \Omega^\bullet(M^A, A) \to \Omega^\bullet(M^A, A)$$

the cohomology operator associated with the following representation

$$\mathfrak{X}(M^A) \to \mathcal{D}er_A[C^\infty(M^A, A)], X \mapsto \widetilde{X}.$$

The mapping d^A is A-linear and

$$d^A(\omega^A) = (d\omega)^A,\ \omega \in \Omega^\bullet(M).$$

3. TANGENT q-VECTORS ON WEIL BUNDLE

Let $\pi : M^A \longrightarrow M$ be the map which assigns any infinitely near point ξ of M^A to its origin $x \in M$, and U be an open neighborhood of M with coordinate system $(x_1, \ldots, x_n)$.

Hence, $(x_{i,\alpha}/i = 1, \ldots, n; \alpha = 1, \ldots, \dim A = r)$ is coordinate system of $\pi^{-1}(U)$ where

$$x_{i,\alpha} : \pi^{-1}(U) \longrightarrow \mathbb{R}, \xi \longmapsto x_{i,\alpha}(\xi)$$

is a map such that $\xi(x_i) = x_i^A(\xi) = \sum_\alpha x_{i,\alpha}(\xi) a_\alpha$ for any $x_i \in C^\infty(U)$, and for all $i = 1, \ldots, n$ [6].

Definition 3.1. *A tangent q-vector at $\xi \in M^A$ is a skew-symmetric $\mathbb{R}$-multilinear map of degree q*

$$Q_1(\xi) : \overbrace{C_\xi^\infty(M^A) \times \ldots \times C_\xi^\infty(M^A)}^{q\text{-times}} \longrightarrow \mathbb{R}, (F_1, \ldots F_q) \longmapsto Q_1(\xi)(F_1, \ldots F_q)$$

such that for any $i = 1, \ldots, q$, the $\mathbb{R}$-linear map

$$Q_1^i(\xi) : C_\xi^\infty(M^A) \longrightarrow \mathbb{R}, F_i \longmapsto Q_1(\xi)(F_1, \ldots, F_i, \ldots, F_q)$$

satisfies for any $G \in C_\xi^\infty(M^A)$, the Leibniz rule

$$Q_1^i(\xi)(F_i \cdot G) = Q_1^i(\xi)(F_i) \cdot G(\xi) + F_i(\xi) \cdot Q_1^i(\xi)(G). \tag{3.1}$$

We denote by $\Lambda^q T_\xi M^A$ the vector space of tangent q-vectors at $\xi \in M^A$. This q-vector is locally given by,

$$Q_1(\xi) = \sum_{\substack{1\leq i_1<\ldots<i_q\leq m \\ 1\leq \alpha_1<\ldots<\alpha_q\leq r}} (Q_1)_{(i_1,\alpha_1)\ldots(i_q,\alpha_q)} \frac{\partial}{\partial x_{i_1,\alpha_1}} |_\xi \wedge \ldots \wedge \frac{\partial}{\partial x_{i_q,\alpha_q}} |_\xi$$

where $(Q_1)_{(i_1,\alpha_1)\ldots(i_q,\alpha_q)} = Q_1\left(x_{i_1,\alpha_1}, \ldots x_{i_q,\alpha_q}\right)$. Let $\Lambda^q T_\xi M^A = \mathcal{D}er^q_{\mathbb{R}}[C^\infty_\xi(M^A), \mathbb{R}]$ be the set, of q-derivations from $C^\infty_\xi(M^A)$ into $\mathbb{R}$.

A q-derivation from $C^\infty_\xi(M^A, A)$ into A is a skew-symmetric A-multilinear map of degree q

$$Q_2(\xi) : \overbrace{C^\infty_\xi(M^A, A) \times \ldots \times C^\infty_\xi(M^A, A)}^{q\text{-times}} \longrightarrow A$$

such that for any $i = 1, \ldots, q$, the A-linear map

$$Q^i_2(\xi) : C^\infty_\xi(M^A, A) \longrightarrow A, \varphi_i \longmapsto Q_2(\xi)(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_q)$$

satisfies the Leibniz rule

$$Q^i_2(\xi)(\varphi_i \cdot \psi) = Q^i_2(\xi)(\varphi_i) \cdot \psi(\xi) + \varphi_i(\xi) \cdot Q^i_2(\xi)(\psi), \tag{3.2}$$

for any $\psi \in C^\infty_\xi(M^A, A)$. Locally,

$$Q_2(\xi) = \sum_{1\leq i_1<\ldots<i_q\leq l} Q_2\left(x^A_{i_1}, \ldots, x^A_{i_q}\right) \left(\frac{\partial}{\partial x_{i_1}}\right)^A |_\xi \wedge \ldots \wedge \left(\frac{\partial}{\partial x_{i_q}}\right)^A |_\xi).$$

A q-derivation from $C^\infty(M)$ into A is a skew-symmetric $\mathbb{R}$-multilinear map of degree q

$$Q_3(\xi) : \overbrace{C^\infty(M) \times \cdots \times C^\infty(M)}^{q\text{-times}} \longrightarrow A$$

such that for any $i = 1, \ldots, q$, the $\mathbb{R}$-linear map

$$Q^i_3(\xi) : C^\infty(M) \longrightarrow A, f_i \longmapsto Q_3(\xi)(f_1, \ldots, f_i, \ldots, f_q)$$

satisfies for any $g \in C^\infty(M)$ the Leibniz rule

$$Q_3(\xi)(f_i \cdot g) = Q_3(\xi)(f_i) \cdot g^A(\xi) + f^A_i(\xi) \cdot Q_3(\xi)(g). \tag{3.3}$$

And locally

$$Q_3(\xi) = \sum_{1 \le i_1 < \ldots < i_q \le m} Q_3\left(x_{i_1}, \ldots, x_{i_q}\right) \left(\frac{\partial}{\partial x_{i_1}}\right)^A |_\xi \wedge \ldots \wedge \left(\frac{\partial}{\partial x_{i_q}}\right)^A |_\xi \,.$$

Theorem 3.1. *The following statements are equivalent:*

(1) *A tangent q-vector at $\xi \in M^A$ is a q-derivation from $C_\xi^\infty(M^A)$ into $\mathbb{R}$.*
(2) *A tangent q-vector at $\xi \in M^A$ is a q-derivation from $C_\xi^\infty(M^A, A)$ into A.*
(3) *A tangent q-vector at $\xi \in M^A$ is a q-derivation from $C^\infty(M)$ into A.*

Proof.

(1)⇒(2) Let $Q_1(\xi)$ be a tangent q-vector at $\xi \in M^A$, that means $Q_1(\xi)$ is a q-derivation from $C_\xi^\infty(M^A)$ into $\mathbb{R}$. Let $Q_2(\xi) : C_\xi^\infty(M^A, A) \times \ldots \times C_\xi^\infty(M^A, A) \to A$ be the map defined by,

$$Q_2(\xi)(\varphi_1, \ldots, \varphi_q) = \sum_{\alpha_1 \ldots \alpha_q} Q_1(\xi)\left(a_{\alpha_1}^* \circ \varphi_1, \ldots, a_{\alpha_q}^* \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}$$

for any $\varphi_1, \ldots, \varphi_q \in C_\xi^\infty(M^A, A)$ with $a_{\alpha_1 \ldots \alpha_q} = a_{\alpha_1} \times \cdots \times a_{\alpha_q}$. The map Q_2^i such that

$$Q_2^i(\xi)(\varphi_i) = \sum_{\alpha_1 \ldots \alpha_q} Q_1(\xi)\left(a_{\alpha_1}^* \circ \varphi_1, \ldots, a_{\alpha_i}^* \circ \varphi_i, \ldots, a_{\alpha_q}^* \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}$$

is A-linear and satisfies Leibniz law. Indeed, for all $\varphi_1, \ldots, \varphi_q, \psi \in C_\xi^\infty(M^A, A)$ and $a \in A$,

$$Q_2^i(\xi)(a\varphi_i) = \sum_{\alpha_1 \ldots \alpha_q} Q_1(\xi)\left(a_{\alpha_1}^* \circ \varphi_1, \ldots, a_{\alpha_i}^* \circ (a\varphi_i), \ldots, a_{\alpha_q}^* \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}.$$

By direct calculation using (2.2), we get

$$Q_2^i(\xi)(a\varphi_i) = aQ_2^i(\xi)(\varphi_i);$$

and

$$Q_2^i(\xi)(\varphi_i \cdot \psi) = \sum_{\alpha_1 \ldots \alpha_q} Q_1(\xi)\left(a_{\alpha_1}^* \circ \varphi_1, \ldots, a_{\alpha_i}^* \circ (\varphi_i \cdot \psi), \ldots, a_{\alpha_q}^* \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}.$$

Since $Q_1(\xi)$ is a q-derivation, from (2.3) and by straightforward calculations, we get

$$
\begin{aligned}
& Q_2^i(\xi)(\varphi_i \cdot \psi) \\
= & \left(\sum_{\alpha_1 \ldots \alpha_q} Q_1(\xi) \left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_i} \circ \varphi_i, \ldots, a^*_{\alpha_q} \circ \varphi_q \right) a_{\alpha_1 \ldots \alpha_q} \right) \cdot \psi(\xi) \\
& + \varphi_i(\xi) \cdot \left(\sum_{\alpha_1 \ldots \alpha_q} Q_1(\xi) \left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_i} \circ \psi, \ldots, a^*_{\alpha_q} \circ \varphi_q \right) a_{\alpha_1 \ldots \alpha_q} \right)
\end{aligned}
$$

i.e.,

$$Q_2^i(\xi)(\varphi_i \cdot \psi) = Q_2^i(\xi)(\varphi_i) \cdot \psi(\xi) + \varphi_i(\xi) \cdot Q_2^i(\xi)(\psi).$$

Thus, $Q_2^i(\xi)$ is a derivation from $C^\infty_\xi(M^A, A)$ into A. Therefore, $Q_2(\xi)$ is a q-derivation from $C^\infty_\xi(M^A, A)$ into A.

(2)⇒(3) Assume that a tangent q-vector at $\xi \in M^A$ is a q-derivation from $C^\infty_\xi(M^A, A)$ into A. Let $Q_3(\xi) : C^\infty(M) \times \ldots \times C^\infty(M) \longrightarrow A$ be the map defined by,

$$Q_3(\xi)(f_1, \ldots, f_q) = Q_2(\xi)\left(f_1^A, \ldots, f_q^A\right).$$

for any $f_1, \ldots, f_q \in C^\infty(M)$. The map

$$Q_3^i(\xi) : C^\infty(M) \longrightarrow A, f_i \longmapsto Q_3^i(\xi)(f_i) = Q_2(\xi)\left(f_1^A, \ldots f_{i,}^A, , , f_q^A\right)$$

satisfies

$$Q_3^i(\xi)(\lambda_1 f_i + \lambda_2 g) = \lambda_1 Q_3^i(\xi)(f_i) + \lambda_2 Q_3^i(\xi)(g);$$

$$Q_3^i(\xi)(f_i \cdot g) = Q_3^i(\xi)(f_i) \cdot g^A(\xi) + f_i^A(\xi) \cdot Q_3^i(\xi)(g),$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $g \in C^\infty(M)$.

(3)⇒(1) Assume that a tangent q-vector $Q_3(\xi)$ at $\xi \in M^A$ is a q-derivation from $C^\infty_\xi(M^A, A)$ into A.

Let $s_{M^A} : M \longrightarrow M^A$ be the section of Weil bundle $\left(M^A, \pi_{M^A}, M\right)$ and let

$$s^*_{M^A} : C^\infty(M^A) \longrightarrow C^\infty(M), F \longmapsto F \circ s_{M^A}$$

be the *pull back map*. Consider the map

$$Q_1(\xi) : C^\infty_\xi(M^A) \times \cdots \times C^\infty_\xi(M^A) \longrightarrow \mathbb{R}$$

such that

$$Q_1\left(\xi\right)\left(F_1,\ldots,F_q\right) = 1^*\Big[Q_3\left(\xi\right)\left(F_1\circ s_{M^A},\ldots,F_q\circ s_{M^A}\right)\Big],$$

for any $F_1,\ldots,F_q \in C^\infty(M^A)$. The map

$$Q_1^i\left(\xi\right): C^\infty(M^A) \longrightarrow \mathbb{R}, F_i \longmapsto Q_1^i\left(\xi\right)\left(F_i\right)$$

such that

$$Q_1^i\left(\xi\right)\left(F_i\right) = 1^*\Big[Q_3\left(\xi\right)\left(F_1\circ s_{M^A},\ldots,F_i\circ s_{M^A}\ldots,F_q\circ s_{M^A}\right)\Big]$$

satisfies, for all $G \in C^\infty(M^A)$ and $\lambda \in \mathbb{R}$,

$$Q_1^i\left(\xi\right)\left(F_i+\lambda G\right) = Q_1^i\left(\xi\right)\left(F_i\right)+\lambda Q_1^i\left(\xi\right)\left(G\right)$$

and moreover

$$\begin{aligned}&Q_1^i\left(\xi\right)\left(F_i\cdot G\right)\\ &= 1^*\Big[Q_3\left(\xi\right)\left(F_1\circ s_{M^A},\ldots,\left(F_i\circ s_{M^A}\right)\cdot\left(G\circ s_{M^A}\right),\ldots,F_q\circ s_{M^A}\right)\Big],\end{aligned}$$

since $Q_3\left(\xi\right)$ is a q-derivation from $C^\infty(M)$ into A, we have

$$\begin{aligned}&Q_1^i\left(\xi\right)\left(F_i\cdot G\right)\\ =\ & 1^*\Big[Q_3\left(\xi\right)\left(F_1\circ s_{M^A},\ldots,\left(F_i\circ s_{M^A}\right),\ldots,F_q\circ s_{M^A}\right)\Big]\cdot 1^*[(G\circ s_{M^A})^A\left(\xi\right)]\\ &+1^*[(F_i\circ s_{M^A})^A\left(\xi\right)]\cdot 1^*\Big[Q_3\left(\xi\right)\left(F_1\circ s_{M^A},\ldots,G\circ s_{M^A},\ldots,F_q\circ s_{M^A}\right)\Big].\end{aligned}$$

As $1^*[(G\circ s_{M^A})^A\left(\xi\right)] = G\left(\xi\right)$, we get

$$Q_1^i\left(\xi\right)\left(F_i\cdot G\right) = Q_1^i\left(\xi\right)\left(F_i\right)\cdot G\left(\xi\right)+F_i\left(\xi\right)\cdot Q_1^i\left(\xi\right)\left(G\right).$$

Therefore, a tangent q-vector $Q_1(\xi)$ at $\xi \in M^A$ is a q-derivation from $C^\infty_\xi(M^A)$ into $\mathbb{R}$.

□

Corollary 3.1. *We have the following isomorphisms:*

(i) $\mathcal{D}er^q_{\mathbb{R}}[C^\infty(M^A),\mathbb{R}] \simeq \mathcal{D}er^q_A[C^\infty(M^A,A),A]$.
(ii) $\mathcal{D}er^q_A[C^\infty(M^A,A),A] \simeq \mathcal{D}er^q_{\mathbb{R}}[C^\infty(M),A]$.
(iii) $\mathcal{D}er^q_{\mathbb{R}}[C^\infty(M),A] \simeq \mathcal{D}er^q_{\mathbb{R}}[C^\infty(M^A),\mathbb{R}]$.

4. Multivector fields on Weil bundle

A q-vector field on M^A is a section of the bundle $\left(\Lambda^q TM^A, \pi_q, M^A\right)$ i.e., a smooth map

$$Q : M^A \longrightarrow \Lambda^q TM^A$$

such that $Q(\xi) \in \Lambda^q T_\xi M^A$, for any $\xi \in M^A$. We denote by $\mathfrak{X}^q\left(M^A\right)$ the set of all q-vector fields on M^A. The set $\mathfrak{X}^q\left(M^A\right)$ is a $C^\infty(M^A)$-module of q-vector fields on M^A. Locally, for $Q \in \mathfrak{X}^q\left(U^A\right)$, we have

$$Q_{|_{U^A}} = \sum_{\substack{1\leq i_1<\ldots<i_q\leq m \\ 1\leq \alpha_1<\ldots<\alpha_q\leq r}} Q_{(i_1,\alpha_1)\ldots(i_q,\alpha_q)} \frac{\partial}{\partial x_{i_1,\alpha_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_q,\alpha_q}}$$

where $Q_{(i_1,\alpha_1)\ldots(i_q,\alpha_q)} = Q\left(x_{i_1,\alpha_1}, \ldots x_{i_q,\alpha_q}\right) \in C^\infty(U^A)$.

A q-derivation of $C^\infty(M^A)$ is a skew-symmetric $\mathbb{R}$-multilinear map of degree q,

$$D_1 : \overbrace{C^\infty(M^A) \times \cdots \times C^\infty(M^A)}^{q\text{-times}} \longrightarrow C^\infty(M^A), (F_1, \ldots F_q) \longmapsto D_1(F_1, \ldots F_q)$$

satisfying the Leibniz rule in each of its arguments i.e., for any $i = 1, \ldots, q$, then

$$D_1^i : C^\infty(M^A) \longrightarrow C^\infty(M^A), F_i \longmapsto D_1(F_1, \ldots F_i, \ldots, F_q)$$

is $\mathbb{R}$-linear and satisfies, for any $G \in C^\infty(M^A)$,

$$D_1^i\left(F_i \cdot G\right) = D_1^i\left(F_i\right) \cdot G + F_i \cdot D_1^i\left(G\right).$$

A q-derivation of $C^\infty(M^A, A)$ is a skew-symmetric A -multilinear map of degree q,

$$D_2 : C^\infty(M^A, A) \times \cdots \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), (\varphi_1, \ldots, \varphi_q) \longmapsto D_2(\varphi_1, \ldots, \varphi_q)$$

satisfying the Leibniz rule in each of its arguments i.e., for any $i = 1, \ldots, q$, the map

$$D_2^i : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), F_i \longmapsto D_2(\varphi_1, \ldots \varphi_i, \ldots, \varphi_q)$$

is A-linear and satisfies, for any $\psi \in C^\infty(M^A, A)$,

$$D_2^i\left(\varphi_i \cdot \psi\right) = D_2^i\left(\varphi_i\right) \cdot \psi + \varphi_i \cdot D_2^i\left(\psi\right).$$

A q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$ is a skew-symmetric $\mathbb{R}$-multilinear map of degree q,

$$D_3 : C^\infty(M) \times \cdots \times C^\infty(M) \longrightarrow C^\infty(M^A, A), (f_1, \ldots f_q) \longmapsto D_3(f_1, \ldots f_q)$$

satisfying the Leibniz rule in each of its arguments i.e., for any $i = 1, \ldots, q$, the map

$$D_3^i : C^\infty(M) \longrightarrow C^\infty(M^A, A), f_i \longmapsto D_3(f_1, \ldots f_i, \ldots, f_q)$$

is $\mathbb{R}$-linear and satisfies, for any $g \in C^\infty(M)$,

$$D_3^i\left(f_i \cdot g\right) = D_3^i\left(f_i\right) \cdot g^A + f_i^A \cdot D_3^i\left(g\right).$$

Theorem 4.1. *The following statements are equivalent:*

(1) *A q-vector field on M^A is a section of the bundle $\left(\Lambda^q TM^A, \pi_q, M^A\right)$.*
(2) *A q-vector field on M^A is a q-derivation of $C^\infty(M^A)$.*
(3) *A q-vector field on M^A is a q-derivation of $C^\infty(M^A, A)$ which is skew-symmetric A-multilinear.*
(4) *A q-vector fields on M^A is a q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$.*

Proof.

(1)⇒(2) Assume that Q_1 is section of $\left(\Lambda^q TM^A, \pi_q, M^A\right)$, then for any $\xi \in M^A$, $Q_1(\xi)$ is a q-derivation from $C^\infty(M)$ into $\mathbb{R}$. Let $Q_2 : C^\infty(M^A) \times \cdots \times C^\infty(M^A) \longrightarrow C^\infty(M^A)$ be the map such that

$$\left[Q_2\left(F_1, \ldots, F_q\right)\right](\xi) = Q_1(\xi)\left(F_1, \ldots, F_q\right),$$

for any $F_1, \ldots, F_q \in C^\infty(M^A)$. The map

$$Q_2^i : C^\infty(M^A) \longrightarrow C^\infty(M^A), F_i \longmapsto Q_2\left(F_1, \ldots, F_i, \ldots, F_q\right)$$

such that, for any $\xi \in M^A$,

$$\left[Q_2^i\left(F_i\right)\right](\xi) = Q_1(\xi)\left(F_1, \ldots, F_i, \ldots, F_q\right)$$

satisfies

$$\left[Q_2^i\left(F_i \cdot G_i\right)\right](\xi) = Q_1(\xi)\left(F_1, \ldots, F_i \cdot G_i, \ldots, F_q\right).$$

Since $Q_1(\xi)$ is a derivation, we get

$$\left[Q_2^i\left(F_i \cdot G_i\right)\right](\xi) = \left[Q_2^i\left(F_i\right) \cdot G_i + F_i \cdot Q_2^i\left(G_i\right)\right](\xi)$$

for any $\xi \in M^A$, that is

$$Q_2^i(F_i \cdot G_i) = Q_2^i(F_i) \cdot G_i + F_i \cdot Q_2^i(G_i).$$

(2)⇒(3) Assume that a q-vector fields Q_2 on M^A is q-derivation of $C^\infty(M^A)$. Let $Q_3 : C^\infty(M^A, A) \times \ldots \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$ be the map such that, for any $\varphi_1, \ldots, \varphi_q \in C^\infty(M^A, A)$,

$$Q_3(\varphi_1, \ldots, \varphi_q) = \sum_{\alpha_1 \ldots \alpha_q} Q_2\left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_q} \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}.$$

The map

$$Q_3^i : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A), \varphi_i \longmapsto Q_3(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_q)$$

satisfies

$$\begin{aligned} Q_3^i(a \cdot \varphi_i) &= \sum_{\alpha_1 \ldots \alpha_q} Q_2\left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_q} \circ (a \cdot \varphi_i), \ldots, a^*_{\alpha_q} \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q} \\ & a \cdot \left(\sum_{\alpha_1 \ldots \alpha_q} Q_2\left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_q} \circ (\varphi_i), \ldots, a^*_{\alpha_q} \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}\right), \end{aligned}$$

$$\begin{aligned} Q_3^i(\varphi_i + \psi) &= \sum_{\alpha_1 \ldots \alpha_q} Q_2\left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_q} \circ (\varphi_i + \psi), \ldots, a^*_{\alpha_q} \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q} \\ & Q_3^i(\varphi_i) + Q_3^i(\psi), \end{aligned}$$

$$\begin{aligned} Q_3^i(\varphi_i \cdot \psi) &= \sum_{\alpha_1 \ldots \alpha_q} Q_2\left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_q} \circ (\varphi_i \cdot \psi), \ldots, a^*_{\alpha_q} \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q} \\ & \left(\sum_{\alpha_1 \ldots \alpha_q} Q_2\left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_i} \circ \varphi_i, \ldots, a^*_{\alpha_q} \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}\right) \cdot \psi \\ & + \varphi_i \cdot \left(\sum_{\alpha_1 \ldots \alpha_q} Q_2\left(a^*_{\alpha_1} \circ \varphi_1, \ldots, a^*_{\alpha_i} \circ \psi, \ldots, a^*_{\alpha_q} \circ \varphi_q\right) a_{\alpha_1 \ldots \alpha_q}\right) \\ &= Q_3^i(\varphi_i \cdot \psi) = Q_3^i(\varphi_i) \cdot \psi + \varphi_i \cdot Q_3^i(\psi). \end{aligned}$$

(3)⇒(4) Assume that a q-vector fields Q_3 on M^A is a q-derivation of $C^\infty(M^A, A)$ which is skew-symmetric A-multilinear. Let $Q_4 : C^\infty(M) \times \cdots \times C^\infty(M) \longrightarrow$

$C^{\infty}(M^A, A)$ be the map such that, for any $f_1, \ldots, f_q \in C^{\infty}(M)$,

$$Q_4(f_1, \ldots, f_q) = Q_3\left(f_1^A, \ldots, f_q^A\right).$$

The map

$$Q_4^i : C^{\infty}(M) \longrightarrow C^{\infty}(M^A, A), f_i \longmapsto Q_3\left(f_1^A, \ldots, f_i^A, \ldots, f_q^A\right)$$

satisfies

$$\begin{aligned} Q_4^i(f_i \cdot g) &= Q_3\left(f_1^A, \ldots, (f_i \cdot g)^A, \ldots, f_q^A\right) \\ &= Q_3\left(f_1^A, \ldots, f_i^A \cdot g^A, \ldots, f_q^A\right) \\ &= Q_3\left(f_1^A, \ldots, f_i^A, \ldots, f_q^A\right) \cdot g^A + f_i^A \cdot Q_3\left(f_1^A, \ldots, g^A, \ldots, f_q^A\right) \\ &= Q_4^i(f_i) \cdot g + f_i \cdot Q_4^i(g). \end{aligned}$$

(4)⇒(1) Assume that a q-vector fields Q_4 on M^A is a q-derivation from $C^{\infty}(M)$ into $C^{\infty}(M^A, A)$. Let $Q_1(\xi) : C^{\infty}(M^A) \times \cdots \times C^{\infty}(M^A) \longrightarrow \mathbb{R}$ the map such that, for any $F_1, \ldots, F_q \in C^{\infty}(M^A)$,

$$Q_1(\xi)(F_1, \ldots, F_q) = [1^* \circ Q_4(F_1 \circ s_{M^A}, \ldots, F_q \circ s_{M^A})](\xi).$$

The map

$$Q_1^i(\xi) : C^{\infty}(M^A) \longrightarrow \mathbb{R}, F_i \longmapsto Q_1^i(F_i)$$

such that, for any $\xi \in M^A$

$$\left[Q_1^i(F_i)\right](\xi) = [1^* \circ Q_4(F_1 \circ s_{M^A}, \ldots, F_i \circ s_{M^A}, \ldots, F_q \circ s_{M^A})](\xi)$$

satisfies

$$\left[Q_1^i(F_i + G)\right](\xi) = \left[Q_1^i(F_i)\right](\xi) + \left[Q_1^i(G)\right](\xi),$$

$$\left[Q_1^i(F_i \cdot G)\right](\xi) = \left[Q_1^i(F_i)\right](\xi) \cdot G(\xi) + F_i(\xi) \cdot \left[Q_1^i(G)\right](\xi).$$

Then $Q_1(\xi) \in \Lambda^q T_\xi M^A$ for any $\xi \in M^A$, i.e, Q_1 is section the q-tangent bundle $(\Lambda^q TM^A, \pi_q, M^A)$ what completes the proof.

□

Corollary 4.1. *We have the following isomorphims:*

(i) $\mathfrak{X}^q\left(M^A\right) \simeq \mathcal{D}er_{\mathbb{R}}^q[C^{\infty}(M^A)]$.

(ii) $\mathcal{D}er_{\mathbb{R}}^q[C^{\infty}(M^A)] \simeq \mathcal{D}er_A^q[C^{\infty}(M^A, A)]$.

(iii) $\mathcal{D}er_A^q[C^\infty(M^A,A)] \simeq \mathcal{D}er_{\mathbb{R}}^q\left[C^\infty(M), C^\infty(M^A,A)\right]$.
(iv) $\mathcal{D}er_{\mathbb{R}}^q\left[C^\infty(M), C^\infty(M^A,A)\right] \simeq \mathfrak{X}^q\left(M^A\right)$.

Proof.

(i) Let Q be a section of $\left(\Lambda^q TM^A, \pi_q, M^A\right)$. Then, the linear map

$$\Phi_1 : \mathfrak{X}^q\left(M^A\right) \longrightarrow \mathcal{D}er_{\mathbb{R}}^q[C^\infty(M^A)], Q \longmapsto \widetilde{Q}$$

such that

$$\left[\widetilde{Q}\left(F_1,\ldots,F_q\right)\right](\xi) = Q(\xi)\left(F_1,\ldots,F_q\right),$$

is an isomorphism of modules, for all $F_1,\ldots,F_q \in C^\infty(M^A)$ and $\xi \in M^A$.

(ii) The linear map

$$\Phi_2 : \mathcal{D}er_{\mathbb{R}}^q[C^\infty(M^A)] \longrightarrow \mathcal{D}er_A^q[C^\infty(M^A,A)], D_1 \longmapsto \widetilde{D}_1$$

such that

$$\widetilde{D}_1\left(\varphi_1,\ldots,\varphi_q\right) = \sum_{\alpha_1\ldots\alpha_q} D_1\left(a^*_{\alpha_1}\circ\varphi_1,\ldots,a^*_{\alpha_q}\circ\varphi_q\right) a_{\alpha_1\ldots\alpha_q},$$

is an isomorphism, for any $\varphi_1,\ldots,\varphi_q \in C^\infty(M^A,A)$.

(iii) The linear map

$$\Phi_3 : \mathcal{D}er_A^q[C^\infty(M^A,A)] \longrightarrow \mathcal{D}er_{\mathbb{R}}^q\left[C^\infty(M), C^\infty(M^A,A)\right], D_2 \longmapsto \widetilde{D}_2$$

such that,

$$\widetilde{D}_2\left(f_1,\ldots,f_q\right) = D_2\left(f_1^A,\ldots,f_q^A\right),$$

is an isomorphism, for any $f_1,\ldots,f_q \in C^\infty(M)$.

(iv) The linear map

$$\Phi_4 : \mathcal{D}er_{\mathbb{R}}^q\left[C^\infty(M), C^\infty(M^A,A)\right] \longrightarrow \mathfrak{X}^q\left(M^A\right), D_3 \longmapsto P$$

such that,

$$P(\xi)\left(F_1,\ldots,F_q\right) = \left[1^*\circ D_3\left(F_1\circ s_{M^A},\ldots,F_q\circ s_{M^A}\right)\right](\xi)$$

is an isomorphism, for all $F_1,\ldots,F_q \in C^\infty(M^A)$ and $\xi \in M^A$.

$\square$

Corollary 4.2. *The set $\mathfrak{X}^q\left(M^A\right)$ is a $C^\infty(M^A,A)$-module of q-vector fields on M^A.*

Proposition 4.1. *If $\mu : A \times \cdots \times A \longrightarrow A$ is a q-linear map and Q is a q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$, then:*

(1) *for all $f_1, \dots, f_{2q-1} \in C^\infty(M)$, we have*

$$Q(f_1 \cdots f_q, f_{q+1}, \dots, f_{2q-1}) = \sum_{i=1}^{q} f_1^A \cdots \widehat{f_i^A} \cdots f_q^A Q(f_i, f_{q+1}, \dots, f_{2q-1});$$

(2) *for all $f_1, \dots, f_q \in C^\infty(M)$, we have*

$$\widetilde{Q}(a_\alpha^* \circ f_1^A, f_2^A, \dots, f_q^A) = a_\alpha^* \circ \widetilde{Q}(f_1^A, \dots, f_q^A),$$

where $\widetilde{Q}$ is a unique q-derivation of $C^\infty(M^A, A)$ such as $\widetilde{Q}(f_1^A, \dots, f_q^A) = Q(f_1, \dots, f_q)$.

(3) *for all $f_1, \dots, f_{2q-1} \in C^\infty(M)$, we have*

$$\begin{aligned}&\widetilde{Q}(\mu \circ (f_1^A, \dots, f_q^A), f_{p+1}^A, \dots, f_{2q-1}^A)\\ &= \sum_{i=1}^{q} \mu \circ (f_1^A, \dots, f_{i-1}^A, Q(f_i, f_{q+1}, \dots, f_{2q-1}), f_{i+1}^A, \dots, f_q^A);\end{aligned}$$

the term $\widehat{f_i^A}$ means that f_i^A is omitted.

5. Schouten bracket of multivector fields on Weil bundle

In what follows, we consider $\mathfrak{X}^p(M^A)$ as the set of all p-derivations of $C^\infty(M^A, A)$. We denote by

$$\mathfrak{X}^\bullet(M^A) := \bigoplus_{p \in \mathbb{N}} \mathfrak{X}^p(M^A)$$

the $C^\infty(M^A, A)$-graded module of multivector fields on M^A.

For $p, q \in \mathbb{N}$, a (p,q)-shuffle is a permutation σ of the set $\{1, \dots, p+q\}$, such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. One denotes $\mathfrak{S}_{p,q}$ the set of all (p,q)-shuffles. For a shuffle $\sigma \in \mathfrak{S}_{p,q}$, we denote $\varepsilon(\sigma)$, the signature of σ. It also convenient to define $\mathfrak{S}_{p,-1} := \emptyset$ and $\mathfrak{S}_{-1,q} := \emptyset$, for $p, q \in \mathbb{N}$ (see [3]).

For $P \in \mathfrak{X}^p\left(M^A\right)$ and $Q \in \mathfrak{X}^q(M^A)$, their wedge product $P \wedge Q \in \mathfrak{X}^{p+q}(M^A)$ is the skew-symmetric $(p+q)$-derivation of $C^\infty(M^A, A)$, defined by

$$P \wedge Q(\varphi_1, \dots, \varphi_{p+q}) = \sum_{\sigma \in \mathfrak{S}_{p,q}} \varepsilon(\sigma) P(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(p)}) Q(\varphi_{\sigma(p+1)}, \dots, \varphi_{\sigma(p+q)}),$$

for any $\varphi_1, \dots, \varphi_{p+q} \in C^\infty(M^A,\ A)$.

Notice that, in particular, $\varphi \wedge P = P \wedge \varphi = \varphi P$, for $\varphi \in \mathfrak{X}^0(M^A) = C^\infty(M^A, A)$.

Proposition 5.1. *The wedge product*

$$\begin{aligned} \wedge : \quad \mathfrak{X}^\bullet(M^A) \times \mathfrak{X}^\bullet(M^A) & \to \mathfrak{X}^\bullet(M^A) \\ (P, Q) \qquad & \mapsto P \wedge Q \end{aligned}$$

endows $\mathfrak{X}^\bullet(M^A)$ with an associative graded $C^\infty(M^A, A)$-algebra structure. Moreover, it is graded commutative: for all $P \in \mathfrak{X}^p(M^A)$ and $Q \in \mathfrak{X}^q(M^A)$, we have

$$P \wedge Q = (-1)^{pq} Q \wedge P.$$

$\mathfrak{X}^\bullet(M^A)$ is called the exterior algebra or graded algebra.

Theorem 5.1. *Let P and Q both be p and q-vector fields on M^A, respectively. Then, the bracket $[P, Q]_S$ of P and Q, defined by:*

$$\begin{aligned} (5.1) \qquad & [P, Q]_S(\varphi_1, \varphi_2 \ldots, \varphi_{p+q-1}) \\ = & \sum_{\sigma \in \mathfrak{S}_{q,p-1}} \varepsilon(\sigma) P\left(Q(\varphi_{\sigma(1)}, \ldots, \varphi_{\sigma(q)}), \varphi_{\sigma(q+1)}, \ldots, \varphi_{\sigma(p+q-1)}\right) \\ & -(-1)^{(p-1)(q-1)} \sum_{\sigma \in \mathfrak{S}_{p,q-1}} \varepsilon(\sigma) Q\Big(P(\varphi_{\sigma(1)}, \ldots, \varphi_{\sigma(p)}), \varphi_{\sigma(p+1)}, \ldots, \varphi_{\sigma(p+q-1)}\Big) \end{aligned}$$

for any $\varphi_1, \varphi_2, \ldots, \varphi_{p+q-1} \in C^\infty(M^A, A)$, is a $(p+q-1)$-vector field on M^A.

We call the above bracket, the Schouten-Nijenhuis bracket of multivector fields on weil bundle.

Theorem 5.2. *Let P, Q, R be p, q and r-vector fields on M^A, respectively. Let φ, ψ both be smooth functions on M^A with values in A. The Schouten-Nijenhuis bracket verifies the following equalities:*

(1) $[\varphi, \psi]_S = 0$;
(2) $[P, Q]_S = -(-1)^{(p-1)(q-1)}[P, Q]_S$;
(3) $[P, Q \wedge R]_S = [P, Q]_S \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]_S$.

Theorem 5.3. *Endowed with Schouten-Nijenhuis bracket, $\mathfrak{X}^\bullet(M^A)$ is a Lie graded algebra over A and its graded Jacobi identity is given by:*

$$\begin{aligned} (5.2) \qquad & (-1)^{(p-1)(r-1)}[P, [Q, R]_S]_S + (-1)^{(q-1)(p-1)}[Q, [R, P]_S]_S \\ & +(-1)^{(r-1)(q-1)}[R, [P, Q]_S]_S = 0 \end{aligned}$$

for all P, Q, R multivector fields on M^A of degree p, q and r, respectively.

6. Prolongations to M^A of multivector fields on M

Proposition 6.1. *If $\theta : C^\infty(M) \times \cdots \times C^\infty(M) \longrightarrow C^\infty(M)$ is a q-vector field on M, then there exists a unique q-derivation $\theta^A : C^\infty(M^A, A) \times \cdots \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$ which skew-symmetric A-multilinear, such that,*

$$\theta^A \left(f_1^A, \ldots, f_q^A\right) = \left[\theta\left(f_1, \ldots, f_q\right)\right]^A \tag{6.1}$$

for any $f_1, \ldots, f_q \in C^\infty(M)$.

Proof. If θ is a q-dreivation, then the map $C^\infty(M) \times \cdots \times C^\infty(M) \longrightarrow C^\infty(M^A, A), (f_1, \ldots, f_q) \longmapsto \left[\theta\left(f_1, \ldots, f_q\right)\right]^A$ is a q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$. According to the Theorem 4.1, there exists a unique skew-symmetric A-multilinear derivation of degree q,

$$\theta^A : C^\infty(M^A, A) \times \cdots \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

such that, for any $f_1, \ldots, f_q \in C^\infty(M)$,

$$\theta^A \left(f_1^A, \ldots, f_q^A\right) = \left[\theta\left(f_1, \ldots, f_q\right)\right]^A.$$

□

Proposition 6.2. *If $\theta, \theta_1, \theta_2 \in \mathfrak{X}^q(M)$ and ,$f \in C^\infty(M)$, then*

(i) $(\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A$;

(ii) $(f \cdot \theta)^A = f^A \cdot \theta^A$.

Moreover, the map

$$\mathfrak{X}^q(M) \longrightarrow \mathcal{D}er_A^q[C^\infty(M^A, A)], \theta \longmapsto \theta^A$$

is an injective morphism of $\mathbb{R}$-vector spaces.

Proposition 6.3. *For all $\theta, \theta_1, \theta_2 \in \mathfrak{X}^\bullet(M)$ and $f \in C^\infty(M)$,*

$$(\theta_1 \wedge \theta_2)^A = \theta_1^A \wedge \theta_2^A \text{ and } (f \cdot \theta)^A = f^A \cdot \theta^A.$$

Proof. If $\theta_1 \in \mathfrak{X}^p(M)$ and $\theta_2 \in \mathfrak{X}^q(M)$, then $\theta_1 \wedge \theta_2 \in \mathfrak{X}^{p+q}(M) \simeq \mathcal{D}er_{\mathbb{R}}^{p+q}[C^\infty(M)]$ and $(\theta_1 \wedge \theta_2)^A$ is the unique skew-symmetric A-multilinear $(p+q)$-derivation such that,

$$(\theta_1 \wedge \theta_2)^A \left(f_1^A, \ldots, f_{p+q}^A\right) = \left[(\theta_1 \wedge \theta_2)(f_1, \ldots, f_{p+q})\right]^A,$$

for any $f_1, \ldots, f_{p+q} \in C^\infty(M)$. On the other hand,

$$\left(\theta_1^A \wedge \theta_2^A\right)\left(f_1^A, \ldots, f_{p+q}^A\right) = \sum_{\sigma \in \mathfrak{S}_{p,q}} \varepsilon(\sigma)\, \theta_1^A \left(f_{\sigma(1)}^A, \ldots, f_{\sigma(p)}^A\right) \theta_2^A \left(f_{\sigma(p+1)}^A, \ldots, f_{\sigma(p+q)}^A\right).$$

From (6.1) and by straightforward calculations, we get

$$\left(\theta_1^A \wedge \theta_2^A\right)\left(f_1^A, \ldots, f_{p+q}^A\right) = \left[(\theta_1 \wedge \theta_2)(f_1, \ldots, f_{p+q})\right]^A$$

for any $f_1, \ldots, f_{p+q} \in C^\infty(M)$. Therefore $(\theta_1 \wedge \theta_2)^A = \theta_1^A \wedge \theta_2^A$. □

We denote by, $[\theta_1, \theta_2]_M$, the Schounten bracket on M (see [3], [8]).

Proposition 6.4. *For any* $\theta_1, \theta_2 \in \mathfrak{X}^\bullet(M)$, *we have*

$$[\theta_1, \theta_2]_M^A = \left[\theta_1^A, \theta_2^A\right]_S.$$

Proof. If $\theta_1 \in \mathfrak{X}^p(M)$ and $\theta_2 \in \mathfrak{X}^q(M)$, then $[\theta_1, \theta_2]_M \in \mathfrak{X}^{p+q-1}(M) \simeq \mathcal{D}er_{\mathbb{R}}^{p+q-1}[C^\infty(M)]$. According to the Proposition 6.1, $[\theta_1, \theta_2]_M^A$ is the unique skew-symmetric A-multilinear $(p+q-1)$-derivation such that,

$$[\theta_1, \theta_2]_M^A \left(f_1^A, \ldots, f_{p+q-1}^A\right) = \left([\theta_1, \theta_2]_M (f_1, \ldots, f_{p+q-1})\right)^A,$$

for any $f_1, \ldots, f_{p+q-1} \in C^\infty(M)$. On the other hand, according to the Theorem 5.1,

$$\begin{aligned} &\left[\theta_1^A, \theta_2^A\right]_S \left(f_1^A, \ldots, f_{p+q-1}^A\right) \\ = &\sum_{\sigma \in S_{q,p-1}} \varepsilon(\sigma)\, \theta_1^A \left[\theta_2^A \left(f_{\sigma(1)}^A, \ldots, f_{\sigma(q)}^A\right), f_{\sigma(q+1)}^A, \ldots, f_{\sigma(p+q-1)}^A\right] \\ &- (-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{q,p-1}} \varepsilon(\sigma)\, \theta_2^A \left[\theta_1^A \left(f_{\sigma(1)}^A, \ldots, f_{\sigma(p)}^A\right), f_{\sigma(p+1)}^A, \ldots, f_{\sigma(p+q-1)}^A\right]. \end{aligned}$$

From (6.1) and by straightforward calculations, we get

$$\left[\theta_1^A, \theta_2^A\right]_S \left(f_1^A, \ldots, f_{p+q-1}^A\right) = \left([\theta_1, \theta_2]_M (f_1, \ldots, f_{p+q-1})\right)^A$$

for any $f_1, \ldots, f_{p+q-1} \in C^\infty(M)$. Therefore $[\theta_1, \theta_2]_M^A = \left[\theta_1^A, \theta_2^A\right]_S$. □

Proposition 6.5. *For any* $P \in \mathcal{D}er_A^p[C^\infty(M^A, A)]$, *there is the unique* $\widetilde{P} \in \mathfrak{L}_{sks}^p(\Omega^1(M^A, A)$,

$C^\infty(M^A, A))$ *such that*

$$P(\varphi_1, \dots, \varphi_p) = \widetilde{P}\left(d^A(\varphi_1), \dots, d^A(\varphi_p)\right)$$

for any $\varphi_1, \dots, \varphi_p \in C^\infty(M^A, A)$.

Proposition 6.6. *If* $\eta \in \mathcal{D}er^p_{\mathbb{R}}[C^\infty(M)]$*, then* η^A *is the unique* p*-derivation of* $\mathcal{D}er^p_A[C^\infty(M^A, A)]$ *such that*

$$\eta^A\left(d^A\left(f_1^A\right), \dots, d^A\left(f_p^A\right)\right) = \left[\eta\left(d(f_1), \dots, d(f_p)\right)\right]^A,$$

for any $f_1, \dots, f_p \in C^\infty(M)$.

For each integer $2 \leq r \leq n$, a Leibniz bracket of order r or a r-derivation on M is the map $\{,\} : \overbrace{C^\infty(M) \times \cdots \times C^\infty(M)}^{r-times} \longrightarrow C^\infty(M), (f_1, \dots f_r) \longmapsto \{f_1, \dots, f_r\}$ such that

i) $\{,\}$ is skew-symmetric $\mathbb{R}$-multilinear,
ii) $\{,\}$ satisfies the Leibniz rule, that is, for any $f, g, f_2, \dots, f_r \in C^\infty(M)$

$$\{fg, f_2, \dots, f_r\} = f\{g, f_2, \dots, f_r\} + g\{f, f_2, \dots, f_r\}.$$

For any Leibniz bracket $\{,\}_M$ on M, there is the unique Leibniz bracket $\{,\}_{M^A}$ on M^A and such that

$$\{f_1, \dots, f_r\}_M^A = \{f_1^A, \dots, f_r^A\}_{M^A} = \eta^A\left(d^A\left(f_1^A\right), \dots, d^A\left(f_r^A\right)\right).$$

In particular, if M is a Poisson manifold [2], then there exits $\pi \in \mathfrak{X}^2(M)$ and $A_J \in \mathfrak{X}^3(M)$ such that

$$\{f, g\}_M^A = \{f^A, g^A\}_{M^A} = \pi^A\left(d^A\left(f^A\right), d^A\left(g^A\right)\right)$$
$$J^A(f^A, g^A, h^A) = [J(f, g, h)]^A = A_J^A\left(d^A\left(f^A\right), , d^A\left(g^A\right), d^A\left(h^A\right)\right)$$

where the Jacobiator of $\{,\}$ is the map $J : C^\infty(M) \times C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ such that

$$J(f, g, h) = \{f, \{g, h\}\} + \{g, \{h, g\}, f\} + \{h, \{f, g\}\}$$

for any $f, g, h \in C^\infty(M)$.

7. Multivector fields on M^A which coming from multi-derivations of A

Proposition 7.1. *If $d : A \times \cdots \times A \longrightarrow A$ is a q-derivation of A, then the map*

$$\bar{d} : C^\infty(M^A, A) \times \cdots \times C^\infty(M^A, A) \to C^\infty(M^A, A), (\varphi_1, \dots \varphi_q) \mapsto d \circ (\varphi_1, \dots \varphi_q)$$

is the unique q-derivation which is A-multilinear such that for all $f_1, \dots f_q \in C^\infty(M)$ the q-derivation

$$d^* : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M^A, A), (f_1, \dots f_q) \mapsto d \circ (f_1^A, \dots f_q^A)$$

verifies

$$d^*(f_1, \dots f_q) = \bar{d}(f_1^A, \dots f_q^A).$$

Proof. There is no difficulty to verify this result. □

Proposition 7.2. *Let $\theta : C^\infty(M) \times \ldots \times C^\infty(M) \to C^\infty(M)$ be q-derivation of $C^\infty(M)$. For all $d, d_1, d_2 \in Der^q_{\mathbb{R}}(A)$, $a \in A$ and $\varphi_1, \dots, \varphi_{2q-1} \in C^\infty(M^A, A)$, we have*

(1) $\overline{(d_1 + d_2)} = \overline{d_1} + \overline{d_2}$.
(2) $\overline{[d_1, d_2]_S} = [\overline{d_1}, \overline{d_2}]_S$
(3) $\overline{(a \cdot d)} = a \cdot \bar{d}$.
(4) $[\bar{d}, \theta^A]_S(\varphi_1, \dots, \varphi_{2q-1}) = \sum_{\sigma \in \mathfrak{S}_{q,q-1}} \varepsilon(\sigma) d \circ \Big[\theta^A\big(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(q)}\big), \varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}\big) + (-1)^{q^2} \sum_{i=1}^{q} \big(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(i-1)}, \theta^A\big(\varphi_{\sigma(i)}, \varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}\big), \varphi_{\sigma(i+1)}, \dots, \varphi_{\sigma(q)}\big)\Big]$.

In particular for $q = 1$, we get $[\bar{d}, \theta^A]_S = 0$.

Proof. The first three assertions can be verified easily.

(4) For any $\varphi_1, \dots, \varphi_{2q-1} \in C^\infty(M^A, A)$, we have:

$$\begin{aligned} &[\bar{d}, \theta^A]_S(\varphi_1, \dots, \varphi_{2q-1}) \\ = &\sum_{\sigma \in \mathfrak{S}_{q,q-1}} \varepsilon(\sigma) \bar{d}(\theta^A(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(q)}), \varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}) \\ &-(-1)^{(q-1)^2} \sum_{\sigma \in \mathfrak{S}_{q,q-1}} \varepsilon(\sigma) \theta^A\big(\bar{d}(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(q)}), \varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}\big). \end{aligned} \tag{7.1}$$

Thus, when we use the definition of d^* and the third assertion of the Proposition 4.1, we obtain:

$$\begin{aligned}
&[\bar{d}, \theta^A]_S(\varphi_1, \dots, \varphi_{2q-1}) \\
&= \sum_{\sigma \in \mathfrak{S}_{q,q-1}} \varepsilon(\sigma) d \circ \big(\theta^A(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(q)}), \varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}\big) \\
&- (-1)^{q^2} \sum_{\sigma \in \mathfrak{S}_{q,q-1}} \varepsilon(\sigma) \sum_{i=1}^{q} d \circ \big(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(i-1)}, \theta^A(\varphi_{\sigma(i)}, \\
&\varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}), \varphi_{\sigma(i+1)}, \dots, \varphi_{\sigma(q)}\big).
\end{aligned}$$

After a simple computation, we get

$$\begin{aligned}
&[\bar{d}, \theta^A]_S(\varphi_1, \dots, \varphi_{2q-1}) \\
&= \sum_{\sigma \in \mathfrak{S}_{q,q-1}} \varepsilon(\sigma) d \circ \Big[\big(\theta^A(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(q)}), \varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}\big) \\
&+ (-1)^{q^2} \sum_{i=1}^{q} \big(\varphi_{\sigma(1)}, \dots, \varphi_{\sigma(i-1)}, \theta^A(\varphi_{\sigma(i)}, \varphi_{\sigma(q+1)}, \dots, \varphi_{\sigma(2q-1)}), \\
&\varphi_{\sigma(i+1)}, \dots, \varphi_{\sigma(q)}\big)\Big].
\end{aligned}$$

In particular, for $q = 1$, we have $[\bar{d}, \theta^A]_S = d \circ (\theta^A(\varphi_1)) - d \circ (\theta^A(\varphi_1)) = 0.$

□

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DEPARTMENT OF MATHEMATICS
ECOLE NORMALE SUPÉRIEURE (ENS), MARIEN NGOUABI UNIVERSITY
AVENUE DES PREMIERS JEUX AFRICAINS, BRAZZAVILLE,
CONGO.
Email address: vannborhen@gmail.com

DEPARTMENT OF MATHEMATICS
ECOLE NORMALE SUPÉRIEURE (ENS), MARIEN NGOUABI UNIVERSITY
AVENUE DES PREMIERS JEUX AFRICAINS, BRAZZAVILLE,
CONGO.
Email address: nmahomouk1@gmail.com

DEPARTMENT OF MATHEMATICS
ECOLE NORMALE SUPÉRIEURE (ENS), MARIEN NGOUABI UNIVERSITY
AVENUE DES PREMIERS JEUX AFRICAINS, BRAZZAVILLE,
CONGO.
Email address: angemalokog@gmail.com